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DEFORMATIONS OF THE TANGENT BUNDLE AND THEIR RESTRICTION TO STANDARD RATIONAL CURVES

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Abstract. We study the deformations of the tangent bundle of a uniruled manifold X by restricting them to standard rational curves on X. By employing an idea from holomorphic symplectic geometry, we prove that, if $H^{2i}(X, \Omega^1_X) = 0$ for all $i \geq 0$, the splitting type on a standard rational curve remains unchanged under small deformations of the tangent bundle.

1. Introduction

By Grothendieck's decomposition theorem, vector bundles on $\mathbf{P}_1$ are sums of line bundles. An important tool for studying vector bundles on a uniruled manifold X is to study the restrictions of the vector bundles to rational curves on X. Of particular interest is the restriction to rational curves of simple type. For example, for vector bundles on the projective space, the study of their restrictions to lines has been very successful (e.g. [OSS]). Lines on the projective space are examples of standard rational curves. Recall that a rational curve $f : \mathbf{P}_1 \to X$ is *standard* if

$$f^*T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$$

for some non-negative integers p, q. This is the 'simplest' decomposition type for $f^*T(X)$ as long as it is semi-positive on $\mathbf{P}_1$. It is natural in the study of vector bundles on uniruled manifolds to examine their restrictions to standard rational curves. In this regard, the following question naturally arises in the study of deformations of the tangent bundle.

Question 1.1. Let X be a uniruled projective manifold and $f : \mathbf{P}_1 \to X$ be a standard rational curve. Denote by Δ the unit disc. Let $\{V_t, t \in \Delta\}$ be deformations of the tangent bundle of X, i.e., a holomorphic family of vector bundles with $V_0 \cong T(X)$. Is there some $0 < \epsilon < 1$, such that for any $|t| < \epsilon$, $f^*V_t \cong f^*V_0$?

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The most simple-minded deformation of the tangent bundle $T(X)$ arises from $\mathrm{Pic}^0(X)$. For a family of line bundles $\{L_t, t \in \Delta\}$ with $L_0 \cong \mathcal{O}$, the tensor product $V_t := T(X) \otimes L_t$ yields a deformation of $T(X)$. Since $\mathrm{Pic}^0(\mathbf{P}_1) \cong \{1\}$, a deformation of this type does not affect the isomorphism type of f^*V_t, giving an affirmative example for Question 1.1.

However, a simple example shows us that the answer is not always affirmative. In fact, on $\mathbf{P}_1$, we have a family of vector bundles $\{W_t, t \in \Delta\}$ with $W_0 \cong \mathcal{O}(2) \oplus \mathcal{O}$ and $W_t \cong \mathcal{O}(1)^2$ for $t \neq 0$. Let $X := \mathbf{P}_1 \times E$ for some elliptic curve E and let $\pi : X \to \mathbf{P}_1$ be the projection. The family of vector bundles $\{V_t := \pi^* W_t, t \in \Delta\}$ is a deformation of the tangent bundle $V_0 \cong T(X)$. The $\mathbf{P}_1$-factors in $X = \mathbf{P}_1 \times E$ are standard rational curves in X. The splitting type of V_t on these standard rational curves changes, yielding a negative answer to Question 1.1.

So the interesting question is what additional conditions on X are needed to guarantee an affirmative answer to Question 1.1. In this article, we will show that the following Hodge-theoretic conditions on X work:

Theorem 1.2. *Let X be a uniruled projective manifold with $H^{2i}(X, \Omega^1_X) = 0$ for all $i \geq 0$. Then for any standard rational curve $f : \mathbf{P}_1 \to X$ and any deformation $\{V_t, t \in \Delta\}$ of $V_0 \cong T(X)$, there exists some $0 < \epsilon < 1$ with $f^*V_t \cong f^*V_0$ for every $|t| < \epsilon$.*

This will be proved in Section 2. The key idea of the proof comes from holomorphic symplectic geometry. One can see that the behavior of V_t on standard rational curves can be interpreted as that of certain rational curves in the underlying manifold of the dual bundle V_t^*. Since the cotangent bundle V_0^* is a holomorphic symplectic manifold, we can employ an argument of Wierzba [Wi] on the deformation of rational curves in holomorphic symplectic manifolds. Wierzba's argument relies on a construction of symplectic deformations due to Kaledin-Verbitsy [KV]. The construction of [KV] does not apply to our situation directly. We will give an analogous construction for the cotangent bundle. The additional vanishing conditions $H^{2i}(X, \Omega^1_X) = 0$ are needed to extend this construction to the underlying manifolds of the dual bundles V_t^*.

