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REMARKS ON THE KOBAYASHI HYPERBOLICITY OF COMPLEX SPACES

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ABSTRACT. The purpose of this article is threefold. The first is to show the hyperbolicity and the tautness of certain Hartogs type domains. The second is to investigate the hyperbolic imbeddedness of an unbounded domain in a complex space. The third is to consider the uniform convergence of Green functions with multiple poles (Lempert functions with multiple poles, Caratheodory functions with multiple poles, respectively) on decreasing domains.

1. INTRODUCTION

Let X be a complex space and $H : X \times \mathbb{C}^m \to [-\infty, \infty)$ be an upper semicontinuous mapping such that $H(z, w) \geq 0, H(z, \lambda w) = |\lambda| H(z, w), \lambda \in \mathbb{C}, z \in X, w \in \mathbb{C}^m$. Put

$$\Omega_H(X) := \{(z.w) \in X \times \mathbb{C}^m : H(z, w) < 1\}.$$

The domain $\Omega_H(X)$ is said to be a Hartogs type domain. Over the last few years, there have been several results for studying the hyperbolicity (complete hyperbolicity, tautness, respectively) of Hartogs type domains in special cases of H. For instance, Thomas, Thai, Duc and Dieu [15, 16, 19] studied the hyperbolicity (complete hyperbolicity, tautness, respectively) when $H(z, w) = |w|e^{u(z)}$, where u is an upper semicontinuous function on X. Recently, Park [13] also investigated the above problems when $H(z, w) = h(w)e^{u(z)}$, where $h : \mathbb{C}^m \to [-\infty, \infty)$ is upper semicontinuous, $h \not\equiv 0, h(\lambda w) = |\lambda| h(w), \lambda \in \mathbb{C}, w \in \mathbb{C}^m$ and u is an upper semicontinuous function on X.

The first aim of this paper is to show the necessary and sufficient conditions of the hyperbolicity and the tautness of the Hartogs type domains in the general case of H. Namely, we prove the following

Theorem 1.1. *$\Omega = \Omega_H(X)$ is hyperbolic if and only if X is hyperbolic and the function H satisfies the following condition:*

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If $\{z_k\}_{k\geq 1}$ *is a sequence in* X *with* $\lim\limits_{k\to\infty} z_k = z_0 \in X$ *and* $\{w_k\}_{k\geq 1}$ *is a sequence in* $\mathbb{C}^m$ *with* $\lim\limits_{k\to\infty} w_k = w_0 \neq 0$, *then* $\limsup\limits_{k\to\infty} H(z_k, w_k) \neq 0(*)$

Theorem 1.2. *Let* X *be a complex space. Then* $\Omega_H(X)$ *is taut if and only if* X *is taut, the fiber* $\Omega_H(z)$ *is taut for any* $z \in X$ *and* $\log H$ *is a continuous plurisubharmonic function.*

The hyperbolic imbeddedness in the sense of Kobayashi is one of the most important problems of hyperbolic complex analysis. Much attention has been given to this problem, and the results on this problem can be applied to many areas of mathematics, in particular to the extension of holomorphic mappings. For details see [8, 10].

Recall that a complex subspace M of a complex space X is hyperbolically imbedded in X if for distinct $p, q \in \overline{M}$, the closure of M, there are open sets U_p, U_q in X such that $p \in U_p, q \in U_q$ and $d_M(M \cap U_p, M \cap U_q) > 0$, where d_M is the Kobayashi pseudodistance on M.

This problem is studied recently by Thai, Duc and Minh [18]. Namely, they proved the following

Theorem [18, Theorem 3] *Let* M *be a hyperbolic domain in complex space* X. *Assume that for each* $p \in \partial M$, *there are local peak and antipeak plurisubharmonic functions at* p, *both defined on a neighborhood of* p *in* X. *Then* M *is hyperbolically imbedded in* X.

However, in our opinion, the conditions in the above-mentioned theorem are rather strong. The second aim of this work is to give somewhat weaker conditions for the hyperbolic imbeddedness of M. Namely, we prove the following

Theorem 1.3. *Let* X *be a complex space that is an increasing union of hyperbolic domains. Let* M *be an unbounded domain in* X. *Assume that there are local peak and antipeak plurisubharmonic functions at* ∞, *both defined on a neighborhood* U *of* ∞ *in* X *and* $U \cap M$ *is hyperbolic. Then* M *is hyperbolically imbedded in* X.

The convergent behavior of the Kobayashi distances on decreasing domains are studied recently by M. Kobayashi [9]. Namely, he proved the following

Theorem [9, Theorem 1.5] *Let* D *be a bounded domain in* $\mathbb{C}^n$ *with* $\mathcal{C}^1$-*boundary such that there exists a weak peak function for each point of* ∂D. *Let* $\{D_j\}_{j=1}^{\infty}$ *be a decreasing sequence of complete hyperbolic domains converging to* D. *Then the sequence* $\left\{k_{D_j}\right\}_{j=1}^{\infty}$ *converges to* k_D *uniformly on compact sets.*

Here a holomorphic function P in a neighborhood of $\overline{D}$ is called a *weak peak function* for a boundary point $\xi \in \partial D$, if $|P(\xi)| = 1$ and $|P(z)| < 1$ for all $z \in D$.

Thus, the following question arises naturally at this point: Does the similar assertion hold for Green functions with multiple poles (Lempert functions with multiple poles, Caratheodory functions with multiple poles, respectively)?

