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ON UNIQUE RANGE SETS FOR HOLOMORPHIC MAPS SHARING HYPERSURFACES WITHOUT COUNTING MULTIPLICITY

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Abstract. In 1975, Fujimoto showed a result on the unique range set counting multiplicity for meromorphic maps from $\mathbb{C}^m$ to $\mathbb{P}^n(\mathbb{C})$ with hyperplanes. Here we will prove some sufficient conditions of unique range sets ignoring multiplicity for algebraically non-degenerate holomorphic maps with hypersurfaces.

1. Introduction

In 1926, Nevanlinna proved that two non-constant meromorphic functions of one complex variable which attain same five distinct values at the same points, must be identical. In 1975, Fujimoto (see [5]) generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Since that time, this problem has been studied intensively. In this paper by using the second main theorem with ramification of An-Phuong (see [1]) we give some uniqueness results for algebraically non-degenerate holomorphic curves sharing sufficiently many non-linear hypersurfaces in projective space. To state our results, we first introduce some notations.

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function, we say that $a \in \mathbb{C}$ is a zero of f with multiplicity α if there exists a nowhere vanishing holomorphic function g in a neighborhood U of a such that

$$f(z) = (z - a)^{\alpha} g(z).$$

Then, we write $\mathrm{ord}_f(a) = \alpha$.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and $f = (f_0, \ldots, f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q be a homogeneous polynomial of degree d in $n + 1$ variables with coefficients in $\mathbb{C}$ defining D, we

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define

$$\overline{E}_f(D) := \{z \in \mathbb{C} \mid Q \circ f(z) = 0 \text{ ignoring multiplicity } \};$$
$$E_f(D) := \{(z, m) \in \mathbb{C} \times \mathbb{N} \mid Q \circ f(z) = 0 \text{ and } \mathrm{ord}_{Q \circ f}(z) = m\}.$$

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of hypersurfaces, we define

$$\overline{E}_f(\mathcal{D}) := \bigcup_{D \in \mathcal{D}} \overline{E}_f(D) \quad \text{and} \quad E_f(\mathcal{D}) := \bigcup_{D \in \mathcal{D}} E_f(D).$$

Definition. A collection of hypersurfaces $\mathcal{D} = \{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$ is said to be a *separated unique range set ignoring multiplicity*, denoted by SURSIM (or *separated unique range set counting multiplicity,* denoted by SURSCM) for a family of holomorphic maps $\mathcal{F}$ if for any pair of holomorphic maps $f, g \in \mathcal{F}$, the condition $\overline{E}_f(D_j) = \overline{E}_g(D_j)$ (resp. $E_f(D_j) = E_g(D_j)$), for $j = 1, \ldots, q$, implies $f \equiv g$. A collection $\mathcal{D}$ is said to be a *unique range set ignoring multiplicity*, denoted by URSIM (or *unique range set counting multiplicity,* denoted by URSCM) for a family of holomorphic maps $\mathcal{F}$ if for any pair of holomorphic maps $f, g \in \mathcal{F}$, the condition $\overline{E}_f(\mathcal{D}) = \overline{E}_g(\mathcal{D})$ (resp. $E_f(\mathcal{D}) = E_g(\mathcal{D})$) implies $f \equiv g$. The SURSIM, SURSCM, URSIM, URSCM are called the unique range set for a family $\mathcal{F}$ to the same.

Obviously, if $\mathcal{D} = \{D_1, \ldots, D_q\}$ is a URSIM (resp. URSCM) then $\mathcal{D}$ will be a SURSIM (resp. SURSCM), but the converse is not true.

Recall that a collection of $q > n$ hypersurfaces $\mathcal{D} = \{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position if for any distinct $i_1, \ldots, i_{n+1} \in \{1, \ldots, q\}$,

$$\bigcap_{k=1}^{n+1} D_{i_k} = \emptyset.$$

In 1975, Fujimoto (see [5]) showed that

Theorem A. *Let $\mathcal{H} = \{H_1, \ldots, H_{3n+2}\}$ be a collection of $3n + 2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$, and $f, g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be meromorphic maps such that $f(\mathbb{C}^m) \not\subset H$ and $g(\mathbb{C}^m) \not\subset H$ for any $H \in \mathcal{H}$. If*

$$E_f(H_j) = E_g(H_j) \quad \textit{for any } H_j \in \mathcal{H}$$

then $f \equiv g$.