Standard rational curves play a particularly important role in the geometry of Fano manifolds of Picard number 1. Theorem 1.2 suggests the following question.

Question 1.3. Let X be a Fano manifold of Picard number 1 with $H^{2i}(X, \Omega^1_X) = 0$ for all $i \geq 0$. For any deformation $\{V_t, t \in \Delta\}$ of $V_0 \cong T(X)$, does there exist some $0 < \epsilon < 1$ with $V_t \cong V_0$ for every $|t| < \epsilon$?

As an illustration of how Theorem 1.2 can be used in the study of deformation of the tangent bundle, we will use it in Theorem 3.1 to answer Question 1.3 in some special cases.

2. Proof of Theorem 1.2

For a complex manifold X, Ω^1_X denotes the locally free sheaf of differentials on X. By abuse of notation, we will use Ω^1_X to denote also the complex manifold underlying the cotangent bundle of X. For a vector bundle W, denote by $\mathbf{P}W$ its projectivization as the set of 1-dimensional subspaces in the fibers of W.

Put $\Delta \times \mathbf{C} := \{(t,s) \in \mathbf{C} \times \mathbf{C}, |t| < 1\}$. A key ingredient of the proof of Theorem 1.2 is the following construction, inspired by [KV].

Proposition 2.1. *In the setting of Theorem 1.2, there exists a complex manifold $\mathcal{M}$ with a smooth morphism $\varphi : \mathcal{M} \to \Delta \times \mathbf{C}$ such that, denoting by $M_{t,s}$ the fiber $\varphi^{-1}((t,s))$,*
(i) *$M_{t,0}$ is biholomorphic to the underlying complex manifold of the dual bundle V_t^* of V_t (in particular, $M_{0,0}$ is biholomorphic to the underlying complex manifold of the cotangent bundle of X), and*
(ii) *given a rational curve $g_0 : \mathbf{P}_1 \to \mathcal{M}$ with $g_0(\mathbf{P}_1) \subset M_{0,0}$, any deformation $\{g_u : \mathbf{P}_1 \to \mathcal{M}, u \in \Delta\}$, has images in $\mathcal{M}_0 := \cup_{t\in\Delta} M_{t,0}$.*

Proof. Let $\pi : X \times \Delta \to \Delta$ be the natural projection. Denote by $\mathcal{U}$ the vector bundle on $X \times \Delta$ such that

$$\mathcal{U}|_{\pi^{-1}(t)} \cong V_t^* \text{ for each } t \in \Delta.$$

From the vanishing $H^{2i}(X, \Omega_X^1) = 0$ for all $i \geq 0$, the semi-continuity of $\dim H^{2i}(X, V_t^*)$ implies that there exists some $\epsilon > 0$ such that $H^{2i}(X, V_t^*) = 0$ for all $i \geq 0$ for any t with $|t| < \epsilon$. Since the Euler characteristic $\chi(X, V_t^*)$ is constant, we see that

$$\dim H^i(X, V_t^*) = \dim H^i(X, \Omega_X^1) \text{ for each } i \geq 0 \text{ and each } |t| < \epsilon.$$

In particular, $H^1(X, V_t^*) \cong H^1(X, \Omega_X^1) \neq 0$ and $R^1\pi_*\mathcal{U}$ is a vector bundle on $\{|t| < \epsilon\}$. By rescaling, we will assume that $\epsilon = 1$.