In fact, Nivoche [12] proved that if $\{D_j\}_{j=1}^{\infty}$ is an increasing sequence of domains converging to a domain $D \subset \mathbb{C}^n$, then the sequence of the Green functions with single pole $\{g_{D_j}\}_{j=1}^{\infty}$ converges uniformly to g_D on compact subsets of $\overline{D}$.

The last part of this work is to study the uniform convergence of Green functions with multiple poles (Lempert functions with multiple poles, Caratheodory functions with multiple poles, respectively) on decreasing domains. Namely, we prove the following

Theorem 1.4. *Let D be a strictly hyperconvex domain in $\mathbb{C}^n$, and $S = \{a_1, a_2, ..., a_N\} \subset D$. Assume that D_k is a hyperconvex domain defined by $D_k = \{z \in \Omega : \varrho(z) < \frac{1}{k}\}$ for every $k \geq 1$. Then the sequence $\{g_{D_k}\}_{k=1}^{\infty}$ converges uniformly to g_D on compact subsets of $\overline{D}$.*

Theorem 1.5. *Let D be a strongly pseudoconvex domain with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$ and $\{D_k\}_{k=1}^{\infty}$ be a decreasing sequence of domains in $\mathbb{C}^n$ converging to D. Then the sequence $\{C_{D_k}\}_{k=1}^{\infty}$ of the Caratheodory distances converges to C_D uniformly on compact subsets.*

Theorem 1.6. *Let D be a bounded convex domain in $\mathbb{C}^n$ and $\{D_k\}_{k\geq 1}$ be a decreasing sequence of domains in $\mathbb{C}^n$ converging to D. Then, for every $a_1, a_2, ..., a_N \in D$ and $z \in D$, the following holds*

$$\lim_{k\to\infty} \ell_{D_k}(S, z) = \ell_D(D, z).$$

2. Hyperbolicity and tautness of certain Hartogs type domains

Thoughout this section, assume that X is a complex space and $H : X \times \mathbb{C}^m \to [-\infty, \infty)$ is upper semicontinuous such that $H(z, w) \geq 0, H(z, \lambda w) = |\lambda| H(z, w)$ for all $\lambda \in \mathbb{C}, z \in X, w \in \mathbb{C}^m$.

Definition 2.1. Put $\Omega_H(X) := \{(z.w) \in X \times \mathbb{C}^m : H(z, w) < 1\}$ and for each $z \in X$, $\Omega_H(z) := \{w \in \mathbb{C}^m : H(z, w) < 1\}$.

Using the same argument as in the proof of Remark 3.1.7 and Proposition 3.1.10 in [5](see also [9]), it is easy to get the following

Lemma 2.2. *Let $\Omega = \Omega_H(X)$ and $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Then $\ell_\Omega((z, 0), (z, w)) \leq p(0, H(z, w))$ for any $(z, w) \in \Omega$, where p is the Poincaré distance, and $\ell_\Omega(a, b) = \inf\{p(0, \lambda) : \exists \varphi \in Hol(\Delta, \Omega), \varphi(0) = a, \varphi(\lambda) = b\}$ is the Lempert function. Here, the equality holds if $H \in PSH(X \times \mathbb{C}^m)$.*

Proof of Theorem 1.1. ($\Longrightarrow$) Suppose $\Omega_H(X)$ is hyperbolic. Since X is isomorphic to a closed complex subspace of $\Omega_H(X)$, we deduce that X is hyperbolic. Next, we will show that H verifies the property $(*)$. Otherwise, there would exist $\{z_k\}_{k\geq 1} \subset X$ with $\lim\limits_{k\to\infty} z_k = z_0 \in X, \{w_k\}_{k\geq 1} \subset \mathbb{C}^m$ with $\lim\limits_{k\to\infty} w_k = w_0 \neq 0$ such that $\limsup\limits_{k\to\infty} H(z_k, w_k) = 0$. Without loss of generality, we may assume that $(z_k, w_k) \in \Omega_H(X)$. Then by Lemma 2.2, we have

$$0 \leq k_\Omega((z_k, 0), (z_k, w_k)) \leq p(0, H(z_k, w_k)), \forall k \geq 1.$$

By letting k go to ∞, we find that $k_\Omega((z_0,0),(z_0,w_0))=0$. This contradicts the hyperbolicity of $\Omega_H(X)$.

($\Longleftarrow$) To proof the converse, we consider the projection $\pi:\Omega_H(X)\to X$ given by $\pi(z,w)=z$. Let U be a hyperbolic neighborhood of z_0 in X. Then $\underset{z\in U}{\cup}\Omega_H(z)$ is a bounded set in $\mathbb{C}^m$.

