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A METHOD OF EXTENDING RANDOM OPERATORS

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Abstract. In this paper, we introduce a method of extending the domain of a random operator to a class of random inputs. This method is based on the convergence of certain random series.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X, Y be separable Banach spaces. By a random operator A from X into Y we mean a linear continuous mapping from X into the Frechet space $L_0^Y(\Omega, \mathcal{F}, P) = L_0^Y(\Omega)$ of all Y-valued random variables. Random operators can be regarded as a random generalization of deterministic linear continuous operators and as well as a natural framework for stochastic integrals. Some results on random operators can be found in [6, 8, 9, 10].

A random operator A from X into Y may be considered as an action which transforms linearly and continuously each deterministic input $x \in X$ into a random output Ax. This original definition of random operator cannot be applied to X-valued random variables (r.v.'s). Taking into account many circumstances in which the inputs are also subject to the influence of a random environment, one needs to define the action of A on some random outputs, i.e. to extend the domain of A to some classes of X-valued r.v.'s. A method of extending the domain of a Gaussian random operator on a Hilbert space H to a class of H-valued r.v.'s was introduced by Dorogovtsev in [1].

In this paper, we propose another method of extending the domain of A to some class $\mathcal{D}(A)$ of X-valued r.v.'s. This method is based on the convergence of certain random series provided that X is a Banach space with the Schauder basis. We shall show that $\mathcal{D}(A)$ is a dense linear subspace of L_0^X and $\mathcal{D}(A) = L_0^X$ if and only if A is a bounded random operator. We also determine some conditions for an X-valued r.v. to be in the $\mathcal{D}(A)$.

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2. The domain of extension of a random operator

Let X, Y be separable Banach spaces. $L_0^X = L_0^X(\Omega)$ and $L_0 = L_0^R$ stand for the set of all X-valued random variables (r.v.'s) and the set of all real-valued r.v.'s, respectively. The set L_0^X equipped with the topology of convergence in probability is a Fréchet space. By a random operator from X into Y we mean a linear continuous mapping A from X into L_0^Y. For examples of random operators, we refer to [10].

Throughout this paper, X is a Banach space with the Schauder basis $e = (e_n)_{n=1}^{\infty}$. The conjugate basis is denoted by $e^* = (e_n^*)_{n=1}^{\infty}$. Then for each $x \in X$ we have

$$x = \sum_{n=1}^{\infty} (x, e_n^*) e_n.$$

Since A is linear and continuous, we get

$$Ax = \sum_{n=1}^{\infty} (x, e_n^*) A e_n,$$

where the series converges in probability.

Denote by $\mathcal{D}(A)$ the set of all X-valued r.v. u for which the series

$$\sum_{n=1}^{\infty} (u, e_n^*) A e_n \tag{1}$$

converges in probability. Clearly, $X \subset \mathcal{D}(A) \subset L_0^X$.

Definition 2.1. $\mathcal{D}(A)$ is called the domain of extension of A.

If $u \in \mathcal{D}(A)$ then the sum (1) is denoted by Φu and it is understood as the action of A on the random variable u.

In general, the domain $\mathcal{D}(A)$ as well as the values $\Phi u, u \in \mathcal{D}(A)$, depend on the basis $e = (e_n)$.

Proposition 2.1. *The following properties are valid:*

(i) *$\mathcal{D}(A)$ is a linear subspace of L_0^X and $\Phi : \mathcal{D}(A) \to L_0^Y$ is linear.*

(ii) *If $\alpha \in L_0$ and $u \in \mathcal{D}(A)$ then $\alpha u \in \mathcal{D}(A)$ and*

$$\Phi(\alpha u) = \alpha \Phi u.$$

In particular, if u is of the form $u = \sum_{i=1}^{n} \xi_i x_i$, $x_i \in X$, $\xi_i \in L_0$ then $u \in \mathcal{D}(A)$ and

$$\Phi(u) = \sum_{i=1}^{n} \xi_i A x_i.$$

(iii) *If u is a countably-valued r.v.*

$$u = \sum_{i=1}^{\infty} 1_{E_i} x_i,$$

then $u \in \mathcal{D}(A)$ and

$$\Phi(u) = \sum_{i=1}^{\infty} 1_{E_i} A x_i = A(u(\omega))(\omega)$$

which does not depend on the basis (e_n). In particular, $\mathcal{D}(A)$ is dense in L_0^X.

