ACTA MATHEMATICA VIETNAMICA
Volume 34, Number 3, 2009, pp. 299–304

PERSISTENCE AND GLOBAL ATTRACTIVITY IN THE MODEL $A_{n+1} = A_n F_n(A_n, A_{n-1}, \cdots, A_{n-m})$

DANG VU GIANG

Abstract. First, we prove the uniform persistence for discrete model $A_{n+1} = A_n F_n(A_n, A_{n-1}, \cdots, A_{n-m})$ of population growth, where $F_n : (0, \infty)^{m+1} \to (0, \infty)$ are continuous all. Second, we investigation the effect of delay m on the global attractivity of the unique positive equilibrium.

1. Introduction

Consider the model

$$A_{n+1} = A_n F_n(A_n, A_{n-1}, \cdots, A_{n-m}), \qquad n = 0, 1, \cdots, \tag{1.1}$$

where $F_n : (0, \infty)^{m+1} \to (0, \infty)$ are continuous all. This model is potentially appeared in medicine (for example, the population of blood cells) and was investigated by several authors [Graef, Liz, Tkachenko et al.] with more restrictions on F_n. If $F_n(x, y) = \exp(\gamma - \alpha x - \beta y)$ with $\alpha, \beta > 0$ we get back a model investigated by Tkachenko et al. (But they found no explicit conditions for the global attractivity of the positive equilibrium.) A positive solution $\{A_n\}_{n=-m}^{\infty}$ is called persistent if

$$0 < \liminf_{n\to\infty} A_n \leqslant \limsup_{n\to\infty} A_n < \infty.$$

The following theorem gives a sufficient condition for persistent (non-extinctive) model.

Theorem 1. *Assume that*

$$F_n(x_0, x_1, \cdots, x_m) \leqslant b < \infty \tag{1.2}$$

for all $n = 0, 1, \cdots,$ *and* $x_0, x_1, \cdots, x_m \geq 0,$

$$\liminf_{n\to\infty} \min_{x_0, x_1, \cdots, x_m \in [0,K]} F_n(x_0, x_1, \cdots, x_m) > 0 \tag{1.3}$$

for every $K > 0$*, and*

$$\limsup_{n, x_0, x_1, \cdots, x_m \to\infty} F_n(x_0, x_1, \cdots, x_m) < 1, \tag{1.4}$$

Received September 10, 2007; in revised form November 17, 2008.
2000 *Mathematics Subject Classification.* 39A12.
Key words and phrases. Positive equilibrium, oscillated solution, 3/2-stability.

$$\liminf_{n\to\infty, x_0, x_1, \cdots, x_m \to 0+0} F_n(x_0, x_1, \cdots, x_m) > 1. \tag{1.5}$$

Then every solution $\{A_n\}_{n=-m}^{\infty}$ *of* (1.1) *is persistent.*

Proof. First, we prove that $\{A_n\}_{n=-m}^{\infty}$ is bounded from above. Assume, for the sake of a contradiction, that $\limsup_{n\to\infty} A_n = \infty$. For each integer $n \geq m$, we define

$$k_n := \max\{\rho : -m \leqslant \rho \leqslant n, A_\rho = \max_{-m\leqslant i\leqslant n} A_i\}.$$

Observe that $k_{-m} \leqslant k_{-m+1} \leqslant \cdots \leqslant k_n \to \infty$ and that

$$\lim_{n\to\infty} A_{k_n} = \infty. \tag{1.6}$$

But $A_{k_n} \leqslant bA_{k_n-1}$, so

$$\lim_{n\to\infty} A_{k_n-1} = \infty. \tag{1.7}$$

Let $n_0 > 0$ such that $k_{n_0} > 0$. We have for $n > n_0$,

$$A_{k_n-1}F_{k_n-1}(A_{k_n-1}, A_{k_n-2}, \cdots, A_{k_n-1-m}) = A_{k_n} \geq A_{k_n-1}$$

and therefore,

$$F_{k_n-1}(A_{k_n-1}, A_{k_n-2}, \cdots, A_{k_n-1-m}) \geq 1.$$

By (1.4) and (1.7), this implies that

$$\limsup_{n\to\infty} \min\{A_{k_n-2}, \cdots, A_{k_n-1-m}\} < \infty. \tag{1.8}$$

On the other hand,

$$\begin{aligned} A_{k_n} &= A_{k_n-1}F_{k_n-1}(A_{k_n-1}, \cdots, A_{k_n-1-m}) = \cdots \\ &= A_{k_n-1-m}F_{k_n-1-m}(A_{k_n-1-m}, \cdots, A_{k_n-1-2m}) \times \cdots \times \\ &\quad \times F_{k_n-1}(A_{k_n-1}, \cdots, A_{k_n-1-m}) \\ &\leqslant \min\{A_{k_n-2}b^2, \cdots, A_{k_n-1-m}b^{m+1}\}. \end{aligned}$$

Now take lim sup on both sides we have $\limsup_{n\to\infty} A_{k_n} < \infty$ which contradicts (1.6). Thus, $\{A_n\}_{n=-m}^{\infty}$ is bounded from above. Let K be an upper bound of $\{A_n\}_{n=-m}^{\infty}$.