In fact, suppose that this property does not hold. Then $\exists\{z_k\}_{k\geq 1}\subset U,\{w_k\}_{k\geq 1}\subset \mathbb{C}^m$ such that $\lim\limits_{k\to\infty}\|w_k\|=\infty$ and $H(z_k,w_k)<1$. Put $w_k:=r_ku_k$ with $\|u_k\|=1,\forall k\geq 1$, and then $|r_k|\to\infty$ as $k\to\infty$. Without loss of generality, we may assume that $z_k\to z_0$ and $u_k\to u_0\neq 0$ as $k\to\infty$. Since $H(z_k,w_k)=|r_k|H(z_k,u_k)<1$, so we have $\limsup\limits_{k\to\infty}H(z_k,u_k)=0$. This is a contradiction to the property $(*)$. So, there exists $R>0$ such that $\underset{z\in U}{\cup}\Omega_H(z)\subset\mathbb{B}(0,R)$. It is then easy to see that $\pi^{-1}(U)\subset U\times\underset{z\in U}{\cup}\Omega_H(z)\subset U\times\mathbb{B}(0,R)$. Therefore, $\pi^{-1}(U)$ is hyperbolic too. By Eastwood's theorem [3] we conclude the proof. □

Proposition 3.2 in [13] is a direct consequence of Theorem 1.1 when X is a domain $G\subset\mathbb{C}^n, H(z,w)=h(w)e^{u(z)}$, where $h:\mathbb{C}^m\to[-\infty,\infty)$ is upper semicontinuous, $h\not\equiv 0, h(\lambda w)=|\lambda|h(w),\lambda\in\mathbb{C},w\in\mathbb{C}^m$ and u is an upper semicontinuous function on G.

Corollary 2.3. [13, Proposition 3.2] *The domain $\Omega=\Omega_H(G)$ is hyperbolic if and only if G is hyperbolic, $D_h:=\{w\in\mathbb{C}^m:h(w)<1\}\Subset\mathbb{C}^m$ and u is locally bounded on G.*

Proof. It suffices to prove that the property $(*)$ is equivalent to the hypothesis that $D_h:=\{w\in\mathbb{C}^m:h(w)<1\}\Subset\mathbb{C}^m$ and u is locally bounded on G.

Suppose $H(z,w)=h(w)e^{u(z)}$ satisfies the condition $(*)$

First of all, we show that D_h is bounded in $\mathbb{C}^m$. Indeed, let us suppose the contrary. Then there exists a sequence $\{w_k\}_{k\geq 1}\subset D_h$ such that $\|w_k\|\to\infty$. Set $w_k=r_ku_k$ with $\|u_k\|=1$. Then $r_k\to\infty$. It is clear that $0\leq H(z,w_k)=h(w_k)e^{u(z)}=|r_k|h(u_k)e^{u(z)}<e^{u(z)}<\infty$. Letting $k\to\infty$, we have $\limsup\limits_{k\to\infty}H(z,u_k)=0$. This is a contradiction to the property $(*)$.

We now show that u is locally bounded on G. Otherwise, there would exist $z_0\in G$ and a sequence $\{z_k\}_{k\geq 1}$ converging to z_0 such that $u(z_k)\to-\infty$. Fix $w\in\mathbb{C}^m, w\neq 0$. We have $\lim\limits_{k\to\infty}H(z_k,w)=\lim\limits_{k\to\infty}h(w)e^{u(z_k)}=0$. This is a contradiction to the property $(*)$ of the function H.

Now we assume that $H(z,w)=h(w)e^{u(z)}, D_h:=\{w\in\mathbb{C}^m:h(w)<1\}\Subset\mathbb{C}^m$, u is locally bounded on G. Suppose that there exist sequences $\{z_k\}\subset G,\{w_k\}\subset\mathbb{C}^m$ such that $z_k\to z_0\in G, w_k\to w_0\neq 0$ and $\limsup\limits_{k\to\infty}H(z_k,w_k)=0$. It is easy to see that $\limsup_{k\to\infty} h(w_k)e^{u(z_k)} = 0$. Since u is locally bounded on G, we have $\limsup_{k\to\infty} h(w_k) = 0$. Choose a sequence $\{r_k\} \subset \mathbb{R}$ such that:

(i) $r_k \to \infty$,

(ii) $\limsup_{k\to\infty} r_k h(w_k) = \limsup_{k\to\infty} h(r_k w_k) = 0$.

Set $v_k := r_k w_k$. Then $\{v_k\} \subset D_h$ for k large enough and $\|v_k\| \to \infty$. This is a contradiction. □

Proof of Theorem 1.2. 1) *Necessity:* Since X is isomorphic to a closed complex subspace of $\Omega_H(X)$, we deduce that X is taut. We now show that H is continuous on $X \times \mathbb{C}^m$. Otherwise, there would exist $r > 0, \{(z_k, w_k)\}_{k\geq 1} \subset X \times \mathbb{C}^m$ such that

$$\{(z_k, w_k)\} \to (z_0, w_0) \in X \times \mathbb{C}^m \text{ and } H(z_k, w_k) < r < H(z_0, w_0), \forall k \geq 1.$$

For each $k \geq 1$, we define the holomorphic mapping $f_k : \Delta \to \Omega_H(X)$ given by $f_k(\lambda) = (z_k, \frac{\lambda w_k}{r})$. It is clear that $f_k(0) = (z_k, 0) \to (z_0, 0) \in \Omega_H(X)$. Since $\Omega_H(X)$ is taut, by passing to a subsequence if necessary, we may assume that f_k converges locally uniformly on Δ to a holomorphic mapping $f \in Hol(\Delta, \Omega_H(X))$. It is easy to see that $f(\lambda) = (z_0, \frac{\lambda w_0}{r})$. Hence

$$\frac{|\lambda|}{r} H(z_0, w_0) = H(z_0, \frac{\lambda w_0}{r}) < 1, \forall \lambda \in \Delta.$$

This implies that $H(z_0, w_0) < \frac{r}{|\lambda|}, \forall \lambda \in \Delta$, and hence $H(z_0, w_0) \leq r$. This is a contradiction.

We now prove that $\Omega_H(z)$ is taut for each $z \in X$.