Choose a holomorphic section $\xi_t \in H^1(X, V_t^*)$ of the vector bundle $R^1\pi_*\mathcal{U}$ on Δ such that $\xi_0 \in H^1(X, \Omega_X^1)$ is the class of an ample line bundle on X. Let $L \subset R^1\pi_*\mathcal{U}$ be the line subbundle generated by ξ_t. The underlying 2-dimensional complex manifold of L is biholomorphic to $\Delta \times \mathbf{C}$ with the coordinate (t,s) given by

$$L = \{(t, s \cdot \xi_t), t \in \Delta, s \in \mathbf{C}\}.$$

For each (t,s), let $W_{t,s}$ be the isomorphism class of the vector bundle on X defined by the extension

$$0 \longrightarrow V_t^* \longrightarrow W_{t,s} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

corresponding to the extension class $s \cdot \xi_t \in H^1(X, V_t^*)$.

On $X \times L$, we have a vector bundle W such that

$$W|_{X\times(t,0)} \cong \mathcal{O}_X \oplus V_t^* \text{ for each } t \in \Delta$$

$$W|_{X\times(t,s)} \cong W_{t,s} \text{ for each } t \in \Delta, s \in \mathbf{C} \setminus \{0\}.$$

For each (t,s), $W|_{X\times(t,s)}$ contains the subbundle of corank 1 given by V_t^*. Thus $\mathbf{P}V_t^* \subset \mathbf{P}W_{t,s}$ defines a natural hypersurface $H \subset \mathbf{P}W$. Define $\mathcal{M} := \mathbf{P}W \setminus H$ and set $\varphi : \mathcal{M} \to \Delta \times \mathbf{C}$ to be the restriction of the composition

$$\mathbf{P}W \longrightarrow X \times L \longrightarrow L \cong \Delta \times \mathbf{C}.$$

The property (i) is obvious and it remains to check (ii).

Denote by $(\alpha(u), \beta(u)) \in \Delta \times \mathbf{C}$ the image $\varphi \circ g_u(\mathbf{P}_1)$. We need to show that $\beta(u) \equiv 0$. Denote by $M_{t,s} \subset \mathcal{M}$ the fiber $\varphi^{-1}((t,s))$. Then $M_{t,s} = \mathbf{P}W_{t,s} \setminus \mathbf{P}V_t^*$ and there is a natural affine bundle structure $M_{t,s} \to X$. Let $h_u : \mathbf{P}_1 \to X$ be the composition of $g_u : \mathbf{P}_1 \to M_{\alpha(u),\beta(u)}$ with the natural projection $M_{\alpha(u),\beta(u)} \to X$. For each u, h_u is a non-constant morphism. The pull-back bundle $h_u^* W_{\alpha(u),\beta(u)}$ on $\mathbf{P}_1$ has a line subbundle $\mathcal{L}_u$ corresponding to $g_u(\mathbf{P}_1) \subset \mathbf{P}W_{\alpha(u),\beta(u)}$. Since $g_u(\mathbf{P}_1)$ is disjoint from the hypersurface $\mathbf{P}V^*_{\alpha(u)} \subset \mathbf{P}W_{\alpha(u),\beta(u)}$, $\mathcal{L}_u$ is complementary to $h_u^* V_t^* \subset h_u^* W_{\alpha(u),\beta(u)}$, splitting the sequence

$$0 \longrightarrow h_u^* V^*_{\alpha(u)} \longrightarrow h_u^* W_{\alpha(u),\beta(u)} \longrightarrow \mathcal{O}_{\mathbf{P}_1} \longrightarrow 0.$$

It follows that

$$0 = h_u^*(\beta(u) \cdot \xi_{\alpha(u)}) = \beta(u) \cdot h_u^* \xi_{\alpha(u)} \text{ for all } u \in \Delta.$$

Suppose that $\beta(u) \not\equiv 0$. Then $h_u^* \xi_{\alpha(u)} = 0$ for all u. But $h_0^* \xi_{\alpha(0)} = h_0^* \xi_0$ is the pull-back of the ample class $\xi_0 \in H^1(X, \Omega^1_X)$ by a non-constant morphism h_0, a contradiction. This shows that $\beta(u) \equiv 0$. □

The proof of the next proposition is borrowed from [Wi, Proof of Theorem 1.1].