Proof. (i) The linearity of Φ is obvious.

(ii) We need the following claim, which is easy to prove.

Claim 1. If $\alpha \in L_0, X_n \in L_0^X, X_n \xrightarrow{\mathrm{P}} X$ then $\alpha X_n \xrightarrow{\mathrm{P}} \alpha X$. If $\alpha_n \in L_0, X \in L_0^X$ and $\alpha_n \xrightarrow{\mathrm{P}} \alpha$, then $\alpha_n X \xrightarrow{\mathrm{P}} \alpha X$.

Now put $Y_n = \sum_{i=1}^{n} (\alpha u, e_i^*) A e_i, X_n = \sum_{i=1}^{n} (u, e_i^*) A e_i$. We have $Y_n = \alpha X_n$. Because $X_n \xrightarrow{\mathrm{P}} \Phi(u)$ by the above claim $Y_n = \alpha_n X \xrightarrow{\mathrm{P}} \alpha \Phi(u)$. Hence $\alpha u \in \mathcal{D}(A)$ and $\Phi(\alpha u) = \alpha \Phi(u)$.

(iii) Put

$$Z_n = \sum_{k=1}^{n} (u, e_k^*) A e_k,\ Z = \sum_{i=1}^{\infty} 1_{E_i} A x_i = A(u(\omega))(\omega).$$

We want to show that $Z_n \xrightarrow{\mathrm{P}} Z$. For each i we have $p\text{-}\lim_n 1_{E_i} Z_n = 1_{E_i} A x_i = 1_{E_i} Z$. Hence

$$\begin{aligned} P(\|Z_n - Z\| > t) &= \sum_{i=1}^{\infty} P(\|Z_n - Z\| > t, E_i) \\ &\leqslant \sum_{i=1}^{N} P(\|1_{E_i} Z_n - 1_{E_i} Z\| > t) + \sum_{i=N+1}^{\infty} P(E_i) \end{aligned}$$

Letting $n \to \infty$ and $N \to \infty$ we get $\lim_n P(\|Z_n - Z\| > t) = 0$. □

Example 2.1. Let $X = l_p, Y = l_t$ and (α_n) be the standard r-stable sequence $(1 < r < 2)$, where $1 < p < r < t < 2p$ and $e_n = (0, \ldots, 0, 1, \ldots)$. We claim that

(a) For each $x \in X$ the series

$$\sum_{n=1}^{\infty} \alpha_n (x, e_n^*) e_n \tag{2}$$

converges a.s. in $Y = l_t$ and defines a random operator A from X into Y.

(b) For each sequence $c = (c_n) \in l_p$, the series

$$\sum_{n=1}^{\infty} \alpha_n c_n e_n$$

converges in $X = l_p$ and defines an X-valued r.v. u.

(c) $u \in \mathcal{D}(A)$ if and only if $(c_n) \in l_{r/2}$.

(One has $l_{r/2} \subset l_p$ because $r < 2p$).

We shall need the following lemma due to L. Schwartz, see [5].

Lemma 1. *Let (α_n) be the standard r-stable sequence $(1 < r < 2)$, (c_n) be a sequence of real numbers, $1 \leqslant s < \infty, s \neq r$ and $e_n = (0, \ldots, 0, 1, \ldots)$. For the series*

$$\sum_{n=1}^{\infty} \alpha_n c_n e_n$$

to be convergent in l_s, it is necessary and sufficient that

(i) $(c_n) \in l_s$ *for the case* $s < r$,

(ii) $(c_n) \in l_r$ *for the case* $s > r$.

Now we are ready to prove the claims (a)-(c) of Example 2.1.

(a) $\sum |(x, e_n^*)|^p < \infty$ and $p < r$ imply that $\sum |(x, e_n^*)|^r < \infty$. Because $t > r$ by Lemma 1, we see that the series (2) converges a.s. in $Y = l_t$.

The formula

$$Ax = \sum_{n=1}^{\infty} \alpha_n (x, e_n^*) e_n \tag{3}$$

defines a random operator A from X into Y.

(b) Since $p < r$, by Lemma 1 the series

$$\sum_{n=1}^{\infty} \alpha_n c_n e_n$$

converges in $X = l_p$.

