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A LOGARITHMIC QUADRATIC REGULARIZATION METHOD FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract. We use the logarithmic quadratic function to develop two iterative algorithms for solving equilibrium problems. We first use the Bregman distance function to solve a pseudomonotone equilibrium problem satisfying a certain Lipschitz condition. Next, to avoid the Lipschitz condition we combine this technique with line search technique to obtain a convergent algorithm for pseudomonotone equilibrium problems.

1. Introduction

Let C be a nonempty closed convex set in the real Euclidean space $\mathbb{R}^n$ and $f : C \times C \to \mathbb{R}$ be such that $f(x, x) = 0$ for every $x \in C$. We consider the following equilibrium problem:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in C. \tag{EP}$$

Problem (EP) is very general in the sense that it includes, as special cases, optimization problem, variational inequality, saddle point problem, Nash equilibrium problem and others [20, 22, 29]. Monotone equilibrium problems have been considered by a lot of authors [21, 24, 26, 28]. Conditions for existence of solutions can be found, for example, in [11] and recently in [1, 17, 18, 19, 33]. It is well known that the logarithmic quadrative regularization technique is a powerfull tool for analyzing and solving optimization problems. Recently this technique has been used to develop iterative algorithm for variational inequalities [6, 10].

In some recent papers [3, 4, 8], authors have investigated contraction and nonexpansiveness properties for mixed multivalued monotone variational inequalities and developed algorithms for solving them.

In this paper we extend our results in [3] to pseudomonotone equilibrium problem (EP). Namely, we first develop a linearly convergent algorithm for (EP) with f being pseudomonotone bifunction satisfying a certain Lipschitz type condition on C by using the Bregman distance function. Next, in order to avoid the Lipschitz condition we will use the line search and the Bregman distance function to

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obtain a convergent algorithm for solving equilibrium problem (EP) with pseudomonotone equilibrium bifunction f.

The paper is organized as follows. In the next section we list some examples and summarize some basic properties used in this paper. In Section 3, we present a linearly convergent algorithm for pseudomonotone and Lipschitz equilibrium problems. In the fourth section we modify the algorithm by combining a line search technique with the Bregman distance function, which allows avoiding the Lipschitz condition. Applications to nonlinear complementarity problems are discussed in the last section.

2. Preliminaries

We list some important results which will be required in our following analysis. Let C be a closed convex set in $\mathbb{R}^n$ with the Euclidean norm $||.||$, we denote the projection on C by $P_C(.)$, i.e,

$$P_C(x) = \text{argmin}\{||y-x||: \ y \in C\} \ \ \forall x \in \mathbb{R}^n.$$

From the above definition and the convexity of C, it follows that

$$||P(x) - y|| \leq ||x-y|| \ \ \forall y \in C, x \in \mathbb{R}^n.$$

Definition 2.1. Let C be a convex set in $\mathbb{R}^n$, and let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$. The bifunction f said to be
(i) monotone on C if for each $x, y \in C$, if we have

$$f(x,y) + f(y,x) \leq 0;$$

(ii) pseudomonotone on C if for each $x, y \in C$, it holds

$$f(x,y) \geq 0 \ \text{ implies } \ f(y,x) \leq 0;$$

(iii) Lipschitz with two constants $\bar{c_1} > 0$ and $\bar{c_2} > 0$ on C, if we have

$$f(x,y) + f(y,z) \geq f(x,z) - \bar{c_1}||y-x||^2 - \bar{c_2}||z-y||^2 \ \ \forall x,y,z \in C. \tag{2.1}$$

We note that when $x = z$ and $f(x,x) = 0$, this condition deduces to

$$f(x,y) + f(y,x) \geq -(\bar{c_1} + \bar{c_2})||y-x||^2 \ \ \forall x, y \in C.$$

Problem (EP) includes the following problems:

1. *Optimization Problem.* Let $C = \mathbb{R}^n_+$ and $\varphi : C \to \mathbb{R}$. We consider the optimization problem:

$$\min\{\varphi(x): \ x \in C\}. \tag{OP}$$

Setting $f(x,y) = \varphi(y) - \varphi(x)$, it is easy to see that (OP) becomes a case of (EP). Note that, if φ is convex, then $f(x,.)$ is convex for each $x \in C$.

2. *Nonlinear Complementarity Problem.* Let $C = \mathbb{R}^n_+$ and $\varphi : C \to \mathbb{R}^n$. The following problem is called a nonlinear complementarity problem [23]:

$$\textit{Find } x^* \in C \textit{ and } \varphi(x^*) \in C \textit{ such that } \langle x^*, \varphi(x^*)\rangle = 0. \tag{NCP}$$

It is easy to see that Problem (NCP) becomes a case of the following variational inequality:

$$Find\ x^* \in C\ such\ that\ \langle \varphi(x^*), x - x^* \rangle \geq 0 \ \ \forall x \in C. \qquad (VIP)$$

Then, setting

$$f(x, y) := \langle \varphi(x), y - x \rangle,$$

we can easily see that (VIP) becomes a case of (EP).

We recall the following well known definition [30].
Let $C \subseteq \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.
• φ is said to be Lipschitz on C with constant $L > 0$ (shortly, L-Lipschitz) if

$$||\varphi(x) - \varphi(y)|| \leq L||x - y|| \ \ \forall x, y \in C.$$

• φ is said to be pseudomonotone on C if

$$\langle \varphi(y), x - y \rangle \geq 0 \text{ implies } \langle \varphi(x), x - y \rangle \geq 0 \ \ \forall x, y \in C.$$

Some relations between the function φ of problem (NCP) and the function f of problem (EP) are formulated in the following lemma [14, 28].