Next, we prove that $\liminf_{n\to\infty} A_n > 0$. Assume, for the sake of a contradiction, that $\liminf_{n\to\infty} A_n = 0$. For each integer $n \geq m$, we define

$$s_n := \max\{\rho : -m \leqslant \rho \leqslant n, A_\rho = \min_{-m\leqslant i\leqslant n} A_i\}.$$

Clearly, $s_{-m} \leqslant s_{-m+1} \leqslant \cdots \leqslant s_n \to \infty$ and that

$$\lim_{n\to\infty} A_{s_n} = 0. \tag{1.9}$$

But $A_{s_n} \geq aA_{s_n-1}$, where

$$a = \inf_{N\geq s_n-1-m} \min_{x_0, x_1, \cdots, x_m \in [0,K]} F_N(x_0, \cdots, x_m) > 0,$$

so

$$\lim_{n\to\infty} A_{s_n-1} = 0. \tag{1.10}$$

Let $n_0 > 0$ such that $s_{n_0} > 0$. We have for any $n > n_0$,

$$A_{s_n} = A_{s_n-1}F_{s_n-1}(A_{s_n-1}, \cdots, A_{s_n-1-m}) \geq A_{s_n}F_{s_n-1}(A_{s_n-1}, \cdots, A_{s_n-1-m})$$

and therefore,

$$F_{s_n-1}(A_{s_n-1}, \cdots, A_{s_n-1-m}) \leqslant 1.$$

By (1.5) and (1.10), this implies that

$$\liminf_{n\to\infty} \max\{A_{s_n-2}, \cdots, A_{s_n-1-m}\} > 0.$$

On the other hand,

$$\begin{aligned} A_{s_n} &= A_{s_n-1}F_{s_n-1}(A_{s_n-1}, \cdots, A_{s_n-1-m}) = \cdots \\ &= A_{s_n-1-m}F_{s_n-1-m}(A_{s_n-1-m}, \cdots, A_{s_n-1-2m}) \times \cdots \times \\ &\quad \times \cdots F_{s_n-1}(A_{s_n-1}, \cdots, A_{s_n-1-m}) \\ &\geq \max\{A_{s_n-2}a^2, \cdots, A_{s_n-1-m}a^{m+1}\}. \end{aligned}$$

Now take lim inf as $n \to \infty$ on both sides we have $\liminf_{n\to\infty} A_{s_n} > 0$ which contradicts (1.9) The proof is complete. □

2. The global attractivity

In this section we assume that there is a unique positive equilibrium $\bar{x}$ of (1.1) and

$$1 = F_n(\bar{x}, \cdots, \bar{x}), \tag{2.1}$$

for every $n = 0, 1, 2, \cdots$. Suppose further that if

$$F_n(x_0, x_1, \cdots, x_m) < 1,$$

then $\max\{x_0, x_1, \cdots, x_m\} > \bar{x}$, and if

$$F_n(x_0, x_1, \cdots, x_m) > 1,$$

then $\min\{x_0, x_1, \cdots, x_m\} < \bar{x}$.

A solution $\{A_n\}_{n=-m}^{\infty}$ is called nonoscillated, if

$$\limsup_{n\to\infty} A_n \leqslant \bar{x} \text{ or } \liminf_{n\to\infty} A_n \geq \bar{x}.$$

Lemma. *Every nonoscillated solution of* (1.1) *converges to* $\bar{x}$.

Proof. Without loss of generality we assume that

$$A_{n_0}, A_{n_0+1}, \cdots \geq \bar{x}$$

all. Then $F_{n_0}(A_{n_0}, A_{n_0-m+1}, \cdot, A_{n_0}) \leqslant 1$, so $A_{n_0+1} \leqslant A_{n_0}$. Similarly, $A_{n+1} \leqslant A_n$ for all $n \geq n_0, \cdots$. Therefore, there is a limit of $\{A_n\}_{n=-m}^{\infty}$. This limit is exactly $\bar{x}$.