By Theorem A, we have a SURSCM having $3n + 2$ hyperplanes in general position for the family of linearly nondegenerate meromorphic maps. In 1983, Smiley (see [10]) proved a result on the unique range set for a special collection of linearly nondegenerate meromorphic maps, which was given again in 1998 by Fujimoto (see [6]) and was considered again by Dethloff and Tan (see [4]) in 2005. In 2002 and 2003, An and Manh (see [2] and [9]) showed some results for SURSIM for linearly nondegenerate meromorphic maps in hyperplanes. Recently, many mathematicians study two following problems: finding properties of unique range sets, and finding out a unique range set with the smallest number of elements as possible. Our contribution is to give unicity results for algebraically non-degenerate holomorphic maps sharing sufficiently many hypersurfaces in general position in projective space.

Now let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of q hypersurfaces. For $j = 1, \ldots, q$, denote the degree of D_j by d_j, and let d be the least common multiple of the d_j for $j = 1, \ldots, q$. We define the *minimal index of degrees of* $\mathcal{D}$ by

$$\delta := \min\{d_1, \ldots, d_q\},$$

and the *bound of truncated level of* $\mathcal{D}$ by

$$B_n(\mathcal{D}) = 2^{n+1} d \Big[2^n (n+1) n (d+1)\Big]^n.$$

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map. We denote by $\mathcal{F}(\mathcal{D}, f)$ the family of all algebraically nondegenerate holomorphic maps $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ satisfying the condition

$$g(z) = f(z) \text{ for any } z \in \overline{E}_g(\mathcal{D}).$$

Furthermore, we define $\mathcal{F}^*(\mathcal{D}, f) \subseteq \mathcal{F}(\mathcal{D}, f)$ to be the set of those maps g in $\mathcal{F}(\mathcal{D}, f)$ such that

$$\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$$

for every $i \neq j \in \{1, \ldots, q\}$.

In this paper, we obtained the following results

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map and $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q \geqslant n + 2 + \dfrac{2nB_n(\mathcal{D})}{\delta}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. Then, $\mathcal{D}$ is a URSIM for the family $\mathcal{F}(\mathcal{D}, f)$.*

Theorem 2. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map and $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q \geqslant n + 2 + \dfrac{2B_n(\mathcal{D})}{\delta}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. Then, $\mathcal{D}$ is a URSIM for the family $\mathcal{F}^*(\mathcal{D}, f)$.*

Note that, Theorem 1 and Theorem 2 have shown the sufficient conditions of the URSIM for a collection of algebraically non-degenerate holomorphic maps $\mathcal{F}(\mathcal{D}, f)$ and $\mathcal{F}^*(\mathcal{D}, f)$ in the case hypersurfaces. But the number of hypersurfaces in the URSIM is large. It would be interesting if one can find a URSIM with the smallest number of hypersurfaces or show other sufficient conditions. The proofs of our theorems base on the second main theorem with ramification of An-Phuong, which is shown in [1], and the technique of An-Manh (see [2]) to the case of hypersurfaces.

2. Some notations and results in Nevanlinna-Cartan theory

In this section, we introduce some notations in Nevanlinna-Cartan theory and recall some results, which are necessary for the proofs of the our main results.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and $f = (f_0, \ldots, f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial of degree d defining D. The proximity function of f is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q \circ f(re^{i\theta})|} d\theta.$$

Let $n_f(r, D)$ be the number of zeros of $Q \circ f$ in the disk $|z| \leqslant r$, counting multiplicity. For any positive integer k, let $n_f(r, D, \leqslant k)$ be the number of zeros having multiplicity $\leqslant k$ of $Q \circ f$ in the disk $|z| \leqslant r$, counting multiplicity and let $n_f(r, D, > k)$ be the number of zeros having multiplicity $> k$ of $Q \circ f$ in the disk $|z| \leqslant r$, counting multiplicity. The integrated counting functions are defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r;$$

$$N_{f, \leqslant k}(r, D) = \int_0^r \frac{n_f(t, D, \leqslant k) - n_f(0, D, \leqslant k)}{t} dt + n_f(0, D, \leqslant k) \log r;$$

$$N_{f, > k}(r, D) = \int_0^r \frac{n_f(t, D, > k) - n_f(0, D, > k)}{t} dt + n_f(0, D, > k) \log r.$$