(c) We have

$$\sum_{n=1}^{\infty} (u, e_n^*) A e_n = \sum_{n=1}^{\infty} \alpha_n^2 c_n e_n.$$

Consequently, $u \in \mathcal{D}(A)$ if and only if $\sum\limits_{n=1}^{\infty} \alpha_n^{2t} |c_n|^t < \infty$, i.e., the series

$$\sum_{n=1}^{\infty} \alpha_n \sqrt{|c_n|} e_n$$

converges in l_{2t}. Since $2t > r$, by Lemma 1 we conclude that $u \in \mathcal{D}(A)$ if and only if $(\sqrt{|c_n|}) \in l_r$, that is, $(c_n) \in l_{r/2}$.

The following example shows that $\mathcal{D}(A)$ needs not be a closed subspace of L_0^X and the mapping $\Phi : \mathcal{D}(A) \to L_0^Y$ needs not be continuous.

Example 2.2. Let $X = L_2[0;1]$ and A be a random operator from X into R defined by the Wiener stochastic integral

$$Ax = \int_0^1 x(t)dW(t),$$

where $W(t)$ is a Wiener process. Let (e_n) be an orthonormal basis of X. Put $\xi_n = Ae_n$. It is well-known that (ξ_n) is a sequence of Gaussian i.i.d. random variables $N(0,1)$. Put

$$u_n = \sum_{k=1}^{n} \frac{\xi_k}{k} e_k, \; u = \sum_{k=1}^{\infty} \frac{\xi_k}{k} e_k.$$

The latter series converges a.s. in the norm of X since

$$\sum_{i=1}^{\infty} \|\frac{e_k}{k}\|^2 = \sum_{i=1}^{\infty} \frac{1}{k^2} < \infty$$

so $u_n \xrightarrow{\text{P}} u$. By Proposition 2.1 $u_n \in \mathcal{D}(A)$. We now prove $u \notin \mathcal{D}(A)$ with the help of the following claim.

Claim 2. Let (α_n) be a sequence of real-valued independent Gaussian random variables with $E\alpha_n = 0$. If $\sum_n \alpha_n^2 < \infty$ a.s, then $\sum_n E\alpha_n^2 < \infty$.

Indeed, put $\alpha = (\alpha_n)_{n=1}^{\infty}$. As $\sum_n \alpha_n^2 < \infty$ a.s, α defines a random variable Gaussian with values in the Hilbert space l_2. By a theorem of Fernique (see [2]) we get $\sum_n E\alpha_n^2 = E\|\alpha\|^2 < \infty$ as desired.

Put

$$\alpha_n = \frac{\xi_n}{\sqrt{n}}.$$

Because $\sum_n E\alpha_n^2 = \sum_n \frac{1}{n} = \infty$, by Claim 2, we infer that

$$\sum_{i=1}^{\infty} (u, e_n) A e_n = \sum_{i=1}^{\infty} \frac{\xi_n^2}{n} = \sum_{i=1}^{\infty} \alpha_n^2 = \infty \quad \text{a.s.}$$

Hence $u \notin \mathcal{D}(A)$ as desired. Next, we show that the mapping $\Phi : \mathcal{D}(A) \to L_0$ is not continuous. Put

$$a_k = (a_{ki})_{i \geq 1} = \Big(\underbrace{\frac{1}{k}, ..., \frac{1}{k}}_{k}, 0, ..., 0, ...\Big), \; k \geq 1,$$

$$\xi_i = Ae_i, \; \alpha_{ki} = a_{ki}\xi_i, \; v_k = \sum_{i=1}^{\infty} \alpha_{ki} e_i = \sum_{i=1}^{k} \alpha_{ki} e_i.$$

Then (ξ_i) is a sequence of i.i.d. random variables $N(0,1)$. By Proposition 2.1, $v_k \in \mathcal{D}(A)$. From the law of large numbers it follows that

$$\|v_k\|^2 = \sum_{i=1}^{k} \alpha_{ki}^2 = \frac{1}{k^2}\sum_{i=1}^{k} \xi_i^2 \to 0 \text{ a.s. as } k \to \infty;$$

so $v_k \to 0$ in L_0^X. But, again by the law of large numbers,

$$\Phi(v_k) = \sum_{i=1}^{\infty}(v_k, e_i)Ae_i = \sum_{i=1}^{\infty} \alpha_{ki}\xi_i = \frac{1}{k}\sum_{i=1}^{k} \xi_i^2 \to 1 \text{ a.s. as } k \to \infty.$$

Therefore, Φ is not a continuous mapping from $\mathcal{D}(A)$ into L_0 as claimed.