Fix $z \in X$. Let $\{\varphi_k\}_{k\geq 1} \subset Hol(\Delta, \Omega_H(z))$ be a sequence. For each $k \geq 1$, we define a map $\psi_k : \Delta \to \Omega_H(X)$ by $\psi_k(\lambda) = (z, \varphi_k(\lambda)), \lambda \in \Delta$. Clearly, it is well-define and $\{\psi_k\}_{k\geq 1} \subset Hol(\Delta, \Omega_H(X))$. So, the tautness of $\Omega_H(X)$ implies that there is a sequence $\{\psi_{1k}\}_{k\geq 1} \subset \{\psi_k\}_{k\geq 1}$ which is either normally convergent in $Hol(\Delta, \Omega_H(X))$ or compactly divergent. Consequently, the sequence $\{\varphi_{1k}\}_{k\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ is either normally convergent in $Hol(\Delta, \Omega_H(z))$ or compactly divergent and we get the tautness of $\Omega_H(z)$.

It remains to show that $\log H$ is plurisubharmonic.

According to a theorem of Fornaess and Narasimhan [4], it suffices to show that $u(z) := \log H \circ g(z) = \log H(g_1(z), g_2(z))$ is subharmonic for every $g = (g_1, g_2) \in Hol(\Delta, X \times \mathbb{C}^m) \cap C(\overline{\Delta}, X \times \mathbb{C}^m)$. Suppose the contrary. Then $\exists z_0 \in X, r > 0$, and a harmonic function h such that $h(z) \geq u(z)$ for any $z = z_0 + re^{i\theta}, \forall \theta \in \mathbb{R}$, but $u(z_0) > h(z_0)$.

We have $u(z) - h(z) = \log H\Big(g_1(z), e^{-h(z) - i\tilde{h}(z)} g_2(z)\Big) \leq 0, \forall z = z_0 + re^{i\theta}$ and $u(z_0) - h(z_0) = \varepsilon_0 > 0$.

For any $n \geq 1$, we set $\varphi_n(\lambda) := \Big(g_1(z), e^{-h(z)-i\tilde{h}(z)-\varepsilon_0-\frac{1}{n}} g_2(z)\Big)$, where $z = z_0 + r\lambda$. It is easy to see that $\varphi_n \in Hol(\Delta, \omega_H(X)), \varphi_n(\partial\Delta) \Subset \Omega_H(X)$ as $n \to \infty$, and $\varphi_n(0)$ tend to a boundary point. This contradicts the tautness of $\Omega_H(X)$ and completes the proof.

2) *Sufficiency:* We consider the projection $\pi : \Omega_H(X) \to X$ given by $\pi(z, w) = z$. Let U be a taut neighborhood of $z \in X$. Since H is continuous, $\Omega_H(z)$ is taut for any $z \in X$, there exists a ball $B \Subset \mathbb{C}^m$ such that

$$\pi^{-1}(U) = \Omega_H(U) \subset U \times B.$$

Let $\{\varphi_n\}_{n\geq 1} \subset Hol(\Delta, \Omega_H(U))$. Since $U \times B$ is taut, $\{\varphi_n\}$ is a normal subfamily of $Hol(\Delta, U \times B)$, i.e. there is a sequence $\{\varphi_{1n}\}_{n\geq 1} \subset \{\varphi_n\}_{n\geq 1}$ which is either normally convergent in $Hol(\Delta, U \times B)$ or compact divergent. In the second case, the sequence $\{\varphi_{1n}\}_{n\geq 1}$, as a subfamily of $Hol(\Delta, \Omega_H(U))$, is compactly divergent.

For $n \geq 1$, we put $\varphi_n = (f_n, g_n)$, where $\{f_n\}_{n\geq 1} \subset Hol(\Delta, U), \{g_n\}_{n\geq 1} \subset Hol(\Delta, B)$. From now on, we only suppose that $\{\varphi_{1n}\}_{n\geq 1}$ is normally convergent in $Hol(\Delta, U \times B)$. Take a function $\varphi = (f, g) \in Hol(\Delta, U \times B)$, where $f \in Hol(\Delta, U), g \in Hol(\Delta, B)$ such that $f_{1n} \overset{K}{\Longrightarrow} f, g_{1n} \overset{K}{\Longrightarrow} g$ as $n \to \infty$. It is easy to see that $\varphi(\Delta) \subset \overline{\Omega_H(U)}$, and $f(\Delta) \subset \overline{U}$. Since U is taut, either $f(\Delta) \subset \partial U$ or $f(\Delta) \subset U$. In the first case, it is clear that $\varphi(\Delta) \subset \partial\Omega_H(U)$, which implies that $\{\varphi_{1n}\}_{n\geq 1}$, as a subsequence of $Hol(\Delta, \Omega_H(U))$, is compactly divergent. Now we assume that $f(\Delta) \subset U$ and define $\psi := H \circ \varphi$. Observe that $\psi \in (C \cap SH)(\Delta)$ and $\psi \leq 1$ on Δ. Then the maximum principle for subharmonic functions implies that either $\psi|_\Delta = 1$ or $\psi|_\Delta < 1$. These properties yield that either $\varphi(\Delta) \subset \partial\Omega_H(U)$ or $\varphi(\Delta) \subset \Omega_H(U)$. Consequently, the sequence $\{\varphi_{1n}\}_{n\geq 1}$ is either normally convergent in $Hol(\Delta, \Omega_H(U))$ or compact divergent. And hence, $\pi^{-1}(U) = \Omega_H(U)$ is taut. By a theorem of Thai and Huong [14] we conclude the proof. □

3. Hyperbolic imbeddedness

First, we recall some preminaries. Let X be a complex space and let TX be the Zariski tangent space of X. By a length function on X we mean a function $H : TX \to [0, \infty)$ satisfying

(LF1) $H(v) = 0$ if and only if $v = 0$,

(LF2) For all complex numbers $c \in \mathbb{C}$, we have $H(cv) = |c|H(v)$,

(LF3) H is continuous.