For any positive integers Δ, k, let $n_f^{\Delta}(r, D)$ be the number of zeros of $Q \circ f$ in the disk $|z| \leqslant r$, where any zero is counted with multiplicity if its multiplicity is less than or equal to Δ, and Δ times otherwise. Let $n_f^{\Delta}(r, D, \leqslant k)$ (resp. $n_f^{\Delta}(r, D, > k)$) be the number of zeros having multiplicity $\leqslant k$ (resp. $> k$) of $Q \circ f$ in the disk $|z| \leqslant r$, where any zero is counted with multiplicity if its multiplicity is less than or equal to Δ, and Δ times otherwise, too. The integrated truncated counting functions are defined by

$$N_f^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D) - n_f^{\Delta}(0, D)}{t} dt + n_f^{\Delta}(0, D) \log r;$$

$$N_{f, \leqslant k}^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D, \leqslant k) - n_f^{\Delta}(0, D, \leqslant k)}{t} dt + n_f^{\Delta}(0, D, \leqslant k) \log r;$$

$$N_{f, > k}^{\Delta}(r, D) = \int_0^r \frac{n_f^{\Delta}(t, D, > k) - n_f^{\Delta}(0, D, > k)}{t} dt + n_f^{\Delta}(0, D, > k) \log r.$$

We have the following lemma about properties of integrated counting functions and integrated truncated counting ones.

Lemma 2.1. *With the above notations we have*

$$1) N_f(r,D) = N_{f,\leqslant k}(r,D) + N_{f,>k}(r,D);$$
$$2) N_f^{\Delta}(r,D) = N_{f,\leqslant k}^{\Delta}(r,D) + N_{f,>k}^{\Delta}(r,D);$$
$$3) N_f^{\Delta}(r,D) \leqslant N_f(r,D);$$
$$4) N_f^{1}(r,D) \leqslant N_f^{\Delta}(r,D) \leqslant \Delta N_f^{1}(r,D);$$
$$5) N_{f,\leqslant k}^{1}(r,D) \leqslant N_{f,\leqslant k}^{\Delta}(r,D) \leqslant \Delta N_{f,\leqslant k}^{1}(r,D);$$
$$6) N_{f,>k}^{1}(r,D) \leqslant N_{f,>k}^{\Delta}(r,D) \leqslant \Delta N_{f,>k}^{1}(r,D);$$
$$7) \frac{1}{k+1} N_{f,\leqslant k}^{\Delta}(r,D) + N_{f,>k}^{\Delta}(r,D) \leqslant \frac{\Delta}{k+1} N_f(r,D).$$

Proof. 1), 2), 3), 4), 5) and 6) are obvious by definitions of integrated counting functions and integrated truncated counting functions. We prove 7). We have

$$\begin{aligned}\frac{1}{k+1} N_{f,\leqslant k}^{\Delta}(r,D) + N_{f,>k}^{\Delta}(r,D) &\leqslant \frac{\Delta}{k+1} N_{f,\leqslant k}^{1}(r,D) + \Delta N_{f,>k}^{1}(r,D)\\ &\leqslant \frac{\Delta}{k+1} N_{f,\leqslant k}(r,D) + \frac{\Delta}{k+1} N_{f,>k}(r,D)\\ &= \frac{\Delta}{k+1} N_f(r,D).\end{aligned}$$

□

A consequence of the Poisson-Jensen formula is the following:

First Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < \infty$,*

$$m_f(r,D) + N_f(r,D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

In 2007, An and Phuong (see [1]) proved the following theorem

Theorem 2.2. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the d_j. Let $0 < \varepsilon < 1$ and let*

$$\Delta \geqslant 2d\Big[2^n(n+1)n(d+1)\epsilon^{-1}\Big]^n.$$

Then

$$(q-(n+1)-\varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f^{\Delta}(r,D_j),$$

where inequality holds for all large r outside a set of finite Lebesgue measure.

3. Proofs of Theorem 1 and Theorem 2

To prove our theorems we need the following lemma.