Lemma 2.2. (i) *If φ is L-Lipschitz on C and $f(x, y) = \langle \varphi(x), y - x \rangle$, then*

$$f(x, y) + f(y, z) \geq f(x, z) - \bar{c_1}||x - y||^2 - \bar{c_2}||y - z||^2 \ \ \forall x, y, z \in C,$$

where $\bar{c_1}$ and $\bar{c_2}$ are any positive numbers satisfying $2\sqrt{\bar{c_1}\bar{c_2}} \geq L$.
(ii) *If φ is pseudomonotone on C then f is also pseudomonotone on C.*

3. *Nash Equilibrium Problem.* Let $I := \{1, ..., p\}$ be the set of p players, $C_i := \mathbb{R}^{n_i}_+$ the strategy set of player i $(i \in I)$ and $f_i : C_1 \times ... \times C_p \to \mathbb{R}$ the loss function of player i $(i \in I)$.

By [21], a point $x^* \in C_1 \times ... \times C_p$ is said to be a Nash equilibrium point of $f := (f_1, ..., f_p)$ on $C := C_1 \times ... \times C_p$ if and only if

$$f_i(x^*) \leq f_i(x^*[y^i]) \ \ \forall y^i \in C_i, \ \forall i \in I,$$

where $x[y^i]$ stands for the vector obtained from $x = (x^1, ..., x^p) \in C_1 \times ... \times C_p$ by replacing x^i with y^i. Then we set

$$f(x, y) := \sum_{i=1}^{p} \big(f_i(x[y^i]) - f_i(x)\big).$$

We can see that the problem of finding a Nash equilibrium point of f on C can be formulated equivalently to (EP) (see [27]).

The following lemma can be found, for example in [27].

Lemma 2.3. *Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. Then the following statements are equivalent:*
(i) *x^* is a solution to (EP).*
(ii) *$x^* \in C$ is a solution to the problem:* $\min\{f(x^*, y) : \ y \in C\}$.

See e.g. Proposition 1 in [27].

Throughout this paper we suppose that

$$C = \mathbb{R}^n_+ := \{(x_1, x_2, ..., x_n) : \ x_i \geq 0 \ \ \forall i = 1, 2, ..., n\}$$

and for each fixed $x \in C$, the function $f(x, .)$ is proper closed convex on C and $f(x, x) = 0$ for every $x \in C$.

It is well known that the problem (NCP) can be alternatively formulated as finding the zero point of the operator $T(x) = \varphi(x) + N_C(x)$ where

$$N_C(x) = \begin{cases} \{y \in \mathbb{R}^n : \ \langle y, z - x\rangle \leq 0, \ \forall z \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.2}$$

A classical method to solve this problem is the proximal point algorithm (see [2, 32]), which starting with any point $x^0 \in \mathbb{R}^n_+$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k T(x) + \nabla_x h(x, x^k), \tag{2.3}$$

where

$$h(x, x^k) = \frac{1}{2}||x - x^k||^2.$$

Recently, Auslender et al. [7] have proposed a new type of proximal interior method through replacing function $q(x, x^k)$ by $d_\phi(x, x^k)$ which is defined as

$$d_\phi(x, y) = \sum_{i=1}^n y_i^2 \phi(y_i^{-1} x_i),$$

where

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

with $\nu > \mu > 0$. The fundamental difference here is that the term d_ϕ is used to force the iteratives $\{x^{k+1}\}$ to stay in the interior of $\mathbb{R}^n_+$.

Applying this idea to the equilibrium problem (EP), in this paper we use the following distance of Bregman type

$$d(x, y) = \begin{cases} \frac{1}{2}||x - y||^2 + \mu \sum_{i=1}^n y_i^2 (\frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

with $\mu \in (0, 1)$ and $y \in \mathbb{R}^n_+$. Then we consider the following regularized auxiliary problem:

$$\text{Find} \ \ x^* \in C \ \ \text{such that} \ \ f(x^*, y) + \frac{1}{c} d(y, x^*) \geq 0 \ \ \text{for all} \ \ y \in C, \qquad (Aux\ EP)$$

where $c > 0$ is a regularization parameter.

We denote by $\nabla_1 d(x, y)$ the gradient of $d(., y)$ at x for every $y \in C$. It is easy to see that

$$\nabla_1 d(x, y) = x - y + \mu X_y \log \frac{x}{y},$$

where $X_y = diag(y_1, ..., y_n)$ and $\log\frac{x}{y} = (\log\frac{x_1}{y_1}, ..., \log\frac{x_n}{y_n})^T$.

The equivalence between (EP) and (Aux EP) is due to the following lemma.

Lemma 2.4. *Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction and $x^* \in C$. Then x^* is a solution to (EP) if and only if x^* is a solution to (Aux EP).*

See e.g. Proposition 1 in [27].

3. A logarithmic-quadratic algorithm

Lemma 2.4 shows that the solution of the equilibrium problem (EP) can be approximated by an iterative procedure $x^{k+1} = h(x^k), k = 0, 1, ...$ where x^0 is any starting point in $C := \mathbb{R}^n_+$ and $h(x^k)$ is the unique solution of the strongly convex program

$$\min\{f(x^k, y) + \frac{1}{c}d(y, x^k) : \ y \in C\}.$$

However, generally, the sequence $\{x^k\}$ does not converge to a solution of the equilibrium [20, 21]. In order to avoid this drawback, the extragradient algorithm has been introduced for monotone equilibrium problems [30].

Algorithm 3.1. Step 0. Choose $x^0 \in C, k := 0, c > 0$ and a positive sequence $\{c_k\}$ such that $c_k \to c$ as $k \to +\infty$.
Step 1. Solve the strongly convex program:

$$\min\{f(x^k, y) + \frac{1}{c_k}d(y, x^k) : \ y \in C\} \tag{3.1}$$

to obtain the unique solution y^k.
If $y^k = x^k$, then terminate: x^k is a solution to (EP).
Otherwise go to Step 2.
Step 2. Find x^{k+1} which is the unique solution to the strongly convex program:

$$\min\{f(y^k, y) + \frac{1}{c_k}d(y, x^k) : \ y \in C\}.$$

Step 3. Set $k := k + 1$, and go to Step 1.