The following theorem characterizes random operators A for which $\mathcal{D}(A) = L_0^X$.

Theorem 2.1. *If A is a bounded random operator then $\mathcal{D}(A) = L_0^X$ and Φu does not depend on the basis (e_n). Conversely, if $\mathcal{D}(A) = L_0^X$ then A must be a bounded random operator.*

Proof. Recall (see[10]) that a random operator A is said to be bounded if there exists a positive real-valued random variable $k(\omega)$ such that for each $x \in X$

$$\|Ax(\omega)\| \leqslant k(\omega)\|x\| \quad \text{a.s.}$$

Note that the exceptional set may depend on x.

Suppose that A is bounded, by Theorem 3.1 in [10] there exists a mapping

$$T : \Omega \to L(X, Y)$$

such that for each $x \in X$ it holds

$$Ax(\omega) = T(\omega)x \qquad \text{a.s.}$$

So there is a set D with $P(D) = 1$ such that for each $\omega \in D$ and for all n we have

$$Ae_n(\omega) = T(\omega)e_n.$$

Thus for each $\omega \in D$,

$$\begin{aligned}\sum_{n=1}^{\infty}(u(\omega), e_n^*)Ae_n(\omega) &= \sum_{n=1}^{\infty}(u(\omega), e_n^*)T(\omega)e_n \\ &= T(\omega)\left(\sum_{n=1}^{\infty}(u(\omega), e_n^*)e_n\right) = T(\omega)(u(\omega)).\end{aligned}$$

Hence the series $\sum_{n=1}^{\infty}(u, e_n^*)Ae_n$ converges a.s.; so it converges in probability. Consequently, $u \in \mathcal{D}(A)$ and $\Phi u(\omega) = T(\omega)(u(\omega))$ does not depend on the basis $e = (e_n)$.

To prove the second claim of the theorem, suppose that $\mathcal{D}(A) = L_0^X$. Put

$$\Phi_n u = \sum_{i=1}^{n} (u, e_i^*) A e_i$$

and note that Φ_n is a linear continuous mapping from L_0^X into L_0^Y. By our assumption, $\lim_n \Phi_n u = \Phi u$ for all $u \in L_0^X$. By the Banach-Steinhaus theorem, Φ is a linear continuous mapping from L_0^X into L_0^Y. In addition, we have

$$\Phi(u) = \sum_{i=1}^{n} 1_{E_i} A x_i$$

for $u = \sum_{i=1}^{n} 1_{E_i} x_i$. By Theorem 5.3 in [10] we conclude that A is bounded. □

For each random operator A, let $\mathcal{F}(A)$ denote the σ-algebra generated by the family $\{Ax, x \in X\}$. A random variable $u \in L_0^X$ is said to be independent of A if $\mathcal{F}(u)$ and $\mathcal{F}(A)$ are independent.

Theorem 2.2. *Suppose that u is independent of A. Then $u \in \mathcal{D}(A)$. Moreover, Φu does not depend on the basis (e_n).*

Proof. Let $t > 0$. By the independence of u and the sequence (Ae_n) we have

$$P\left(\|\sum_{i=m}^{n} (u, e_i^*) A e_i\| > t\right) = \int_X P\left(\|\sum_{i=m}^{n} (x, e_i^*) A e_i\| > t\right) d\mu(x), \tag{4}$$

where μ is the distribution of u. Because for each $x \in X$ it holds

$$\lim_{m,n\to\infty} P\left(\|\sum_{i=m}^{n} (x, e_i^*) A e_i\| > t\right) = 0,$$

by the dominated convergence theorem we infer that the series

$$\sum_{i=1}^{\infty} (u, e_i^*) A e_i$$

converges in L_0^Y ,i.e., $u \in \mathcal{D}(A)$.