To investigate the effect of delay, we suppose further that

$$\limsup_{n\to\infty} |\ln F_n(x_0, x_1, \cdots, x_m)| \leqslant L \max\Big\{\Big|\ln \frac{x_0}{\bar{x}}\Big|, \Big|\ln \frac{x_1}{\bar{x}}\Big|, \cdots, \Big|\ln \frac{x_m}{\bar{x}}\Big|\Big\} \tag{2.2}$$

for all $x_0, x_1, \cdots, x_m > 0$. □

Theorem 2. *Assume that* (1.2) – (1.5), (2.1) *and* (2.2) *hold. Suppose further that*

$$(m+\frac{3}{2})L<\frac{3}{2}.$$

Then every solution $\{A_n\}_{n=-m}^{\infty}$ *of* (1.1) *converges to* $\bar{x}$.

Proof. Without loss of generality we assume that $L(m+\frac{3}{2})\geq 1$ (if L is small, we can replace it by $1/(m+\frac{3}{2})$) and $\{A_n\}_{n=-m}^{\infty}$ is an oscillated solution. This means that there is a sequence $t_n\to\infty$ of integers such that $A_{t_n}\leqslant\bar{x}$, $A_{t_n+1}>\bar{x}$ and $t_{n+1}-t_n>2m$ for every $n=1,2,\cdots$. Let

$$\rho_n\geq\big|\ln\frac{A_t}{\bar{x}}\big|\quad\text{for every}\quad t\geq t_n-2m.$$

Then

$$\Big|\ln\frac{A_{t+1}}{A_t}\Big|=|\ln F_t(A_t,\cdots,A_{t-m})|\leqslant L\max\Big\{\big|\ln\frac{A_t}{\bar{x}}\big|,\cdots,\big|\ln\frac{A_{t-m}}{\bar{x}}\big|\Big\}\leqslant L\rho_1$$

for all $t\geq t_1-m$. Indeed, by our assumption, we have for every $\epsilon>0$,

$$|\ln F_t(A_t,\cdots,A_{t-m})|\leqslant(L+\epsilon)\max\Big\{\big|\ln\frac{A_t}{\bar{x}}\big|,\cdots,\big|\ln\frac{A_{t-m}}{\bar{x}}\big|\Big\}$$

if t is large enough. Here, we use L instead of $L+\epsilon$ legally. Let $A_{t_*}\leqslant\bar{x}$ with $t_*\geq t_1$. It follows that

$$\Big|\ln\frac{A_s}{\bar{x}}\Big|\leqslant\sum_{t=s}^{t_*-1}\big|\ln\frac{A_t}{A_{t+1}}\big|\leqslant\sum_{t=s}^{t_*}\big|\ln\frac{A_{t+1}}{A_t}\big|\leqslant L\rho_1(t_*+1-s)$$

for all $s\in[t_1-m,t_*]$. This is right because the last sum is of (t_*+1-s) terms and each of them is $\leqslant L\rho_1$. Furthermore,

$$\begin{aligned}\Big|\ln\frac{A_{t+1}}{A_t}\Big|=|\ln F_t(A_t,\cdots,A_{t-m})|&\leqslant L\max\Big\{\big|\ln\frac{A_t}{\bar{x}}\big|,\cdots,\big|\ln\frac{A_{t-m}}{\bar{x}}\big|\Big\}\\&\leqslant L^2\rho_1(t_*+m+1-t)\end{aligned}$$

for all $t\in[t_1,t_*+m]$. First, we prove that

$$\big|\ln\frac{A_t}{\bar{x}}\big|\leqslant\rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big)\quad\text{for all}\quad t>t_1+m.$$

If this were not so, let

$$T=\min\Big\{t>t_1+m:\quad A_t>\bar{x},\quad\big|\ln\frac{A_t}{\bar{x}}\big|>\rho_1\Big(L(m+\frac{3}{2})-\frac{1}{2}\Big)\Big\}.$$

If $A_{t_*} := \min\{A_{T-1}, \cdots, A_{T-(m+1)}\} \leqslant \bar{x}$ then $t_* + m + 1 \geq T > t_1 + m$ and

$$\begin{aligned} |\rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big)| &< \big|\ln\frac{A_T}{\bar{x}}\big| \leqslant \sum_{t=t_*}^{T-1}\Big|\ln\frac{A_{t+1}}{A_t}\Big| \leqslant \sum_{t=t_*}^{t_*+m}\Big|\ln\frac{A_{t+1}}{A_t}\Big| \\ &\leqslant \sum_{t=t_*}^{t_*+m-[\frac{1}{L}]} L\rho_1 + \sum_{t=t_*+m-[\frac{1}{L}]+1}^{t_*+m} L^2\rho_1(t_*+m+1-t) \\ &= L\rho_1\big(m+1-[\frac{1}{L}]\big) + \frac{1}{2}\rho_1 L^2[\frac{1}{L}]([\frac{1}{L}]+1) \\ &\leqslant \rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big). \end{aligned}$$