Lemma 3.1 *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic map and $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the d_j. Then for any positive integer k and $0 < \varepsilon < 1$, we have*

$$\frac{q(k+1-\Delta) - (n+1+\varepsilon)(k+1)}{k} T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N^{\Delta}_{f,\leqslant k}(r, D_j) + O(1), \tag{3.1}$$

where $\Delta = \Delta(\varepsilon) = 2d\Big[2^n(n+1)n(d+1)\epsilon^{-1}\Big]^n$ and the inequality (3.1) holds for all large r outside a set of finite Lebesgue measure.

Proof. Set $\mathcal{D} = \{D_1, \ldots, D_q\}$, then for any $D_j \in \mathcal{D}$, by Lemma 2.1 and First Main Theorem we have

$$\begin{aligned}
N^{\Delta}_f(r, D_j) &= N^{\Delta}_{f,\leqslant k}(r, D_j) + N^{\Delta}_{f,>k}(r, D_j) \\
&= \frac{k}{k+1} N^{\Delta}_{f,\leqslant k}(r, D_j) + \frac{1}{k+1} N^{\Delta}_{f,\leqslant k}(r, D_j) + N^{\Delta}_{f,>k}(r, D_j) \\
&\leqslant \frac{k}{k+1} N^{\Delta}_{f,\leqslant k}(r, D_j) + \frac{\Delta}{k+1} N_f(r, D_j) \\
&\leqslant \frac{k}{k+1} N^{\Delta}_{f,\leqslant k}(r, D_j) + \frac{\Delta d_j}{k+1} T_f(r) + O(1),
\end{aligned}$$

so

$$\frac{1}{d_j} N^{\Delta}_f(r, D_j) \leqslant \frac{k}{d_j(k+1)} N^{\Delta}_{f,\leqslant k}(r, D_j) + \frac{\Delta}{k+1} T_f(r) + O(1).$$

This implies that

$$\sum_{j=1}^{q} \frac{1}{d_j} N^{\Delta}_f(r, D_j) \leqslant \frac{k}{k+1} \sum_{j=1}^{q} \frac{1}{d_j} N^{\Delta}_{f,\leqslant k}(r, D_j) + \frac{q\Delta}{k+1} T_f(r) + O(1). \tag{3.2}$$

On the other hand, by Theorem 2.2, we have

$$(q - n - 1 - \varepsilon) T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N^{\Delta}_f(r, D_j). \tag{3.3}$$

Combining the formulas (3.2) and (3.3) together, we have

$$\Big(q - \frac{q\Delta}{k+1} - n - 1 - \varepsilon\Big) T_f(r) \leqslant \frac{k}{k+1} \sum_{j=1}^{q} \frac{1}{d_j} N^{\Delta}_{f,\leqslant k}(r, D_j) + O(1).$$

This concludes the proof of the lemma. $\square$

Proof of Theorem 1. We will prove $g \equiv h$ for any pair of maps $g, h \in \mathcal{F}(\mathcal{D}, f)$ such that $\overline{E}_g(\mathcal{D}) = \overline{E}_h(\mathcal{D})$ by the indirect method. Assume for the sake of contradiction that there exist two maps $g, h \in \mathcal{F}(\mathcal{D}, f)$ such that $\overline{E}_g(\mathcal{D}) = \overline{E}_h(\mathcal{D})$ and $g \not\equiv h$. Then there are two numbers $\alpha, \beta \in \{0, \dots, n\}$, $\alpha \neq \beta$ such that $g_\alpha h_\beta \not\equiv g_\beta h_\alpha$. Let d_j be the degree of D_j, $j = 1, \dots, q$, and let d be the least common multiple of the d_j. Let k be a sufficiently large positive integer, ε be a real number such that $0 < \varepsilon < 1$ and $\Delta = \Delta(\varepsilon) = 2d\Big[2^n(n+1)n(d+1)\epsilon^{-1}\Big]^n$. With the hypothesis in Theorem 1 and using Lemma 3.1, we have

$$
\begin{aligned}
&\big(q(k+1-\Delta) - (n+1+\varepsilon)(k+1)\big)T_g(r) \\
&\leqslant k\sum_{j=1}^{q}\frac{1}{d_j}N^{\Delta}_{g,\leqslant k}(r, D_j) + O(1) \\
&\leqslant \Delta k\sum_{j=1}^{q}\frac{1}{d_j}N^{1}_{g,\leqslant k}(r, D_j) + O(1) \\
&\leqslant \frac{\Delta k}{\delta}\sum_{j=1}^{q}N^{1}_{g,\leqslant k}(r, D_j) + O(1).
\end{aligned}
\tag{3.4}
$$