Next, let V be the subset of L_0^X consisting of r.v.'s independent of A and let $V_0 \subset V$ be the linear subspace of simple r.v.'s. It is easy to see that V is a closed subspace of L_0^X and V_0 is dense in V equipped with the topology of L_0^X. For each n we define a mapping $\Phi_n : V \to L_0^Y$ by setting

$$\Phi_n u = \sum_{i=1}^{n} (u, e_i^*) A e_i.$$

It is easy to see that Φ_n is a linear continuous mapping from V into L_0^Y and $\lim_n \Phi_n u = \Phi u$ for all $u \in V$. By the Banach-Steinhaus theorem, $\Phi : V \to L_0^Y$ is again a linear continuous mapping. On the other hand, by Proposition 2.1, if $u \in V_0$ then Φu takes the same values for all the basis e. Since Φ is continuous on V and V_0 is dense in V we conclude that Φu also takes the same values for all the basis e. □

3. The case where Ae_i's are independent

In this section A is always assumed to be a random operator from X into Y such that the sequence of Y-valued r.v.'s (Ae_i) is independent. For example, if A is a random operator from $L_2[0;1]$ into R defined by the Wiener stochastic integral

$$Ax = \int_0^1 x(t)dW(t)$$

then the sequence (Ae_i) is independent, provided that (e_n) is an orthonormal basis of $L_2[0;1]$ (see Example 2.2.)

Theorem 3.1. *Let Y be a Hilbert space. Denote by $\mathcal{F}_n$ the σ-algebra generated by $(Ae_1, ..., Ae_n)$. Then for each $u \in L_0^X$ the condition*

$$(u, e_n^*) \quad \textit{is } \mathcal{F}_{n-1}\textit{-measurable, for each } n > 1,$$

is sufficient for $u \in \mathcal{D}(A)$.

The proof is based on the following lemma

Lemma 2. *Let Y be a Hilbert space and (z_n) be a sequence of r.v.'s taking values in Y. Denote by $\mathcal{F}_n$ the σ-algebra generated by $(z_1, \ldots, z_n)$, and by $\mu_n(\omega)$ the regular conditional distribution of z_n given $\mathcal{F}_{n-1}$. Suppose that for almost ω the sequence (μ_n) is summable in the following sense: If (ξ_n) is a sequence of Y-valued independent r.v.'s defined on another probability space such that the distribution of ξ_n is $\mu_n(\omega)$, then the series $\sum \xi_n$ converges in L_0^Y. Under this condition, the series $\sum_n z_n$ converges in L_0^Y.*

Lemma 2 can be proved by the same argument as given in the proof of Theorem 2 in [3] by using the Kolmogorov three-series theorem for independent r.v.'s taking values in Hilbert spaces (see [7]).

Proof of Theorem 3.1. Let $\mu_n(\omega)$ be the regular conditional distribution of $z_n = (u, e_n^*)Ae_n$ given by $\mathcal{F}_{n-1}$. Since (u, e_n^*) is $\mathcal{F}_{n-1}$- measurable and Ae_n is independent of $\mathcal{F}_{n-1}$, we have

$$\begin{aligned} \mu_n(\omega)(E) &= P\{(u, e_n^*)Ae_n \in E | \mathcal{F}_{n-1}\} \\ &= P\left\{\omega' : (u(\omega), e_n^*)Ae_n(\omega') \in E\right\}. \end{aligned} \tag{5}$$

Let $\nu_n(x)$ be the distribution of the r.v. $(x, e_n^*)Ae_n$. From (5) we get

$$\mu_n(\omega) = \nu_n[u(\omega)]. \tag{6}$$

As for each $x \in X$ the sequence $\{(x, e_n^*)Ae_n\}$ are independent and the series $\sum_n (x, e_n^*)Ae_n$ converges in L_0^Y, from (6) it follows that the sequence (μ_n) is summable. By Lemma 2, we conclude that the series $\sum_n (u, e_n^*)Ae_n$ converges in L_0^Y, i.e., $u \in \mathcal{D}(A)$. □

Recall that a Banach space is said to be p-uniformly smooth $(1 < p \leqslant 2)$ if the modulus of smoothness defined by

$$\rho(t) = \sup_{\|x\|=1, \|y\|=t} \left\{ \frac{\|x+y\| + \|x-y\| - 2}{2} \right\}$$

satisfies the condition $\rho(t) = 0(t^p)$.