Now we shall define jets for holomorphic mappings into a complex space X. Denote by Δ_r the disc of radius $r > 0$ and the unit disc Δ_1 by Δ. Let $x \in X$. We consider germs of holomorphic mappings $f : \Delta_r \to X$ that satisfy $f(0) = x$. In a local holomorphic coordinate system any such f is given by its convergent series

$$f(z) = f^{(0)} + f^{(1)}z + f^{(2)}\frac{z^2}{2!} + f^{(3)}\frac{z^3}{3!} + ...,$$

where $f^{(k)} \in \mathbb{C}^n$ for some $n > 0$ and $f^{(0)} = x$.

Two germs f and $\tilde{f}$ osculate to order k if

$$f^{(0)} = \tilde{f}^{(0)}, f^{(1)} = \tilde{f}^{(1)}, ..., f^{(k)} = \tilde{f}^{(k)}.$$

The equivalence classes of such germs will be called jets of order k at x and denoted by $J_k(X)_x$. We set $J_k(X) = \bigcup_{x \in X} J_k(X)_x$.

Given a holomorphic mapping $f : \Delta_r \to X$ with $f(0) = x$, we denote by $j_k(f)_x \in J_k(X)_x$ the k-jet defined by the germ of f at x.

Next, we will define an action of $\mathbb{C}$ on jets. Let $f : \Delta_r \to X$ be a holomorphic map with $f(0) = x$ and $t \in \mathbb{C}$. We set $f_t(z) = f(tz)$ and define $t \cdot j_k(f)_x = j_k(f_t)_x$.

In general, the jet spaces $J_k(X)$ are holomorphic fibre bundles over X, but for $k \geq 2$ they are not vector bundles. Moreover, a holomorphic mapping $h : X \to Y$ between complex spaces induces a map $h^* : J_k(X) \to J_k(Y)$ on k-jets.

We now define the Kobayashi k-metric of a complex space X. Given a complex space X, a point $x \in X$ and a k-osculating vector $\xi \in J_k(X)_x$, we define the Kobayashi length $K_X^k(x, \xi)$ by the greatest lower bound of the positive real numbers of the form $1/r$ for which there exists a holomorphic mapping $f : \Delta \to X$ satisfying $f(0) = x$ and $j_k(f)_x = r\xi$. The function $K_X^k : J_k(X) \to [0, \infty)$ so defined will be called the Kobayashi k-metric of the complex space X. For the Kobayashi k-metrics the following results hold:

(M1) Given two complex spaces X and Y, and any holomorphic mapping $f \in Hol(X, Y)$, then $K_Y^k(f(x), f_x^*(\xi)) \leq K_X^k(x, \xi)$ for all $x \in X$ and $\xi \in J_k(X)_x$.

(M2) For every $k \geq 1$ the Kobayashi k-metric $K_X^k : J_k(X) \to [0, \infty)$ is a Borel function.

(M3) Let $\gamma : [a, b] \to X, [a, b] \subset \mathbb{R}$, be a real-analytic curve. For every $t \in [a, b]$ there exists just one holomorphic germ $\varphi_t \in Hol(\mathbb{C}, X)$ such that $\varphi_t(0) = \gamma(t)$ and $\gamma(t + s) = \varphi_t(s)$ for sufficiently small $\varepsilon > 0$ and every $s \in (-\varepsilon, \varepsilon)$. This enables us to define, for every $k \geq 1$,

$$j_k\gamma(t) = j_k(\varphi_t)_{\gamma(t)} \in J_k(X)_{\gamma(t)}.$$

We define

$$L_X^k(\gamma) = \int_a^b K_X^k(\gamma(t), j_k\gamma(t))dt.$$

All these definitions extend in an obvious way to continuous, piecewise real-analytic curves. If $\gamma : [a, b] \to X$ is a continuous, piecewise real-analytic curve in a complex space X then $\{L_X^k(\gamma)\}_{k=1}^{\infty}$ is a bounded increasing sequence of nonnegative real numbers.

(M4) (Theorem of Venturini [21]) For every $p, q \in X$ we have

$$d_X(p, q) = \inf \Big\{ \sup_{k \geq 1} \int_0^1 K_X^k(\gamma(t), j_k\gamma(t))dt : \gamma \in \Omega_{p,q} \Big\},$$

where $\Omega_{p,q}$ denotes the set of all continuous piecewise real-analytic curves joining p and q.

For more fundamental properties of this subject, see [21] or [17].

We recall the following:

Let M be a domain in a complex space X, that is, M is a connected open nonempty subset of X. Let $X^+ = X \cup \{\infty\}$ be the 1-point Alexandrov compactification of X [6]. Denote by $Cl_{X^+}M$ the closure of M in X^+. We say that M is bounded if $\infty \notin Cl_{X^+}M$ and M is unbounded if $\infty \in Cl_{X^+}M$. If M is unbounded and φ is a function defined on M and c a complex number, we set $\varphi(\infty) = c$ if $\lim\limits_{z\to\infty} \varphi(z) = c$.