Assume that $z_0 \in \mathbb{C}$ is a zero of $D_j \circ g$ with multiplicity less than or equal to k, then $z_0 \in \overline{E}_g(\mathcal{D}) = \bigcup_{j=1}^{q}\overline{E}_g(D_j)$. Because $g \in \mathcal{F}(\mathcal{D}, f)$, this implies that $g(z_0) = f(z_0)$. Since $\overline{E}_g(\mathcal{D}) = \overline{E}_h(\mathcal{D})$ we have $z_0 \in \overline{E}_h(\mathcal{D}) = \bigcup_{j=1}^{q}\overline{E}_h(D_j)$, so $h(z_0) = f(z_0)$. Hence $g(z_0) = h(z_0)$, so $\dfrac{g_\alpha(z_0)}{g_\beta(z_0)} = \dfrac{h_\alpha(z_0)}{h_\beta(z_0)}$, namely z_0 is a zero of the function $\dfrac{g_\alpha}{g_\beta} - \dfrac{h_\alpha}{h_\beta}$. Note that by the hypothesis that the hypersurfaces in $\mathcal{D}$ are in general position, then there exist at most n hypersurfaces D_j in $\mathcal{D}$ such that $D_j \circ g(z_0) = 0$. This implies that

$$\sum_{j=1}^{q}N^{1}_{g,\leqslant k}(r, D_j) \leqslant nN_{\frac{g_\alpha}{g_\beta} - \frac{h_\alpha}{h_\beta}}(r, 0),$$

where $N_{\frac{g_\alpha}{g_\beta} - \frac{h_\alpha}{h_\beta}}(r, 0)$ is the counting function of zeros of $\frac{g_\alpha}{g_\beta} - \frac{h_\alpha}{h_\beta}$. We have by properties of counting functions,

$$N_{\frac{g_\alpha}{g_\beta} - \frac{h_\alpha}{h_\beta}}(r, 0) \leqslant T_g(r) + T_h(r) + O(1).$$

Therefore, (3.4) becomes

$$
\begin{aligned}
&\big(q(k+1-\Delta) - (n+1+\varepsilon)(k+1)\big)T_g(r) \\
&\leqslant \frac{\Delta nk}{\delta}(T_g(r) + T_h(r)) + O(1).
\end{aligned}
\tag{3.5}
$$

Similarly for the holomorphic map h we have

$$\begin{aligned}(3.6)\qquad &\big(q(k+1-\Delta)-(n+1+\varepsilon)(k+1)\big)T_h(r)\\ &\leqslant \frac{\Delta nk}{\delta}(T_g(r)+T_h(r))+O(1).\end{aligned}$$

Adding the inequalities (3.5) and (3.6) together, we have

$$\begin{aligned}&\big(q(k+1-\Delta)-(n+1+\varepsilon)(k+1)\big)(T_g(r)+T_h(r))\\ &\leqslant \frac{2\Delta nk}{\delta}(T_g(r)+T_h(r))+O(1).\end{aligned}$$

This concludes that

$$q(k+1-\Delta)-(n+1+\varepsilon)(k+1)-\frac{2\Delta nk}{\delta}\leqslant\frac{O(1)}{T_g(r)+T_h(r)}$$

holds for a sufficiently large positive real number r. Let $r\to\infty$ we have

$$q(k+1-\Delta)-(n+1+\varepsilon)(k+1)-\frac{2\Delta nk}{\delta}\leqslant 0.$$

This is equivalent to

$$(3.7)\qquad k(q\delta-(n+1+\varepsilon)\delta-2\Delta n)+(q-q\Delta-(n+1+\varepsilon))\delta\leqslant 0.$$