A Banach space is called p-smoothable if it is isomorphic to a p-uniformly smooth space (see [11]). The spaces L_p, l_p are $\min(2,p)$-smoothable spaces.

Theorem 3.2. *Let $X = l_p$ and Y be a Banach space of p-smoothable $(1 < p \leqslant 2)$ and (e_i) be the standard basis in l_p. Suppose that $E\|Ax\|^p < \infty$ for all $x \in X$ and $EAe_i = 0$ for all i. Then for each $u \in L_0^X$ the condition*

$$(u, e_n^*) \quad \text{is } \mathcal{F}_{n-1}\text{-measurable, for each } n > 1 \tag{7}$$

is sufficient for $u \in \mathcal{D}(A)$.

Proof. Let $t, \epsilon > 0$ be given. Put

$$u_{mn} = \sum_{i=m}^{n} \alpha_i e_i, \quad \alpha_i = (u, e_i^*),$$

$$C_i = \{\omega : \sum_{k=m}^{i} |\alpha_k|^p \leqslant \epsilon^p\}, \quad \xi_i = \alpha_i 1_{C_i}$$

and

$$u_\epsilon = \sum_{i=m}^{n} \xi_i e_i.$$

We have

$$\begin{aligned} P\{\|\Phi u_{mn}\| > t; \|u_{mn}\| \leqslant \epsilon\} &= P\{\| \sum_{k=m}^{n} \alpha_k A e_k\| > t, \|u_{mn}\| \leqslant \epsilon\} \\ &= P\{\| \sum_{k=m}^{n} \xi_k A e_k\| > t, \|u_{mn}\| \leqslant \epsilon\} \\ &\leqslant P\{\|\Phi u_\epsilon\| > t\}, \end{aligned} \tag{8}$$

because the inequality $\|u_{mn}\| \leqslant \epsilon$ implies that $\alpha_i = \xi_i$ for all $m \leqslant i \leqslant n$.

The assumption that α_i is $\mathcal{F}_{i-1}$-measurable implies that ξ_i is $\mathcal{F}_{i-1}$-measurable. Since $EAe_i = 0$, the sequence $(\xi_i A e_i, \mathcal{F}_i)_{i=m}^n$ constitutes an Y-valued martingale difference. As Y is p-smoothable by the Assoad-Pisier inequality (see [11]) there exists a constant $C_1 > 0$ such that

$$E\|\Phi u_\epsilon\|^p = E\left\|\sum_{i=m}^{n} \xi_i A e_i\right\|^p \leqslant C_1 \sum_{i=m}^{n} E\|\xi_i A e_i\|^p.$$

Since $E\|Ax\|^p < \infty$, the random operator A is a mapping from X into L_p^Y. By the closed graph theorem, A is continuous. Hence there is a constant $C_2 > 0$

such that for all $x \in X$ $E\|Ax\|^p \leqslant C_2||x||^p$. In particular, $E\|Ae_k||^p \leqslant C_2$ for all k. Hence

$$E\|\xi_i Ae_i\|^p = E\left\{|\xi_i|^p E\{\|Ae_i\|^p|\mathcal{F}_{i-1}\}\right\} = E|\xi_i|^p E\|Ae_i\|^p \leqslant C_2 E\|\xi_i\|^p. \tag{9}$$

Therefore,

$$E\|\Phi u_\epsilon\|^p \leqslant C_1C_2 \sum_{i=m}^{n} E\|\xi_i\|^p = CE\|u_\epsilon\|^p, \quad \text{where } C = C_1C_2.$$

We have $\|u_\epsilon\|^p = \sum\limits_{k=m}^{n} |\alpha_k|^p 1_{C_k}$. For each fixed ω, if $|\alpha_m(\omega)|^p > \epsilon^p$ then $u_\epsilon(\omega) = 0$. Otherwise, let $i(\omega)$ be the largest index such that $\sum\limits_{k=m}^{i(\omega)} |\alpha_k(\omega)|^p \leqslant \epsilon^p$. Then $\|u_\epsilon(\omega)\|^p = \sum\limits_{k=1}^{i(\omega)} |\alpha_k(\omega)|^p \leqslant \epsilon^p$. Hence, we always have $\|u_\epsilon\|^p \leqslant \epsilon^p$ which implies that

$$E\|\Phi u_\epsilon\|^p \leqslant C\epsilon^p. \tag{10}$$

By Chebyshev's inequality, we have

$$P\{\|\Phi u_\epsilon\| > t\} \leqslant \frac{E\|\Phi u_\epsilon\|^p}{t^p}. \tag{11}$$