Definition 3.1. Let M be a domain in a complex space X.

(i) A function φ is called a local peak plurisubharmonic function at ∞ if there exists a neighborhood U of ∞ such that φ is plurisubharmonic on $U \cap M$ that is continuous up to $U \cap \overline{M}$ and satisfies

$$\begin{cases} \varphi(\infty) = 0, \\ \varphi(z) < 0 \text{ for all } z \in (U \cap \overline{M}) \setminus \{\infty\}. \end{cases}$$

(ii) A function ψ is called a local antipeak plurisubharmonic function at ∞ if there is a neighborhood U of ∞ such that ψ is plurisubharmonic on $U \cap M$ that is continuous up to $U \cap \overline{M}$ and satisfies

$$\begin{cases} \psi(\infty) = -\infty, \\ \psi(z) > -\infty \text{ for all } z \in (U \cap \overline{M}) \setminus \{\infty\}. \end{cases}$$

Lemma 3.2. [18] *Let X be a complex space and let H be a length function on X. Then X is hyperbolic if and only if for each $p \in X$, there are a neighborhood U of p and a constant $C > 0$ such that $F_X(\xi) \geq CH(\xi)$ for all $\xi \in T_xX$ with $x \in U$*

Lemma 3.3. [18] *Let M be an unbounded domain in X. Assume that there are local peak and antipeak plurisubharmonic functions on a neighborhood of ∞. Then for every neighborhood U of ∞ there exists a neighborhood V of ∞ such that $\overline{V} \subset U$ and for every holomorphic mapping $f : \Delta \to M$ satisfying $f(0) \in V$ we have $f(\Delta_{1/2}) \subset U$.*

Proof of Theorem 1.3. Let $p, q \in \partial M, p \neq q$ be given. Take relatively compact neighborhoods U_p, U_q of p, q in X such that $\overline{U_p} \cap \overline{U_q} = \emptyset$.

By Lemma 3.3 and the hypothesis, there exist two neighborhoods U and V of ∞ in X satisfying the following:

i) $\overline{V} \subset U, U \cap M$ is hyperbolic and $(\overline{U_p} \cup \overline{U_q}) \cap \overline{V} = \emptyset$;

ii) For every analytic disc f in M,

$$f(0) \in V \Rightarrow f(\Delta_{1/2}) \subset U \cap M.$$

In particular, for every $z \in \overline{V} \cap M$ and $\xi \in T_zM$ we get $F_M(z,\xi) \geq \frac{1}{2}F_{U\cap M}(z,\xi)$. Thus, $F_M(z,\xi) \geq \frac{C}{2}H(z,\xi)$ for every $z \in \overline{V} \cap M$ and $\xi \in T_zM$.

Let $x, y \in M \setminus \overline{V}$ be arbitrary points. Let $\gamma(t)$ be any continuous piecewise real-analytic curve on M such that $\gamma(0) = x, \gamma(1) = y$. If $\gamma(t) \in M \setminus \overline{V}, \forall t \in [0,1]$, then

$K_M(\gamma(t),\gamma'(t)) \geq K_{M\setminus\overline{V}}(\gamma(t),\gamma'(t)), \forall t \in [0,1]$. If there exist minimal numbers $0 < r < s < 1$ such that $\gamma(r) \in \partial U$ and $\gamma(s) \in \partial V$ then we have

$$\begin{aligned}\int_0^1 K_M^1(\gamma(t),\gamma'(t))dt &\geq \int_r^s K_M^1(\gamma(t),\gamma'(t))dt = \int_r^s F_M(\gamma(t),\gamma'(t))dt \\ &\geq \frac{C}{2}\int_r^s H(\gamma'(t))dt \geq \frac{C}{2} dist(\partial U,\partial V) =: C_1 > 0.\end{aligned}$$

By a result of Venturini [21], it follows that

$$\begin{aligned} d_M(x,y) &= \inf\{\sup_{k\geq 1}\int_0^1 K_M^k(\gamma(t), j_k\gamma(t))dt : \gamma \in \Omega_{x,y}\} \\ &\geq \min\{d_{M\setminus\overline{V}}(x,y); C_1\}, \forall x,y \in M\setminus\overline{V}.\end{aligned}$$

Since X is an increasing union of hyperbolic domains, there exists a hyperbolic domain Ω of M such that $M\setminus\overline{V} \Subset \Omega$. Thus

$$d_{M\setminus\overline{V}}(U_p\cap M, U_q\cap M) \geq d_\Omega(U_p\cap M, U_q\cap M) =: C_2 > 0.$$

This implies that

$$d_M(U_p\cap M, U_q\cap M) \geq \min\{C_1, d_{M\setminus\overline{V}}(U_p\cap M, U_q\cap M)\} \geq \min\{C_1, C_2\} > 0,$$

and hence, M is hyperbolically imbedded in X. □

4. Convergence of invariant functions with multiple poles on decreasing domains

Let us begin this section by recalling some definitions.

Definition 4.1. Let D be a domain in $\mathbb{C}^n$ and $S = \{a_1, ..., a_N\}$ a finite subset of D.

Generalizing the Green function with single pole, Lelong has introduced in [11] the one with multiple poles. More precisely, the Green function of D with the pole set S is defined by

(i) $g_D(S,z) = \sup\big\{u(z) :\ u \in PSH(D),\ u \leq 0,\ u(x) = \log|x-a_j| + O(1)\ (1 \leq j \leq N)\big\}$.

