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ON MINIMAX AND GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M, N be two finitely generated R-modules. Let t be a non-negative integer. It is shown that if the local cohomology module $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$, then the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is minimax for all $i < t$. Also, we prove that if the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is minimax for all $i < t$, then for any minimax module L the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N)/L)$ is finitely generated. In particular, $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M, N)/L)$ is a finite set.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R and M, N two R-modules.

If R is local with maximal ideal $\mathfrak{m}$ and N is a finitely generated R-module, then it is known that the local cohomology module $H^i_{\mathfrak{m}}(N)$ is Artinian and so $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{m}}(N))$ is finitely generated for all i (see [11, Remark 1.3]).

Grothendieck [7] proposed the following conjecture: If $\mathfrak{a}$ is an ideal of R and N is a finitely generated R-module, then $\mathrm{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(N))$ is finitely generated for all i.

Hartshorne [8] showed that this conjecture is false in general. However, it is known that this conjecture is true in many situations, see [5, 11, 13, 16, 20]. On the other hand, an important problem in commutative algebra is determining when the set of associated primes of the local cohomology modules $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$ is finite (see [10, Problem 4]). Katzman in [12] gives a counterexample that this is not true in general. However, it is known that this is true in many situations, for example see [3, 15, 6]. There are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory was introduced by Herzog [9] (see also [19]), which is defined as follows:

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

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It is clear that $H^i_{\mathfrak{a}}(R, N)$ is just the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$.

The finiteness results of generalized local cohomology modules are not well understood. Recently, in [1], it is shown that if M, N are two finitely generated R-modules and the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is finitely generated for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is a finitely generated R-module.

The purpose of this note is to extend [3, Theorem 2.2], [1, Theorem 1.2] and [2, Theorem 2.5] to the class of minimax modules. A module is called a minimax module, when it has a finite submodule, such that the quotient by it is an Artinian module (see [21]). It is clear that every finitely generated module and every Artinian module is minimax. We also show that if M, N are two finitely generated R-modules and the local cohomology module $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$, then $H^i_{\mathfrak{a}}(M, N)$ is minimax for all $i < t$. For the definition of local cohomology and its basic properties, we refer the reader to [4].

2. The results

Lemma 2.1. (i) *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Then M is minimax if and only if L and N are both minimax. Then any subquotient of a minimax module as well as any finite direct sum of minimax modules is minimax.*
(ii) *Let M and N be two R-modules. If M is minimax and N is finitely generated, then $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$ are minimax for all $i \geq 0$.*

Proof. (i) see ([2, Lemma 2.1]).

(ii) We only prove the assertion for the Ext modules, and the proof for the Tor modules is similar. Since R is a Noetherian ring and N is finitely generated, it follows that N possesses a free resolution

$$F_. : \ldots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0,$$

consisting of finitely generated free modules. If $F_i = \oplus^n R$ for some integer n, then $\mathrm{Ext}^i_R(N, M) = H^i(\mathrm{Hom}_R(F_., M))$ is a subquotient of $\oplus^n M$. Therefore, it follows from (i), that $\mathrm{Ext}^i_R(N, M)$ is minimax for all $i \geq 0$. □

Theorem 2.2. *Let M, N be two finitely generated R-modules and that $H^j_{\mathfrak{a}}(N)$ be minimax for all $j < t$. Then $H^j_{\mathfrak{a}}(M, N)$ is minimax for all $j < t$.*

Proof. By [18, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M, N).$$

Since $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $i \geq 2$, by Lemma 2.1 we deduce that $E_i^{p,q}$ is minimax for all $i \geq 2$, $p \geq 0$, and $q < t$. There is a finite filtration

$$0 = \phi^{j+1} H^j \subseteq \phi^j H^j \subseteq \ldots \subseteq \phi^1 H^j \subseteq \phi^0 H^j = H^j_{\mathfrak{a}}(M, N)$$

such that $E_\infty^{i,j-i} \cong \phi^i H^j/\phi^{i+1}H^j$ for all $0 \le i \le j$. Since $E_i^{p,q} \cong E_\infty^{p,q}$ for i sufficiently large, we have that $E_\infty^{p,q}$ is minimax for all $q < t$. Hence, using the exact sequence

$$0 \longrightarrow \phi^{i+1}H^j \longrightarrow \phi^i H^j \longrightarrow E_\infty^{i,j-i} \longrightarrow 0 \quad (0 \le i \le j)$$

we get that $H_{\mathfrak{a}}^j(M,N)$ is minimax for all $j < t$. □

Corollary 2.3. *Let M, N be two finitely generated R-modules and that $H_{\mathfrak{a}}^j(N)$ be Artinian for all $j < t$. Then $H_{\mathfrak{a}}^j(M,N)$ is Artinian for all $j < t$.*

Proof. Apply Theorem 2.2 and the fact that the class of minimax modules includes all Artinian modules. □

The following theorem extends [1, Theorem 1.2], [2, Theorem 2.3] and [14, Corollary 2.4].