Proposition 2.2. *In the setting of Proposition 2.1, let* $g : \mathbf{P}_1 \to M_{0,0}$ *be a rational curve in* $M_{0,0}$. *Then the germ* $\mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0)$ *of the Hom-scheme* $\mathrm{Hom}(\mathbf{P}_1, \mathcal{M}_0)$ *at* g *has dimension* $\geq 2n+2$ *where* $n = \dim X$.

Proof. Since $M_{0,0}$ is a symplectic manifold, $g^* K_{\mathcal{M}}$ is a trivial bundle on $\mathbf{P}_1$. By [Ko, II.1.2],

$$\dim \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}) \geq \dim \mathcal{M} - \deg(g^* K_{\mathcal{M}}) = 2n+2.$$

By Proposition 2.2 (ii),

$$\mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}) = \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0)$$

and the result follows. □

Recall the following elementary fact.

Lemma 2.3. *Let* $\{E_t, t \in \Delta\}$ *be a family of vector bundles on* $\mathbf{P}_1$. *Assume that*

$$E_0 \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n),\ a_1 \geq \cdots \geq a_n$$

and for $t \neq 0$ *with* $|t|$ *small,*

$$E_t \cong \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_n),\ b_1 \geq \cdots \geq b_n.$$

Then

$$\sum_i a_i = \sum_i b_i,\ a_1 \geq b_1 \text{ and } a_n \leq b_n.$$

Proof. The equality is just the equivalence of Chern classes. To see the two inequalities, just use the semi-continuity of $\dim H^0(\mathbf{P}_1, E_t(-b_1))$ and $\dim H^0(\mathbf{P}_1, E_t^*(b_n))$. □

We are ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $f : \mathbf{P}_1 \to X$ be a standard rational curve with $f^* T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$.

If $p = n-1$, the vector bundle $\mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ on $\mathbf{P}_1$ is locally rigid by Lemma 2.3. So there is nothing to prove.

Assume $p < n-1$. It is easy to see by a repeated application of Lemma 2.3 that a non-trivial small deformation of the vector bundle $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$ on $\mathbf{P}_1$ must be of the form $\mathcal{O}(1)^{p+2} \oplus \mathcal{O}^{n-p-2}$. Thus to prove the theorem, it suffices to show that $\dim H^0(\mathbf{P}_1, f^*V_t^*) \geq n-p-1$ for $|t| < \epsilon$.

Let $g : \mathbf{P}_1 \to M_{0,0} \cong V_0^*$ be the rational curve given by a non-zero section of $f^*V_0^*$. We will define a number of morphisms associated to the germ $\mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0)$. From Proposition 2.1 (i), we have natural projections $\psi : M_{0,0} \to X$ and $\tilde{\psi} : \mathcal{M}_0 \to X$, which induce morphisms

$$\psi_* : \mathrm{Hom}_g(\mathbf{P}_1, M_{0,0}) \longrightarrow \mathrm{Hom}_f(\mathbf{P}_1, X) \text{ and } \tilde{\psi}_* : \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0) \longrightarrow \mathrm{Hom}_f(\mathbf{P}_1, X)$$

such that $\tilde{\psi}_*|_{\mathrm{Hom}_g(\mathbf{P}_1, M_{0,0})} = \psi_*$. For $h \in \tilde{\psi}_*^{-1}(f) \subset \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0)$, define $\eta(h) := \bar{\varphi}(h(\mathbf{P}_1))$, where $\bar{\varphi} : \mathcal{M}_0 \to \Delta$ is the morphism sending $M_{t,0} \subset \mathcal{M}_0$ to t. This defines a morphism $\eta : \tilde{\psi}_*^{-1}(f) \to \Delta_0$, where Δ_0 denotes the germ of Δ at 0. Then

$$\eta^{-1}(0) = \tilde{\psi}_*^{-1}(f) \cap \mathrm{Hom}_g(\mathbf{P}_1, M_{0,0}) = \psi_*^{-1}(f).$$