In [2], Coman defined the Lempert function with multiple poles as follows.

(ii) $\ell_D(S,z) = \inf\big\{\sum_{j=1}^{N}\log|\zeta_j|\ :\exists\ \varphi \in Hol(\Delta, D),\ \varphi(0) = z,\ \varphi(\zeta_j) = a_j\ (1 \leq j \leq N)\big\}$.

Now we move to another useful function, the Caratheodory function that will give a lower bound for g. Set

(iii) $C_D(S,z) = \sup\big\{\log|F(z)|\ :\ F \in Hol(D,\Delta),\ F(a_j) = 0\ (1 \leq j \leq N)\big\}$.

Definition 4.2. Let D be a bounded domain in $\mathbb{C}^n$.

(i) D is said to be hyperconvex if there exists a continuous plurisubharmonic exhaustive function $\varrho : D \to (-\infty, 0)$.

(ii) D is said to be strictly hyperconvex if there exist a bounded domain Ω and a function $\varrho \in (\Omega, (-\infty, 1)) \cap PSH(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$, ϱ is exhautive for Ω and for all real number $C \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < C\}$ is connected.

We now prove Theorem 1.4 by using the main ideas in [12].

Proof of Theorem 1.4. Let $R > r > 0$ be given such that $\overline{B}(a_j, r) \subset D \Subset D_k \subset B(a_j, R)$ $(k \geq 1,\ 1 \leq j \leq N)$ and $\overline{B}(a_j, r) \cap \overline{B}(a_i, r) = \emptyset$ $(j \neq i)$.

(i) We now show that $\lim\limits_{k\to\infty} g_{D_k}(S, z) = 0 = \lim\limits_{\substack{\zeta\to z\\ \zeta\in D}} g_D(S, \zeta)$ $(z \in \partial D)$.

Indeed, for each $k \geq 1$, define the function ω_k by putting

$$\omega_k(z) := \begin{cases} \max\Big\{d(\varrho(z) - C_k); \sum\limits_{j=1}^{N} \log\big(\dfrac{\|z - a_j\|}{R}\big)\Big\} \text{ if } z \in D_k \setminus \cup_{j=1}^{N} B(a_j, r)), \\ \sum\limits_{j=1}^{N} \log\big(\dfrac{\|z - a_j\|}{R}\big) \text{ if } z \in \cup_{j=1}^{N} B(a_j, r), \end{cases}$$

where ϱ is the function in the definition of the strict hyperconvexity of the domain D and $C_k \in [0, 1]$ such that $\overline{D} \subset\ D'_k := \{z \in D_k : \varrho(z) \leq C_k\}$ $(k \geq 1)$ and d is a positive constant such that $d\varrho(z) < N \log(\dfrac{r}{R})$ $(z \in \cup_{j=1}^{N} \partial B(a_j, r))$.

It is easy to see that

$$\overline{D} \subset \cap_{k=1}^{\infty} D'_k, \omega_k \in PSH(D_k, [-\infty, 0))$$

and

$$\omega_k(z) = \sum_{j=1}^{N} \log\big(\frac{\|z - a_j\|}{R}\big)\ (z \in \cup_{j=1}^{N} \overline{B}(a_j, r)).$$

Then ω_k belongs to the defining family for $g_{D_k}(S, .)$. Consequently, $\omega_k \leq g_{D_k}(S, .)$ on D_k. Hence $\liminf\limits_{k\to\infty} g_{D_k}(S, z) \geq \lim\limits_{k\to\infty} \omega_k = 0$ $(z \in \partial D)$. On the other hand, we know that $g_{D_k} \leq 0$ $(z \in D)$. So $\lim\limits_{k\to\infty} g_{D_k}(S, z) = 0$ $(z \in \partial D)$.

(ii) For every $k \geq 1$, define $\gamma_k = \inf\limits_{z\in\partial D} g_{D_k}(S, z)$. Since g_{D_k} is a continuous negative real-valued function on the compact set ∂D, it follows that $\gamma_k \in (-\infty, 0)$. Since the sequence $\{g_{D_k}\}_{k\geq1}$ is an increasing sequence of continuous functions on ∂D and converges pointwise to the function $g_D|_{\partial D} \equiv 0$, according to Dini theorem, it follows that the sequence $\{g_{D_k}\}$ converges uniformly on compact subsets of ∂D to 0 and $\lim\limits_{k\to\infty} \gamma_k = 0$.

For $k \geq 1$, define the function g_k on D_k by

$$g_k(z) := \begin{cases} g_{D_k}(S,z) & \text{if } z \in D_k \backslash D, \\ \max\left\{g_{D_k}(S,z); g_D(S,z) + \gamma_k\right\} & \text{if } z \in \overline{D}. \end{cases}$$

Then g_k belongs to the defining family of $g_{D_k}(S,.)$. This implies that $g_k \leq g_{D_k}$ on D_k. In particular, $g_D + \gamma_k \leq g_{D_k}$, and hence, $g_{D_k} \leq g_D \leq g_{D_k} - \gamma_k$ on $\overline{D}$. The theorem is proved. □

Assume that D is a strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-boundary and ρ is a strictly plurisubharmonic function of class $\mathcal{C}^2$ on an open neighborhood of $\overline{D}$ such that

i) $D = \left\{z \in \mathbb{C}^n \ : \ \rho(z) < 0\right\}$,

ii) $\bigtriangledown\rho(\zeta) \neq 0$ $(\zeta \in \partial D)$, where $\bigtriangledown = (\dfrac{\partial}{\partial z_1}, ..., \dfrac{\partial}{\partial z_n})$.