In the next proposition, we justify the stopping criterion.

Proposition 3.2. *If $y^k = x^k$, then x^k is a solution to (EP).*

Proof. If the algorithm terminates at Step 1, then $y^k = x^k$. It means that x^k is the solution of problem (3.1). By Lemma 2.3 and Lemma 2.4 it is a solution of (EP). □

In order to prove the convergence of Algorithm 3.1, we give the following key property of the sequence $\{x^k\}_{k\geq 0}$ generated by the algorithm.

Lemma 3.3. *Suppose that the bifunction $f : C\times C \to \mathbb{R}\cup\{+\infty\}$ is pseudomonotone, Lipschitz with constants c_1 and c_2 on C, and $f(x, .)$ is closed convex subdifferentiable on C for each $x \in C$. Then, if the algorithm does not terminate, then for every solution x^* of (EP) we have*

$$||x^{k+1} - x^*||^2$$
$$\leq ||x^k - x^*||^2 - \frac{1-3\mu-2\bar{c_2}c_k}{1+\mu}||x^{k+1} - y^k||^2 - \frac{1-3\mu-2\bar{c_1}c_k}{1+\mu}||x^k - y^k||^2.$$

Proof. Since y^k is the solution of problem (3.1), from an optimization result in convex programming (Moreau-Rockafellar theorem (see [31])), we have

$$0 \in \partial_2 f(x^k, y^k) + \frac{1}{c_k}\nabla_1 d(y^k, x^k) + N_C(y^k),$$

where $\partial_2 f(x^k, y^k)$ denotes the subdifferential of $f(x^k, .)$ at y^k and N_C denotes the normal cone. We have

$$0 = w_1 + \frac{1}{c_k}\nabla_1 d(y^k, x^k) + w_2, \tag{3.2}$$

where $w_1 \in \partial_2 f(x^k, y^k), w_2 \in N_C(y^k)$.
Since $w_2 \in N_C(y^k)$, we have

$$\langle w_2, y - y^k\rangle \leq 0 \quad \forall y \in C. \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\langle \frac{1}{c_k}\nabla_1 d(y^k, x^k), y - y^k\rangle \geq \langle w_1, y^k - y\rangle \quad \forall y \in C.$$

By the definition of subgradient, we have from the the last inequalities that

$$\frac{1}{c_k}\langle\nabla_1 d(y^k, x^k), y - y^k\rangle \geq f(x^k, y^k) - f(x^k, y) \quad \forall y \in C. \tag{3.4}$$

Replacing y by x^*, we obtain

$$\frac{1}{c_k}\langle\nabla_1 d(y^k, x^k), x^* - y^k\rangle \geq f(x^k, y^k) - f(x^k, x^*).$$

Note that x^* is a solution of (EP), $f(x^*, y) \geq 0$. By pseudomonotonicity of f, it follows that $f(x^k, x^*) \leq 0$. Then

$$\frac{1}{c_k}\langle\nabla_1 d(y^k, x^k), x^* - y^k\rangle \geq f(x^k, y^k).$$

On the other hand, since x^{k+1} is the solution to the convex program

$$\min\{f(y^k, y) + \frac{1}{c_k}d(y, x^k) : \ y \in C\},$$

in the same way, we can show that

$$\frac{1}{c_k}\langle\nabla_1 d(x^{k+1}, x^k), x^* - x^{k+1}\rangle \geq f(y^k, x^{k+1}). \tag{3.5}$$

We recall that

$$\nabla_1 d(x^{k+1}, x^k) = x^{k+1} - x^k + \mu X_{x^k} \log \frac{x^{k+1}}{x^k},$$

where $X_{x^k} = diag(x_1^k, ..., x_n^k)$ and $\log\frac{x^{k+1}}{x^k} = (\log\frac{x_1^{k+1}}{x_1^k}, ..., \log\frac{x_n^{k+1}}{x_n^k})^T$. Then (3.5) can be written as

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle + \mu\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^* - x^{k+1}\rangle \geq c_k f(y^k, x^{k+1}). \tag{3.6}$$

Now applying the Lipschitz condition (2.1) of f with $x = x^k, y = y^k, z = x^{k+1}$ we obtain

$$f(x^k, y^k) + f(y^k, x^{k+1}) \geq f(x^k, x^{k+1}) - \bar{c_1}||y^k - x^k||^2 - \bar{c_2}||x^{k+1} - y^k||^2. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} &\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle \\ &\geq -\mu\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^* - x^{k+1}\rangle + c_k f(x^k, x^{k+1}) - c_k f(x^k, y^k) \\ &\quad - \bar{c_1}c_k||y^k - x^k||^2 - \bar{c_2}c_k||x^{k+1} - y^k||^2. \end{aligned} \tag{3.8}$$

If $y = x^{k+1}$, inequality (3.4) becomes

$$\begin{aligned} &f(x^k, x^{k+1}) - f(x^k, y^k) \\ &\geq \frac{1}{c_k}\langle \nabla_1 d(y^k, x^k), y^k - x^{k+1}\rangle \\ &= \frac{1}{c_k}\langle y^k - x^k, y^k - x^{k+1}\rangle + \frac{1}{c_k}\mu\langle X_k \log\frac{y^k}{x^k}, y^k - x^{k+1}\rangle. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$\begin{aligned} &\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle \\ &\geq -\mu\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^* - x^{k+1}\rangle + \langle y^k - x^k, y^k - x^{k+1}\rangle \\ &\quad + \mu\langle X_{x^k}\log\frac{y^k}{x^k}, y^k - x^{k+1}\rangle - \bar{c_1}c_k||y^k - x^k||^2 - \bar{c_2}c_k||x^{k+1} - y^k||^2. \end{aligned} \tag{3.10}$$