From (8)-(11) we get

$$P\{\|\Phi u_{mn}\| > t; \|u_{mn}\| \leqslant \epsilon\} \leqslant \frac{C\epsilon^p}{t^p}. \tag{12}$$

Consequently,

$$\begin{aligned} P\{\|\Phi u_{mn}\| > t\} &\leqslant P\{\|\Phi u_{mn}\| > t; \|u_{mn}\| \leqslant \epsilon\} + P\{\|u_{mn}\| > \epsilon\} \\ &\leqslant \frac{C\epsilon^p}{t^p} + P\{\|u_{mn}\| > \epsilon\}. \end{aligned}$$

Letting $m, n \to \infty$ we get

$$\limsup_{m,n\to\infty} P\{\|\Phi u_{mn}\| > t\} \leqslant \frac{C\epsilon^p}{t^p}.$$

Taking the limit as $\epsilon \to 0$ we get $\lim_{m,n\to\infty} P\{\|\Phi u_{mn}\| > t\} = 0$, i.e.,

$$\lim_{m,n\to\infty} P\left\{\|\sum_{i=m}^{n}(u, e_i^*)Ae_i\| > t\right\} = 0,$$

that is $u \in \mathcal{D}(A)$. □

Theorem 3.3. *Let Y be a Banach space which is p-smoothable ($1 < p \leqslant 2$). Suppose that $E\|Ax\|^p < \infty$ for all $x \in X$ and $EAe_i = 0$ for all i. Then for each $u \in L_0^X$, the conditions*

$$(u, e_n^*) \quad \textit{is } \mathcal{F}_{n-1}\textit{-measurable for each } n > 1 \tag{13}$$

and

$$\sum_n E|(u, e_n^*)|^p < \infty \tag{14}$$

imply that $u \in \mathcal{D}(A)$.

Proof. Put $\alpha_i = (u, e_k^*)$. From (13), the independence of (Ae_i), and equalities $EAe_i = 0$ it follows that $(\alpha_i Ae_i, \mathcal{F}_i)$ forms a Y-valued martingale difference. Since Y is p-smoothable by the Assoad-Pisier inequality (see [11]), there exists a constant $C_1 > 0$ such that

$$E\|\sum_{i=m}^{n} \alpha_i Ae_i\|^p \leqslant C_1 \sum_{i=m}^{n} E\|\alpha_i Ae_i\|^p. \tag{15}$$

As $E\|Ax\|^p < \infty$, the random operator A is a mapping from X into L_p^Y. By the closed graph theorem, A is continuous. Hence there is a constant $C_2 > 0$ such that for all $x \in X$ it holds $E\|Ax||^p < C_2||x||^p$. In particular, $E\|Ae_k||^p < C_2$ for all k. Therefore,

$$E\|\alpha_i Ae_i\|^p = E\left\{|\alpha_i|^p E\{\|Ae_i\|^p|\mathcal{F}_{i-1}\}\right\} = E|\alpha_i|^p E\|Ae_i\|^p \leqslant C_2 E\|\alpha_i\|^p. \tag{16}$$

From (15) and (16) we get

$$E\|\sum_{i=m}^{n} \alpha_i Ae_i\|^p \leqslant C_1 C_2 \sum_{i=m}^{n} E\|\alpha_i\|^p. \tag{17}$$

From (14) and (17) we conclude that the series $\sum\limits_{i=1}^{\infty} \alpha_i Ae_i$ converges in L_p^Y so in L_0^Y. □

Remark. Without condition (13), condition (14) does not imply that $u \in \mathcal{D}(A)$. Indeed, in Example 2.2, $p = 2$ and the random operator A satisfying $E|Ax|^2 < \infty, EAe_i = 0$ and $Y = R$ is 2 - smoothable. Condition (14) holds for the random variable u because

$$\sum_k E|(u, e_k^*)|^2 = \sum_k \frac{1}{k^2} < \infty,$$

but $u \notin \mathcal{D}(A)$.

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