We define $D_\delta = \left\{z \in \mathbb{C}^n \ : \ \rho(z) < \delta\right\}$ for $\delta \in \mathbb{R}$ sufficiently near zero. Note that D_δ is also a strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-boundary and $\left\{D_\delta\right\}_{\delta>0}$ is an open neighborhood base of $\overline{D}$.

In order to prove Theorem 1.5 we need the following lemmas

Lemma 4.3. [1, Theorem 2.4]. *Let F be a bounded holomorphic function on D. Then there exist functions $\left\{F_j\right\}_{j=1}^{\infty}$ such that*

(i) *F_j is continuous on $\overline{D}$ which is holomorphic in D,*

(ii) *$\|F_j\|_D \leq \|F\|_D$ for all j,*

(iii) *$F_j \to F$ pointwise on D as $j \to \infty$,*

where $\| \ . \ \|_D$ denotes the supremum norm on D.

Lemma 4.4. [7, Theorem 1.4.1]. *There exists an open set $U \subset \mathbb{C}^n$ with $\overline{D} \subset U$ such that any continuous function on $\overline{D}$ which is holomorphic in D can be uniformly approximated on $\overline{D}$ by a holomorphic function G on U.*

Proof of Theorem 1.5. Fix points $a_1, a_2, ..., a_N, z$ in D and an extremal function $F \in Hol(D, \Delta)$ for $C_D(S,z)$ such that $\log|F(z)| = C_D(S,z)$. By Lemma 4.3, for any $\varepsilon > 0$ we can choose a continuous function G on $\overline{D}$ that is holomorphic in D such that $|F(z) - G(z)| < \varepsilon, |F(a_j) - G(a_j)| < \varepsilon$ and $\|G\|_D \leq 1$. By Lemma 4.4, we can choose a holomorphic function H on an open neighborhood of $\overline{D}$ such that $\|H - G\|_{\overline{D}} < \varepsilon$. Since $\{D_\delta\}_{\delta>0}$ forms an open neighborhood base of $\overline{D}$ for some $\delta > 0$ small enough, $|H(z) - F(z)| < 2\varepsilon$ and $|H(a_j) - F(a_j)| < 2\varepsilon$ and $\|H\|_{D_\delta} < 1 + 2\varepsilon$.

Now, we can construct a polynomial $P \in \mathbb{C}[z]$ such that $P(a_j) = -H(a_j)$ $(1 \leq j \leq N)$ and $\|P\|_{L^\infty(D_\delta)} \leq C \cdot \varepsilon$.

Put $L = H + P$ and

$$\tilde{F}(z) := \frac{1}{1 + (2 + C)\varepsilon} L(z).$$

Then $\tilde{F}$ belongs to the defining family of C_{D_δ}, and hence, $\log|F(z)| = C_D(S,z) \geq C_{D_\delta}(S,z) \geq \log|\tilde{F}(z)|$. Since ε is arbitrarily small, we have $C_{D_\delta}(S,z) \to C_D(S,z)$ as $\delta \to 0$. By the continuity of C_{D_δ} for every $\delta \geq 0$ and by the distance decreasing property, it follows that this convergence is uniform on compact sets because of Dini's theorem. □

For Lempert functions with single pole Theorem 1.6 was proved by Jarnicki and Pflug (see [5, Proposition 3.3.5]). We now prove Theorem 1.6 by using the ideas of [5].

Proof of Theorem 1.6. The sequence $\left\{\ell_{D_k}(S,z)\right\}_{k\geq 1}$ is increasing with $\lim\limits_{k\to\infty} \ell_{D_k}(S,z) \leq \ell_D(S,z)$. Now let us suppose that $\lim\limits_{k\to\infty} \ell_{D_k}(S,z) < M < \ell_D(S,z)$. Then we are able to select holomorphic functions $\varphi_k \in Hol(\Delta, D_k)$ with $\varphi_k(0) = z,\ \varphi_k(\zeta_j^k) = a_j,\ \zeta_j^k \in \Delta\ (1 \leq j \leq N, k \geq 1)$ such that

$$\sum_{j=1}^{N} \log|\zeta_j^k| < M \ (k \geq 1).$$

By passing to a subsequence if necessary, we may assume that φ_k converges locally uniformly to some $\varphi \in Hol(\Delta, D)$. Also again by passing to a subsequence if necessary, we may assume for each j that $\zeta_j^k \longrightarrow \zeta_j \in \overline{\Delta}$ as $k \to \infty$. Set

$$J = \{j \in \{1, ..., N\} : \zeta_j \in \Delta\}.$$

Suppose that $J = \emptyset$. Then $\ell_D(S,z) = 0$. This is a contracdiction. Hence $J \neq \emptyset$. It is easy to see that

$$\varphi(0) = z; \varphi(\zeta_j) = a_j \ (j \in J) \ \text{ and } \ell_{D_k}(S,z) \longrightarrow \sum_{j=1}^{N} \log|\zeta_j| \leq M.$$

Put $S_J = \{a_j : j \in J\}$. By [22] (see also [20]), it follows that

$$\ell_D(S,z) \leq \ell_D(S_J,z) \leq \sum_{j\in J} \log|\zeta_j| \leq M < \ell_D(S,z).$$

This is a contradiction. □

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