Substituting

$$\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle = \frac{1}{2}(||x^k - x^*||^2 - ||x^k - x^{k+1}||^2 - ||x^{k+1} - x^*||^2)$$

into (3.10), we obtain the estimation

$$\begin{aligned} ||x^k - x^*||^2 - ||x^k - x^{k+1}||^2 \geq\ &||x^{k+1} - x^*||^2 - 2\mu\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^* - x^{k+1}\rangle \\ &+ 2\langle y^k - x^k, y^k - x^{k+1}\rangle + 2\mu\langle X_{x^k}\log\frac{y^k}{x^k}, y^k - x^{k+1}\rangle \\ &- 2\bar{c_1}c_k||y^k - x^k||^2 - 2\bar{c_2}c_k||x^{k+1} - y^k||^2. \end{aligned} \tag{3.11}$$

Combining the inequality (3.11) with the following equality

$$||x^{k+1}-x^k||^2 = ||x^{k+1}-y^k||^2 + ||x^k-y^k||^2 + 2\langle x^{k+1}-y^k, y^k-x^k\rangle,$$

we have

$$\begin{aligned}
&||x^{k+1}-x^*||^2\\
&\le ||x^k-x^*||^2 - ||x^{k+1}-y^k||^2 - ||x^k-y^k||^2 + 2\bar{c_1}c_k||y^k-x^k||^2\\
&\quad + 2\bar{c_2}c_k||x^{k+1}-y^k||^2 + 2\mu\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^*-x^{k+1}\rangle\\
&\quad - 2\mu\langle X_{x^k}\log\frac{y^k}{x^k}, y^k-x^{k+1}\rangle. \qquad (3.12)
\end{aligned}$$

For each $t > 0$ we have $1-\frac{1}{t} \le \log t \le t-1$, then we obtain after multiplication by $x_i^* \ge 0$ for each $i = 1, ..., n$,

$$x_i^k x_i^* \log\frac{x_i^{k+1}}{x_i^k} \le x_i^*(x_i^{k+1}-x_i^k), \qquad (3.13)$$

and after multiplication by $-x_i^k \le 0$ for each $i = 1, ..., n$,

$$-x_i^k x_i^{k+1}\log\frac{x_i^{k+1}}{x_i^k} \le -x_i^k x_i^{k+1}\Big(1-\frac{x_i^k}{x_i^{k+1}}\Big) = x_i^k(x_i^k - x_i^{k+1}). \qquad (3.14)$$

Adding the two inequalities (3.13) and (3.14), we obtain

$$\begin{aligned}
&2x_i^k\log\frac{x_i^{k+1}}{x_i^k}(x_i^*-x_i^{k+1})\\
&\le 2(x_i^k-x_i^*)(x_i^k-x_i^{k+1})\\
&= |x_i^k-x_i^*|^2 + |x_i^k-x_i^{k+1}|^2 - |x_i^{k+1}-x_i^*|^2 \quad \forall i = 1, ..., n.
\end{aligned}$$

These inequalities deduce that

$$2\langle X_{x^k}\log\frac{x^{k+1}}{x^k}, x^*-x^{k+1}\rangle \le ||x^k-x^*||^2 + ||x^k-x^{k+1}||^2 - ||x^{k+1}-x^*||^2. \qquad (3.15)$$

In the same way, we also have

$$2\langle X_{x^k}\log\frac{y^k}{x^k}, x^{k+1}-y^k\rangle \le ||x^k-x^{k+1}||^2 + ||x^k-y^k||^2 - ||y^k-x^{k+1}||^2. \qquad (3.16)$$

Substituting (3.15) and (3.16) into (3.12), we get

$$\begin{aligned}
&||x^{k+1}-x^*||^2\\
&\le ||x^k-x^*||^2 - ||x^{k+1}-y^k||^2 - ||x^k-y^k||^2 + 2\bar{c_1}c_k||y^k-x^k||^2\\
&\quad + 2\bar{c_2}c_k||x^{k+1}-y^k||^2 + \mu(||x^k-x^*||^2 + ||x^k-x^{k+1}||^2 - ||x^{k+1}-x^*||^2)\\
&\quad + \mu(||x^{k+1}-x^k||^2 + ||y^k-x^k||^2 - ||x^{k+1}-y^k||^2),
\end{aligned}$$

and consequently

$$
\begin{aligned}
&(1+\mu)||x^{k+1}-x^*||^2 \\
&\le (1+\mu)||x^k-x^*||^2 - (1+\mu-2\bar{c_2}c_k)||x^{k+1}-y^k||^2 \\
&\quad -(1+\mu-2\bar{c_1}c_k)||x^k-y^k||^2 + 2\mu||x^{k+1}-x^k||^2. \qquad (3.17)
\end{aligned}
$$

Applying the following inequality

$$||x^{k+1}-x^k||^2 \le 2||x^{k+1}-y^k||^2 + 2||x^k-y^k||^2$$

to the last term in the right hand side of (3.17), we obtain

$$
\begin{aligned}
&(1+\mu)||x^{k+1}-x^*||^2 \\
&\le (1+\mu)||x^k-x^*||^2 - (1-3\mu-2\bar{c_2}c_k)||x^{k+1}-y^k||^2 \\
&\quad -(1-3\mu-2\bar{c_1}c_k)||x^k-y^k||^2,
\end{aligned}
$$

the lemma thus is proved. □

Now we are in a position to prove the convergence of Algorithm 3.1.