If we take $\varepsilon=\dfrac{1}{2}$ and

$$k>\frac{(qB_n(\mathcal{D})-q+n+\frac{3}{2})\delta}{q\delta-(n+\frac{3}{2})\delta-2nB_n(\mathcal{D})},$$

then from the hypothesis that $q\geqslant n+2+\dfrac{2nB_n(\mathcal{D})}{\delta}$ we have a contradiction. Hence $g_ih_j\equiv g_jh_i$ for any $i\neq j\in\{0,\dots,n\}$, namely $g\equiv h$. This is the conclusion of Theorem 1. □

Proof of Theorem 2. We prove Theorem 2 by the indirect method too. Assume for the sake of contradiction that there exist two maps $g,h\in\mathcal{F}(\mathcal{D},f)$ such that $\overline{E}_g(\mathcal{D})=\overline{E}_h(\mathcal{D})$ and $g\not\equiv h$. Then there are two numbers $\alpha,\beta\in\{0,\dots,n\},\ \alpha\neq\beta$ such that $g_\alpha h_\beta\not\equiv g_\beta h_\alpha$. Let d_j be the degree of $D_j,\ j=1,\dots,q$, and let d be the least common multiple of the d_j. Let k be a sufficiently large positive integer, ε be a real number such that $0<\varepsilon<1$ and $\Delta=\Delta(\varepsilon)=2d\Big[2^n(n+1)n(d+1)\epsilon^{-1}\Big]^n$. With the hypothesis in Theorem 2 and the proof of Theorem 1, we have by Lemma 3.1

$$\begin{aligned}(3.8)\qquad &\big(q(k+1-\Delta)-(n+1+\varepsilon)(k+1)\big)T_g(r)\\ &\leqslant\frac{\Delta k}{\delta}\sum_{j=1}^{q}N^1_{g,\leqslant k}(r,D_j)+O(1).\end{aligned}$$

From the hypothesis that

$$\overline{E}_g(D_i)\cap\overline{E}_g(D_j)=\emptyset$$

for any pair $i \neq j \in \{1, \ldots, q\}$ and arguments in the proof of Theorem 1, we have

$$\sum_{j=1}^{q} N^1_{g,\leqslant k}(r, D_j) \leqslant N_{\frac{g_\alpha}{g_\beta}-\frac{h_\alpha}{h_\beta}}(r, 0) \leqslant T_g(r) + T_h(r) + O(1).$$

This implies that

$$\begin{aligned}(3.9)\qquad &\big(q(k+1-\Delta) - (n+1+\varepsilon)(k+1)\big)T_g(r)\\ &\qquad\leqslant \frac{\Delta k}{\delta}(T_g(r) + T_h(r)) + O(1).\end{aligned}$$

Similarly for the holomorphic map h we have

$$\begin{aligned}(3.10)\qquad &\big(q(k+1-\Delta) - (n+1+\varepsilon)(k+1)\big)T_h(r)\\ &\qquad\leqslant \frac{\Delta k}{\delta}(T_g(r) + T_h(r)) + O(1).\end{aligned}$$

From the inequalities (3.9) and (3.10), we have

$$\begin{aligned}&\big(q(k+1-\Delta) - (n+1+\varepsilon)(k+1)\big)(T_g(r) + T_h(r))\\ &\qquad\leqslant \frac{2\Delta k}{\delta}(T_g(r) + T_h(r)) + O(1).\end{aligned}$$

Hence

$$q(k+1-\Delta) - (n+1+\varepsilon)(k+1) - \frac{2\Delta k}{\delta} \leqslant \frac{O(1)}{T_g(r) + T_h(r)}$$

holds for a sufficiently large positive real number r. Letting $r \to \infty$ we have

$$(3.11)\qquad k(q\delta - (n+1+\varepsilon)\delta - 2\Delta) + (q - q\Delta - (n+1+\varepsilon))\delta \leqslant 0.$$

If we take $\varepsilon = \dfrac{1}{2}$ and

$$k > \frac{(qB_n(\mathcal{D}) - q + n + \frac{3}{2})\delta}{q\delta - (n+\frac{3}{2})\delta - 2B_n(\mathcal{D})},$$

then from the hypothesis that $q \geqslant n + 2 + \dfrac{2B_n(\mathcal{D})}{\delta}$ we have a contradiction. Hence $g_i h_j \equiv g_j h_i$ for any $i \neq j \in \{0, \ldots, n\}$, namely $g \equiv h$. This finishes the proof of Theorem 2. □

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