($[a]$ denotes the largest integer $\leqslant a$). This is a contradiction, so we have

$$\min\{A_{T-1}, \cdots, A_{T-(m+1)}\} > \bar{x}$$

and consequently,

$$F_{T-1}(A_{T-1}, \cdots, A_{T-(m+1)}) < 1. \tag{2.3}$$

Hence, $A_{T-1} > A_T$. By the minimality of T we should have $T = t_1 + m + 1$. Therefore,

$$\begin{aligned} |\rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big)| &< \big|\ln\frac{A_T}{\bar{x}}\big| \leqslant \sum_{t=t_1}^{T-1}\Big|\ln\frac{A_{t+1}}{A_t}\Big| \leqslant \sum_{t=t_1}^{t_1+m}\Big|\ln\frac{A_{t+1}}{A_t}\Big| \\ &\leqslant \sum_{t=t_1}^{t_1+m-[\frac{1}{L}]} L\rho_1 + \sum_{t=t_1+m-[\frac{1}{L}]+1}^{t_1+m} L^2\rho_1(t_1+m+1-t) \\ &= L\rho_1\big(m+1-[\frac{1}{L}]\big) + \frac{1}{2}\rho_1 L^2[\frac{1}{L}]([\frac{1}{L}]+1) \\ &\leqslant \rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big). \end{aligned}$$

This is a contradiction, so we have

$$\big|\ln\frac{A_t}{\bar{x}}\big| \leqslant \rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big) \quad \text{for all} \quad t > t_1 + m.$$

This result permits us to choose

$$\rho_2 = \rho_1\big(L(m+\frac{3}{2})-\frac{1}{2}\big).$$

Repeat the above argument (with t_1 and ρ_1 replaced by t_2 and ρ_2) we have

$$\big|\ln\frac{A_t}{\bar{x}}\big| \leqslant \rho_2\big(L(m+\frac{3}{2})-\frac{1}{2}\big) \quad \text{for all} \quad t > t_2 + m.$$

Using the assumption $\big(L(m+\frac{3}{2})-\frac{1}{2}\big) < 1$, we complete the proof. □

3. Application

A tipical example is the equation

$$A_{n+1} = A_n \exp(\gamma - \alpha A_n - \beta A_{n-1}).$$

Here $m = 1$ and we easily compute

$$\bar{x} = \frac{\gamma}{\alpha + \beta}, \quad L = \gamma e^{2\gamma}.$$

Hence, if $\gamma e^{2\gamma} < \frac{3}{5}$ the positive equilibrium is globally attractive.

Another example is the model of blood cells

$$A_{n+1} = \frac{\lambda A_n}{1 + \sum_{j=1}^{m} \alpha_{j,n} A_{n-j}}$$

where

$$\lambda > 1 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_{j,n} = \alpha \quad \text{is fixed.}$$

We easily compute

$$\bar{x} = \frac{\lambda - 1}{\alpha}, \quad L = \frac{\lambda - 1}{\lambda}.$$

Hence, if $(m + \frac{3}{2})\frac{\lambda - 1}{\lambda} < \frac{3}{2}$ the positive equilibrium is globally attractive.

Acknowledgement

The author wish to express his sincere thanks to the referee for various advices and suggestions.

References

[1] D. V. Giang, Persistence and global attractivity in the model $A_{n+1} = A_n F(A_{n-m})$, (*submitted*).
[2] J. R. Graef, C. Qian and P. W. Spikes, Stability in a population model, *Appl. Math. Comput.* **89** (1998), 119–132.
[3] J. R. Graef and C. Qian, Global attractivity in a nonlinear difference equation and application, *Dynam. Systems Appl.* **15** (2006), 89–96.
[4] E. Liz, A sharp global stability result for a discrete population model, *J. Math. Anal. Appl.* **330** (2007), 740–743.
[5] E. Liz, V. Tkachenko and S. Trofimchuk, Global stability in discrete population models with delayed-sensity dependence, *Math. Biosci.* **199** (2006), 26–37.
[6] V. Tkachenko and S. Trofimchuk, A global attractivity criterion for nonlinear non-autonomous difference equation, *J. Math. Anal. Appl.* **322** (2006), 901–912.

Hanoi Institute of Mathematics
18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
E-mail address: dangvugiang@yahoo.com