Theorem 3.4. *Suppose that the bifunction $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ is pseudomonotone, Lipschitz with constants c_1 and c_2 on C, and $f(x,.)$ is closed convex subdifferentiable on C for each $x \in C$. Suppose further that f is lower semicontinuous on $C \times C$ and $f(.,y)$ upper semicontinuous on C for each $y \in C$. Then, if the algorithm does not terminate and $0 < \mu < \frac{1}{3}\min\{1-\epsilon-2\bar{c_1}c_k, 1-\epsilon-2\bar{c_2}c_k\}$ where $\epsilon > 0$, then the sequence $\{x^k\}$ converges to a solution x^* of (EP).*

Proof. The assumptions $0 < \mu < \frac{1}{3}\min\{1-\epsilon-2\bar{c_1}c_k, 1-\epsilon-2\bar{c_2}c_k\}$ and $\epsilon > 0$ imply

$$1-2\bar{c_2}c_k > 0 \ \text{ and } \ 1-2\bar{c_2}c_k > 0 \ \ \forall k = 0,1,....$$

Then, from Lemma 3.3 we have

$$||x^{k+1}-x^*||^2 \le ||x^k-x^*||^2 \ \ \forall k = 0,1,...$$

This inequality shows that the sequence $\{||x^k-x^*||\}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. Then the sequence $\{x^k\}_{k\ge 0}$ is bounded and it has a subsequence $\{x^{k_i}\}$ such that $x^{k_i} \to \bar{x}$ as $i \to +\infty$. From Lemma 3.3, we get

$$\frac{1-3\mu-2\bar{c_1}c_k}{1+\mu}||x^k-y^k||^2 \le ||x^k-x^*||^2 - ||x^{k+1}-x^*||^2 \ \ \forall k = 0,1,...$$

Applying these inequalities iteratively, we obtain

$$\sum_{k=0}^{n} \frac{1-3\mu-2\bar{c_1}c_k}{1+\mu}||x^k-y^k||^2 \le ||x^0-x^*||^2 - ||x^{n+1}-x^*||^2 \ \ \forall n \ge 0.$$

As the sequence $\{||x^k-x^*||\}_{k\ge 0}$ is convergent, passing $n \to +\infty$ we have

$$\lim_{k\to+\infty} \frac{1-3\mu-2\bar{c_1}c_k}{1+\mu}||x^k-y^k||^2 = 0.$$

Using this with the assumption $1 - 3\mu - 2\bar{c_1}c_k > \epsilon > 0$, we get

$$\lim_{k\to+\infty} \epsilon||x^k - y^k|| = 0.$$

It follows that $\lim_{i\to\infty} y^{k_i} = \bar{x}$.

Recall that y^{k_i} is the solution of the problem

$$\min\{f(x^{k_i}, y) + \frac{1}{c_{k_i}} d(y, x^{k_i}) : \ y \in C\}.$$

Then

$$f(x^{k_i}, y^{k_i}) + \frac{1}{c_{k_i}} d(y^{k_i}, x^{k_i}) \le f(x^{k_i}, y) + \frac{1}{c_{k_i}} d(y, x^{k_i}) \ \ \forall y \in C.$$

Using the lower semicontinuity of f, the upper semicontinuity of $f(., y)$ and $d(y, .)$, taking the liminf of the left-hand side and the limsup of the right-hand one we obtain

$$f(\bar{x}, y) + \frac{1}{c} d(y, \bar{x}) \ge 0 \ \ \forall y \in C.$$

So $\bar{x}$ is a solution to (Aux EP). Then, by Lemma 2.4, $\bar{x}$ is a solution to (EP). Replacing x^* by $\bar{x}$ in Lemma 3.3 yields

$$||x^{k+1} - \bar{x}|| \le ||x^k - \bar{x}|| \ \ \forall k = 0, 1, ...,$$

which implies that the sequence $||x^k - \bar{x}||$ is convergent. By the above proof, the sequence $\{x^k\}$ has a subsequense converging to $\bar{x}$, we deduce that the whole sequence $\{x^k\}$ converges to the solution $\bar{x}$ of (EP). □

4. An algorithm without Lipschitz condition

In Section 3, we consider the bifunction f, which satisfies the Lipschitz condition on C. In this section, in order to avoid this requirement, we modify Algorithm 3.1 by using line search. The line search technique has been used widely in descent method for solving this problem (see [25, 30]) and variational inequalities (see [9, 13, 15, 16, 20]).

The algorithm then can be described as follows.

Algorithm 4.1. Step 0. Take $x^0 \in C, k := 0$ and a sequence $\gamma_k \in (0; 2) \ \ \forall k \ge 0$, choose $\bar{c} > 0$ and a sequence $c_k \to \bar{c}$ as $k \to +\infty$.
Step 1. Find y^k which is the solution to the strongly convex program:

$$\min\{f(x^k, y) + \frac{1}{c_k} d(y, x^k) : \ \ y \in C\}. \tag{4.1}$$

If $y^k = x^k$, then stop.
Otherwise go to Step 2.
Step 2. Find $\lambda_k \in (0, 1)$ as the smallest number such that

$$f((1 - \lambda_k)x^k + \lambda_k y^k, y^k) + \frac{1}{2c_k} d(y^k, x^k) \le 0. \tag{4.2}$$

Set $z^k := (1-\lambda_k)x^k + \lambda_k y^k$.
If $0 \in \partial_2 f(z^k, z^k)$, then stop.
Otherwise go to Step 3.
Step 3. Choose $g^k \in \partial_2 f(z^k, z^k)$, set

$$\bar{z}^k := x^k - \gamma_k \frac{-\lambda_k f(z^k, y^k)}{(1-\lambda_k)||g^k||^2} g^k \text{ and } x^{k+1} := P_C(\bar{z}^k),$$

it means that for every $i = 1, ..., n$,

$$x_i^{k+1} = \begin{cases} \bar{z}_i^k & \text{if } \bar{z}_i^k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Step 4. Increase k by 1 and return to Step 1.

Recall that $P_C(x)$ denotes the projection of x on C.

Now we are in a position to prove the following convergence theorem for Algorithm 4.1.