In summary, we have

$$\begin{array}{ccccccc} \mathrm{Hom}_f(\mathbf{P}_1, X) & \overset{\tilde{\psi}_*}{\longleftarrow} & \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0) & \supset & \tilde{\psi}_*^{-1}(f) & \overset{\eta}{\longrightarrow} & \Delta_0 \\ \| & & \cup & & \cup & & \cup \\ \mathrm{Hom}_f(\mathbf{P}_1, X) & \overset{\psi_*}{\longleftarrow} & \mathrm{Hom}_g(\mathbf{P}_1, M_{0,0}) & \supset & \psi_*^{-1}(f) & \longrightarrow & \{0\}. \end{array}$$

Since g comes from a section of V_0^*, an element h of the germ $\mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0)$ comes from a section of $(\tilde{\psi}_*(h))^*V_t^*$ for some t. In particular, if $h \in \tilde{\psi}_*^{-1}(f)$, it comes from a section of $f^*V_t^*$ with $t = \eta(h)$. We conclude that

$$\dim \eta^{-1}(t) = \dim H^0(\mathbf{P}_1, f^*V_t^*) \text{ and } \dim \psi_*^{-1}(f) = \dim H^0(\mathbf{P}_1, f^*V_0^*) = n-p-1.$$

Note that $\dim \mathrm{Hom}_f(\mathbf{P}_1, X) = n+p+2$ because the deformations of standard rational curves are unobstructed. Also, $\dim \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0) \geq 2n+2$ by Proposition 2.2. It follows that

$$\dim \tilde{\psi}_*^{-1}(f) \geq \dim \mathrm{Hom}_g(\mathbf{P}_1, \mathcal{M}_0) - \dim \mathrm{Hom}_f(\mathbf{P}_1, X) \geq (2n+2)-(n+p+2) = n-p.$$

Since $\dim \tilde{\psi}_*^{-1}(f) \geq n-p$ and $\dim \psi_*^{-1}(f) = n-p-1$, we see that η is surjective and $\dim \eta^{-1}(t) \geq n-p-1$ for $t \in \Delta_0$. It follows that $\dim H^0(\mathbf{P}_1, f^*V_t^*) = \dim \eta^{-1}(t) \geq n-p-1$. This completes the proof. □

3. Rigidity of the tangent bundle under additional conditions on varieties of minimal rational tangents

In this section, we will show how Theorem 1.2 can be used in combination with the theory of varieties of minimal rational tangents to study the deformation of the tangent bundle of a Fano manifold of Picard number 1. We refer the reader to [Hw] for an introduction to the theory of varieties of minimal rational tangents.

Recall that a subvariety $Z \subset \mathbf{P}_{n-1}$ is linearly normal if it is not contained in any hyperplane and $\dim H^0(Z, \mathcal{O}(1)) = n$. Let X be a Fano manifold of Picard number 1. We will consider the following condition for a component $\mathcal{K}$ of the space of rational curves RatCurves(X).

(†) $\mathcal{K}$ is complete and each member of $\mathcal{K}$ is a standard rational curve. For each $x \in X$, let $\mathcal{K}_x \subset \mathcal{K}$ be the closed subscheme parametrizing members of $\mathcal{K}$ through x. Then the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbf{P}T_x(X)$, i.e. the union of tangent directions of members of $\mathcal{K}_x$ at x, is a linearly normal non-singular projective subvariety of the projective space $\mathbf{P}T_x(X)$ with $H^1(\mathcal{C}_x, \mathcal{O}) = 0$.