Theorem 2.4. *Let M, N be two finitely generated R-modules such that $H_{\mathfrak{a}}^i(M,N)$ is a minimax R-module for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N))$ is finitely generated. In particular, $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M,N))$ is finite.*

Proof. We will induct on t. The case $t = 0$ is obvious, because $H_{\mathfrak{a}}^0(M,N) \cong \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$ is finitely generated and so is $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^0(M,N))$.
Assume inductively that $t \ge 1$ and the result has been proved for all $i < t$. The exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0$$

induces the long exact sequence

$$\ldots \longrightarrow H_{\mathfrak{a}}^i(M, \Gamma_{\mathfrak{a}}(N)) \xrightarrow{\alpha} H_{\mathfrak{a}}^i(M,N) \xrightarrow{\beta} H_{\mathfrak{a}}^i(M, N/\Gamma_{\mathfrak{a}}(N)) \longrightarrow \ldots.$$

By [1, Lemma 1.1], $H_{\mathfrak{a}}^i(M, \Gamma_{\mathfrak{a}}(N))$ is finitely generated and so is $\operatorname{Im}(\alpha)$. By using the left exact functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ on the following exact sequences

$$0 \longrightarrow \operatorname{Im}(\alpha) \longrightarrow H_{\mathfrak{a}}^i(M,N) \longrightarrow \operatorname{Im}(\beta) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im}(\beta) \longrightarrow H_{\mathfrak{a}}^i(M, N/\Gamma_{\mathfrak{a}}(N)),$$

it is enough for us to show that $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, N/\Gamma_{\mathfrak{a}}(N)))$ is finitely generated. Hence, we can assume that N is an $\mathfrak{a}$-torsion-free R-module and so there exists an element $x \in \mathfrak{a}$ which is N-regular.

Now the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces the long exact sequence
$\ldots \longrightarrow H_{\mathfrak{a}}^i(M,N) \xrightarrow{x} H_{\mathfrak{a}}^i(M,N) \longrightarrow H_{\mathfrak{a}}^i(M,N/xN) \longrightarrow H_{\mathfrak{a}}^{i+1}(M,N) \xrightarrow{x} H_{\mathfrak{a}}^{i+1}(M,N) \longrightarrow \ldots.$
Therefore we deduce the exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^{i-1}(M,N)/xH_{\mathfrak{a}}^{i-1}(M,N) \longrightarrow H_{\mathfrak{a}}^{i-1}(M,N/xN) \longrightarrow 0 :_{H_{\mathfrak{a}}^i(M,N)} x \longrightarrow 0.$$

By Lemma 2.1(i) and the hypothesis, $H_{\mathfrak{a}}^{i-1}(M, N/xN)$ is minimax for all $i < t$, so that, by the inductive hypothesis, $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M, N/xN))$ is finitely generated.

On the other hand, the exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^{t-1}(M,N)/xH_{\mathfrak{a}}^{t-1}(M,N) \longrightarrow H_{\mathfrak{a}}^{t-1}(M,N/xN) \longrightarrow 0 :_{H_{\mathfrak{a}}^t(M,N)} x \longrightarrow 0$$

induces the long exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N)/xH_{\mathfrak{a}}^{t-1}(M,N)) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N/xN))$$
$$\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, 0 :_{H_{\mathfrak{a}}^t(M,N)} x) \longrightarrow \operatorname{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N)/xH_{\mathfrak{a}}^{t-1}(M,N)).$$

$\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N)/xH_{\mathfrak{a}}^{t-1}(M,N))$ is finitely generated, since $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N/xN))$ is finitely generated.

Also, by [17, Proposition 4.3] $\operatorname{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M,N)/xH_{\mathfrak{a}}^{t-1}(M,N))$ is finitely generated. Therefore $\operatorname{Hom}_R(R/\mathfrak{a}, 0 :_{H_{\mathfrak{a}}^t(M,N)} x)$ is finitely generated. Now, as $x \in \mathfrak{a}$ the result follows. □

Corollary 2.5. *Let N be a finitely generated R-module and let t be a non-negative integer such that $H_{\mathfrak{a}}^i(N)$ is minimax for all $i < t$. Then $H_{\mathfrak{a}}^i(N)$ is $\mathfrak{a}$-cofinite for all $i < t$, that is, $\operatorname{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(N))$ is finitely generated for all j and all $i < t$.*

Proof. We proceed by induction on i. The case $i = 0$ is obvious as $H_{\mathfrak{a}}^0(N)$ is finitely generated. So, let $i > 0$ and the result has been proved for smaller values of i. By induction assumption, $H_{\mathfrak{a}}^j(N)$ is $\mathfrak{a}$-cofinite for $j = 0, \ldots, i-1$. Hence by Theorem 2.4, $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(N))$ is finitely generated and by [17, Proposition 4.3] the result follows. □

The following corollary immediately follows by Corollary 2.5.

Corollary 2.6. *Let N be a finitely generated R-module. Then*

$$\inf\{i \mid H_{\mathfrak{a}}^i(N) \text{ is not minimax}\} \leq \inf\{i \mid H_{\mathfrak{a}}^i(N) \text{ is not } \mathfrak{a}\text{-cofinite}\}.$$

The following theorem extends [2, Theorem 2.5] and [3, Theorem 2.2].

Theorem 2.7. *Let t be a non-negative integer such that $H_{\mathfrak{a}}^i(M,N)$ is minimax for all $i < t$ and let L be a minimax submodule of $H_{\mathfrak{a}}^t(M,N)$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N)/L)$ is finitely generated. In particular, the set $\operatorname{Ass}_R(H_{\mathfrak{a}}^t(M,N)/L)$ is finite.*

Proof. By Theorem 2.4, $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N))$ is finitely generated. On the other hand, according to [17, Proposition 4.3], L is $\mathfrak{a}$-cofinite. Now, the exact sequence

$$0 \longrightarrow L \longrightarrow H_{\mathfrak{a}}^t(M,N) \longrightarrow H_{\mathfrak{a}}^t(M,N)/L \longrightarrow 0$$

induces the following exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N)) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N)/L) \longrightarrow \operatorname{Ext}_R^1(R/\mathfrak{a}, L).$$

Consequently $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N)/L)$ is finitely generated. □

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