Theorem 4.2. *Suppose that the sequence $\gamma_k \in (0, 2)$ has* $\liminf \gamma_k(2-\gamma_k) > 0$, *the bifunction $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ is pseudomonotone on $C$and $f(x, .)$ is closed, convex and subdifferentiable on C for each $x \in C$. Suppose further that f is lower semicontinuous on $C \times C$ and $f(., y)$ is upper semicontinuous on C. Then*
(i) *If Algorithm 4.1 terminates at Step 1 or Step 2 then x^k is a solution to (EP).*
(ii) *For all x^* which is a solution to (EP), we have*

$$||x^{k+1} - x^*||^2 \leq ||x^k - x^*||^2 - (2-\gamma_k)\gamma_k \Big(\frac{\lambda_k f(z^k, y^k)}{(1-\lambda_k)||g^k||}\Big)^2.$$

(iii) *If Algorithm 4.1 doesn't terminate at Step 1 or Step 2, then the sequence $\{x^k\}$ converges to x^* which is a solution to (EP).*

Proof. First, we have to show that there always exists $\lambda_k \in (0, 1)$ as the smallest number satisfying (4.2). We suppose on the contrary that for every $\lambda \in (0, 1)$, we have

$$f((1-\lambda)x^k + \lambda y^k, y^k) + \frac{1}{2c_k} d(y^k, x^k) > 0.$$

Passing to the limit in the above inequality (as $\lambda \to 0^+$), by upper semicontinuity of $f(., y)$, we obtain

$$f(x^k, y^k) + \frac{1}{2c_k} d(y^k, x^k) \geq 0. \tag{4.3}$$

Since y^k is a solution to (4.1), it follows that

$$f(x^k, y) + \frac{1}{c_k} d(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k} d(y^k, x^k).$$

Replacing y by x^k in the above inequality, we have

$$0 \geq f(x^k, y^k) + \frac{1}{c_k} d(y^k, x^k). \tag{4.4}$$

Then from (4.3) and (4.4) it follows that $d(y^k, x^k) = 0$, i.e, $x^k = y^k$. This contracdics to $x^k \neq y^k$ in Step 1.

To prove part (i), we suppose that Algorithm 4.1 terminates at Step 1, hence $x^k = y^k$. Then

$$f(x^k, y) + \frac{1}{c_k} d(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k} d(y^k, x^k) = 0 \quad \forall y \in C.$$

This means that x^k is a solution to (Aux EP). From Lemma 2.4, x^k is also a solution to (EP).

If Algorithm 2 terminates at Step 2, from $z^k \in \text{int}C$ it follows that

$$0 \in \partial_2 f(z^k, z^k)$$

can be written as

$$0 \in \partial_2 f(z^k, z^k) + N_C(z^k).$$

It means that z^k is a solution to the following convex problem:

$$\min_{y \in C} f(z^k, y).$$

Then by virtue of Lemma 2.3, z^k is a solution to (EP).

Now we prove part (ii). Note that, by the above proof, at Step 3, from $0 \notin \partial_2 f(z^k, z^k)$ it follows that z^k isn't a solution to the following convex problem:

$$\min_{y \in C} f(z^k, y).$$

It means that z^k isn't a solution to (EP). We set

$$\sigma_k := \frac{-\lambda_k f(z^k, y^k)}{(1 - \lambda_k)||g_k||^2}.$$

Then

$$\begin{aligned} ||x^{k+1} - x^*||^2 &= ||P_k\big(x^k - \gamma_k \sigma_k g^k\big) - x^*||^2 \leq ||x^k - x^* - \gamma_k \sigma_k g^k||^2 \\ &= ||x^k - x^*||^2 - 2\gamma_k \sigma_k \langle g^k, x^k - x^* \rangle + (\gamma_k \sigma_k ||g^k||)^2. \end{aligned} \tag{4.5}$$

Note that, since x^* is a solution to (EP), $f(x^*, y) \geq 0$. Then by pseudomonotonicity, it follows that $-f(z^k, x^*) \geq 0$. Combine this with

$$\begin{aligned} \langle g^k, x^k - x^* \rangle &= \langle g^k, x^k - z^k \rangle + \langle g^k, z^k - x^* \rangle \\ &\geq \langle g^k, x^k - z^k \rangle + f(z^k, z^k) - f(z^k, x^*), \\ &\geq \langle g^k, x^k - z^k \rangle = \frac{\lambda_k}{1 - \lambda_k} \langle g^k, z^k - y^k \rangle \\ &\geq \frac{\lambda_k}{1 - \lambda_k} \big(f(z^k, z^k) - f(z^k, y^k)\big) = \frac{-\lambda_k}{1 - \lambda_k} f(z^k, y^k). \\ &= \sigma_k ||g_k||^2. \end{aligned} \tag{4.6}$$

Since, at Step 2, $d(y^k, x^k) > 0$ and

$$f(z^k, y^k) + \frac{1}{2c_k} d(y^k, x^k) \le 0,$$

we obtain that $f(z^k, y^k) < 0$. Hence

$$\sigma_k = \frac{-\lambda_k f(z^k, y^k)}{(1-\lambda_k)||g^k||^2} > 0. \tag{4.7}$$

Then from (4.5) and (4.6), we have

$$\begin{aligned} ||x^{k+1} - x^*||^2 &\le ||x^k - x^*||^2 - 2\gamma_k \sigma_k^2 ||g^k||^2 + (\gamma_k \sigma_k ||g^k||)^2 \\ &= ||x^k - x^*||^2 - (2-\gamma_k)\gamma_k(\sigma_k ||g^k||)^2, \end{aligned} \tag{4.8}$$

which proves part (ii).