Theorem 3.1. *Let X be a Fano manifold of Picard number 1 satisfying $H^{2i}(X, \Omega^1_X) = 0$ for $i \geq 0$ and admitting a component $\mathcal{K}$ of* RatCurves(X) *with the property (†). For any deformation $\{V_t, t \in \Delta\}$ of $V_0 \cong T(X)$, there exists some $0 < \epsilon < 1$ with $V_t \cong V_0$ for every $|t| < \epsilon$.*

Proof. Let $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$ be the splitting type of $T(X)$ on a member of $\mathcal{K}$. We have seen in the proof of Theorem 1.2 that a non-trivial small deformation of $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$ on $\mathbf{P}_1$ is $\mathcal{O}(1)^{p+2} \oplus \mathcal{O}^{n-p-2}$. Let $U \subset \mathcal{K} \times \Delta$ be the subset consisting of $([C], t) \in \mathcal{K} \times \Delta$ such that for the normalization $f : \mathbf{P}_1 \to C \subset X$,

$$f^* V_t \cong \mathcal{O}(1)^{p+2} \oplus \mathcal{O}^{n-p-2}.$$

Since U is an open subset by Lemma 2.3, U must be empty by Theorem 1.2.

On the other hand, let $E \subset \mathcal{K} \times \Delta$ be the subset consisting of $([C], t) \in \mathcal{K} \times \Delta$ such that for the normalization $f : \mathbf{P}_1 \to C \subset X$,

$$f^* V_t \ncong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1} \text{ or } \mathcal{O}(1)^{p+2} \oplus \mathcal{O}^{n-p-2}.$$

By Lemma 2.3, E is a closed subset. Since U is empty, if $([C], t) \notin E$, then the normalization $f : \mathbf{P}_1 \to C \subset X$ satisfies $f^* V_t \cong V_0$. From the assumption on $\mathcal{K}$, we know that the closed set E is disjoint from $\mathcal{K} \times \{0\}$. Since $\mathcal{K}$ is complete, there exists an $\epsilon > 0$ with $f^* V_t \cong V_0$ for any $|t| < \epsilon$ and any $f : \mathbf{P}_1 \to X$ belonging to $\mathcal{K}$.

For each $x \in X$ and $\alpha \in \mathcal{C}_x$, there exists a unique member C of $\mathcal{K}_x$ and a germ C_α of C at x whose tangent direction is α (e.g. [Hw, Proposition 1.4]). When $\nu : \mathbf{P}_1 \to C$ is the normalization, α corresponds to the $\mathcal{O}(2)$-factor of $\nu^* T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$. Since for each $|t| < \epsilon$,

$$\nu^* V_t \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1},$$

the $\mathcal{O}(2)$-factor of V_t determines a unique point in $\mathbf{P}V_{t,x}$ corresponding to x and the germ C_α of C at x. This induces a morphism

$$\tau_{x,t} : \mathcal{C}_x \longrightarrow \mathbf{P}V_{t,x}.$$

The set $\{\tau_{x,t}, x \in X, |t| < \epsilon\}$ is a holomorphic family of morphisms with $\tau_{x,0} = \mathrm{Id}_{\mathcal{C}_x}$. Since $\tau_{x,0}$ is a non-degenerate embedding, so is $\tau_{x,t}$ for small $|t|$. Since $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ is linearly normal, so is the embedding $\tau_{x,t}$ for small $|t|$. From $H^1(\mathcal{C}_x, \mathcal{O}) = 0$, all these embeddings are given by the same line bundle. This induces an isomorphism $\psi_{x,t} : T_x(X) \to V_{t,x}$ for small t, determined up to a scalar factor. Since $H^1(X, \mathcal{O}) = 0$ for a Fano manifold X, we can choose the scalar factor in the choice of $\psi_{x,t}$ to define a global isomorphism $\psi_t : T(X) \to V_t$. □

It is easy to see that the property (†) holds for the family $\mathcal{K}$ of lines on an irreducible Hermitian symmetric space X (e.g. [Hw, 1.4.5]). Since $H^{2i}(X, \Omega^1_X) = 0$ for $i \geq 0$ is also well-known, we have the proof of the local deformation-rigidity of the tangent bundle of irreducible Hermitian symmetric spaces. This rigidity would follow also from the vanishing $H^1(X, \mathrm{End}(T(X))) = 0$, which may be proved by Borel-Weil-Bott Theorem. However, to our knowledge, this cohomological vanishing has never been checked.

The conditions in Theorem 3.1 hold also for symplectic Grassmannians (e.g [HM, Proposition 3.2.1]).

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