Now we rewrite (4.8) as follows

$$\begin{aligned} \sum_{k=0}^{n} (2-\gamma_k)\gamma_k(\sigma_k ||g^k||)^2 &\le \sum_{k=0}^{n} (||x^k - x^*||^2 - ||x^{k+1} - x^*||^2) \\ &= ||x^0 - x^*||^2 - ||x^{n+1} - x^*||^2 \quad \forall n \ge 0. \end{aligned}$$

On the other hand, since (4.8) deduces that $\{||x^k - x^*||\}$ is a decreasing sequence and is lower bounded by $||x^0 - x^*||$, then it must converge. It means that

$$\sum_{k=0}^{\infty} (2-\gamma_k)\gamma_k(\sigma_k ||g^k||)^2 < +\infty.$$

Hence

$$\lim_{k\to\infty} (2-\gamma_k)\gamma_k(\sigma_k ||g^k||)^2 = 0,$$

which together with $\lim_{k\to\infty} \inf(2-\gamma_k)\gamma_k > 0$ implies

$$\lim_{k\to\infty} \frac{\lambda_k f(z^k, y^k)}{(1-\lambda_k)||g^k||} = 0.$$

From the convergence of $\{||x^k - x^*||\}$, we have that the sequence $\{x^k\}$ is bounded. Then by the maximum theorem [5], we can deduce that the sequence $\{g^k\}$ is bounded too. Thus

$$\lim_{k\to\infty} \frac{\lambda_k f(z^k, y^k)}{1-\lambda_k} = 0. \tag{4.9}$$

According to the rule (4.2), it is easy to see that

$$\frac{1}{2c_k} d(y^k, x^k) \le -f(z^k, y^k). \tag{4.10}$$

We consider two cases:

Case 1: If $\lim_{k\to\infty} \sup \lambda_k > 0$, then we have that there exist a subsequence $\{x^{k_i}\}_{i\ge 0}$

and $\bar{\lambda} \in (0,1]$ such that $\lambda_{k_i} \geq \bar{\lambda}\ \forall i \geq 0$. From (4.9) and inequality (4.10), we have

$$\lim_{i\to\infty} d(y^{k_i}, x^{k_i}) = 0. \tag{4.11}$$

Since the sequence $\{x^{k_i}\}$ is bounded, hence it has a subsequence $\{x^{k_i}:\ i \in M\}$ converging to a point $\bar{x}$. Using the limit (4.11) we see that the subsequence $\{y^{k_i} : i \in M\}$ also converges to $\bar{x}$. Note that y^{k_i} is a solution to (4.1), hence

$$f(x^{k_i}, y) + \frac{1}{c_{k_i}} d(y, x^{k_i}) \geq f(x^{k_i}, y^{k_i}) + \frac{1}{c_{k_i}} d(y^{k_i}, x^{k_i}) \quad \forall i \in M, y \in C.$$

Passing to the limit as $i \to \infty$ and using the upper semicontinuity of $f(., y)$, we have

$$f(\bar{x}, y) + \frac{1}{\bar{c}} d(y, \bar{x}) \geq f(\bar{x}, \bar{x}) + \frac{1}{\bar{c}} d(\bar{x}, \bar{x}) = 0 \quad \forall y \in C.$$

By Lemma 2.4, $\bar{x}$ is a solution to (EP), thus the proof of the theorem in this case is complete.

Case 2: If $\lim\limits_{k\to\infty} \sup \lambda_k = 0$, then since $\{x^k\}$ is bounded, we have some subsequence $\{x^k :\ k \in M\}$ converging to some point $\bar{x}$ as $k \to \infty$. From Step 1 of Algorithm 4.1, by the lower semicontinuity of $f(x^k, .) + \frac{1}{c_k} d(., x^k)$, the sequence $\{y^k\}$ is bounded too [5, 12]. Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^k :\ k \in M\}$ also converges to some point $\bar{y}$. From

$$f(x^k, y) + \frac{1}{c_k} d(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k} d(y^k, x^k) \quad \forall k \in M, y \in C,$$

by the lower semicontinuity of f, d and upper semicontinuity of $f(., y), d(y, .)$, taking the limit as $k \to \infty$, we can write

$$f(\bar{x}, y) + \frac{1}{\bar{c}} d(y, \bar{x}) \geq f(\bar{x}, \bar{y}) + \frac{1}{\bar{c}} d(\bar{y}, \bar{x}). \tag{4.12}$$

Substituting $y = \bar{x}$ we then have

$$0 \geq f(\bar{x}, \bar{y}) + \frac{1}{\bar{c}} d(\bar{y}, \bar{x}). \tag{4.13}$$

On the other hand, by Step 2 in Algorithm 4.1, since $\lambda_k \in (0,1)$ is the smallest number satisfying

$$f\big((1-\lambda_k)x^k + \lambda_k y^k, y^k\big) + \frac{1}{2c_k} d(y^k, x^k) \leq 0,$$

we deduce that

$$f\left(\left(1 - \frac{1}{2}\lambda_k\right) x^k + \frac{1}{2}\lambda_k y^k, y^k\right) + \frac{1}{2c_k} d(y^k, x^k) > 0 \quad \forall k \geq 0.$$

Passing $k \to \infty, k \in M$ in the above inequality, we obtain

$$f(\bar{x}, \bar{y}) + \frac{1}{2\bar{c}} d(\bar{y}, \bar{x}) \geq 0.$$

This together with (4.13) implies $d(\bar{y},\bar{x}) = 0$, hence $\bar{x} = \bar{y}$. Then replacing $\bar{y}$ in (4.12) by $\bar{x}$, we deduce that

$$f(\bar{x}, y) + \frac{1}{\bar{c}} d(y, \bar{x}) \geq 0 \quad \forall y \in C.$$

The proof is complete. □

Remark 4.3. The smallest number $\lambda_k \in (0,1)$ in Step 2 of Algorithm 4.1 can be replaced by the following: With $\beta \in (0,1)$, we find n as the smallest natural number such that

$$f\big(\beta^n x^k + (1-\beta^n) y^k, y^k\big) + \frac{1}{2c_k} d(y^k, x^k) \leq 0,$$

then set $\lambda_k := 1 - \beta^n$.

5. An application to nonlinear complementarity problems

We apply Algorithm 3.1 to the complementarity problem (NCP) when φ is pseudomonotone and L-Lipschitz on $C := \mathbb{R}^n_+$. Note that in this case, the subproblem

$$y^k = \operatorname{argmin}\left\{ f(x^k, y) + \frac{1}{c_k} d(y, x^k) :\ y \in C \right\}$$

takes the form

$$y^k = \operatorname{argmin}\left\{ \langle \varphi(x^k), y - x^k \rangle + \frac{1}{c_k} d(y, x^k) :\ y \in C \right\}$$

where d is defined by (2.5). It is written as

$$y^k = \operatorname{argmin}\Bigg\{ \langle \varphi(x^k), y - x^k \rangle + \frac{1}{2c_k} ||y - x^k||^2 + \frac{\mu}{c_k} \sum_{i=1}^{n} (x_i^k)^2 \Big(\frac{y_i}{x_i^k} \log \frac{y_i}{x_i^k} - \frac{y_i}{x_i^k} + 1 \Big) :\ y \in C_+ \Bigg\},$$

where

$$C_+ := \{x \in R^n :\ x_i > 0 \ \ \forall i = 1, ..., n\}.$$

It is not difficult to see that if we denote $y^k = (y_1^k, ..., y_n^k)$ and $\varphi(x) = \big(\varphi_1(x), ..., \varphi_n(x)\big)$ $\forall x \in C$, then for every $i = 1, ..., n$, we have y_i^k is the unique solution to the strongly convex problem

$$\min\{\frac{1}{2} t^2 + \eta_{ki}\ t + \xi_{ki}\ t\log t :\ t \in (0, +\infty)\},$$

where

$$\eta_{ki} := c_k \varphi_i(x^k) - x_i^k - \mu x_i^k \ \log x_i^k - \mu x_i^k,\ \xi_{ki} := \mu x_i^k \ \ \forall i = 1, ..., n.$$

In the same way, in Algorithm 3.1, we can show that x^{k+1} is the unique solution of the following problem

$$\min\{f(y^k, y) + \frac{1}{c_k} d(y, x^k) :\ y \in C\},$$

which is also defined by

$$x_i^{k+1} := \text{argmin}\{\frac{1}{2}t^2 + \rho_{ki}t + \xi_{ki}\ t\text{log}t :\ t \in (0, +\infty)\},$$

where

$$\rho_{ki} := c_k\varphi_i(y^k) - x_i^k - \mu x_i^k\ \text{log}x_i^k - \mu x_i^k,\ \xi_{ki} := \mu x_i^k\quad \forall i = 1, ..., n.$$

Then the algorithm for the nonlinear complementarity problem (NCP) can be detailed in the following.

Algorithm 5.1. Step 0. Choose $x^0 \in C, k := 0, \mu > 0$ and a positive sequence $\{c_k\}$ such that $c_k \to \bar{c} > 0$ as $k \to +\infty$.
Step 1. For every $i = 1, ..., n$, solve the strongly convex program:

$$\min\{\frac{1}{2}t^2 + \eta_{ki}\ t + \xi_{ki}\ t\text{log}t :\ t \in (0, +\infty)\}, \tag{5.1}$$

to obtain the unique solution y_i^k, where

$$\eta_{ki} := c_k\varphi_i(x^k) - x_i^k - \mu x_i^k\ \text{log}x_i^k - \mu x_i^k,\ \xi_{ki} := \mu x_i^k\quad \forall i = 1, ..., n.$$

If $y^k = x^k$, then terminate: x^k is a solution to (NCP).
Otherwise go to Step 2.
Step 2. For every $i = 1, ..., n$, find

$$x_i^{k+1} := \text{argmin}\{\frac{1}{2}t^2 + \rho_{ki}t + \xi_{ki}\ t\text{log}t :\ t \in (0, +\infty)\}, \tag{5.2}$$

where

$$\rho_{ki} := c_k\varphi_i(y^k) - x_i^k - \mu x_i^k\ \text{log}x_i^k - \mu x_i^k\quad \forall i = 1, ..., n.$$

Step 3. Set $k := k + 1$, and return to Step 1.

Validity and linear convergence of this algorithm are immediate from Algorithm 3.1.

Thus both problems (5.1) and (5.2) are strongly convex programming problems which can be solved efficiently by MATLAB 7.5 Optimization Toolbox. To test the proposed method, we consider the nonlinear complementarity problem (NCP) (see [10]) for $\varphi(x) = D(x) + Mx + q$, where the components of the $D(x)$ are $D_j(x) = d_j * arctan(x_j)\ \forall j \geq 1$, d_j is chosen randomly in $(0, 1)$. The matrix $M = A^TA$ with A being $n \times n$ matrix whose entries are randomly generalized in the interval $(-1, 3)$. The vector q is generated from a uniform distribution in the interval $(-5, 9)$.

In the test we take the logarithmic parameter $\mu = 0.01$, $c_k = 0.01\ \forall k \geq 1$ and the tolerance 10^{-7}. We obtain the following computational results.

The approximate solution obtained after eleven iterations is

$$x^{10} = (1.1317, 0.1882, 0.8906, 0.4829, 0.1277, 0.2159, 0.2640)^T.$$

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k
0	1	1	1	1	1	1	1
1	1.0470	0.8957	1.0302	1.0030	0.9289	0.9641	0.9084
2	1.0886	0.8120	1.0654	1.0099	0.8549	0.9311	0.8199
3	1.1243	0.7436	1.1005	1.0155	0.7793	0.8975	0.7347
4	1.1528	0.6847	1.1296	1.0134	0.7022	0.8584	0.6540
5	1.1735	0.6296	1.1475	0.9976	0.6232	0.8095	0.5793
6	1.1864	0.5710	1.1496	0.9632	0.5422	0.7473	0.5112
7	1.1915	0.5067	1.1344	0.9088	0.4597	0.6706	0.4497
8	1.1887	0.4364	1.1014	0.8340	0.3763	0.5794	0.3946
9	1.1779	0.3601	1.0502	0.7383	0.2926	0.4733	0.3455
10	1.1590	0.2774	0.9801	0.6214	0.2095	0.3522	0.3020
11	1.1317	0.1882	0.8906	0.4829	0.1277	0.2159	0.2640

Table 1. Numerical results: Algorithm 5.1 with $n = 7$.

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