ACTA MATHEMATICA VIETNAMICA
Volume 34, Number 2, 2009, pp. 173–182

NON-EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS IN BOUNDED DOMAINS

TRINH THI MINH HANG AND HOANG QUOC TOAN

Abstract. In the present paper, by using variational arguments, we prove the non-existence, multiplicity of positive solutions to a system of p-Laplace equations of gradient form with nonlinear boundary conditions.

1. Introduction

In a recent paper, [15], K. Perera has studied, by using variational arguments, the existence, multiplicity and non-existence of positive solutions to the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in R^n, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian , $1 < p < \infty$, λ is a positive parameter, and $f(x,u)$ is a Caratheodory function on $\Omega \times [0,\infty)$.

They proved that there are $\underline{\lambda}$ and $\overline{\lambda}$, $0 < \underline{\lambda} < \overline{\lambda}$, such that the problem (1.1) has no positive solution for $\lambda < \underline{\lambda}$ and it has at least two positive solutions for $\lambda \geqslant \overline{\lambda}$.

Recently, in [10], J. Fernandez Bonder has extended these results to the Dirichlet problem for a gradient system of p-Laplace equations:

$$\begin{cases} -\Delta_p u = \lambda f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda g(x,u,v) & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and for the quasilinear elliptic problem with nonlinear boundary condition

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \gamma} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

Received May 19, 2007; in revised form July 25, 2008.
2000 *Mathematics Subject Classification.* 35J20, 35J50, 35J65.
Key words and phrases. Positive solutions, p-Laplacian, quasilinear elliptic problems.

where $\dfrac{\partial}{\partial\gamma}$ is the outer unit normal derivative.
In the present article, we extend the results in [10] to a quasilinear elliptic system with nonlinear boundary conditions as follows

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ -\Delta_q v + |v|^{q-2}v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial\gamma} = \lambda G_u(x,u,v) & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\dfrac{\partial v}{\partial\gamma} = \lambda G_v(x,u,v) & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain in $R^n, n \geqslant 2, 2 \leqslant p, q < \infty, \lambda$ is a positive parameter.

We introduce the following hypotheses

H1) $G(x,u,v)$ is a Caratheodory function on $\Omega \times [0,\infty) \times [0,\infty)$ such that $G(x,\cdot,\cdot)$ is C^1 for a.e. $x \in \Omega$ and
$$G_u(x,u,v) = f(x,u,v), \quad G_v(x,u,v) = g(x,u,v)$$
are Caratheodory functions on $\partial\Omega \times [0,\infty) \times [0,\infty)$.

H2)
$$G(x,0,0) = f(x,0,0) = g(x,0,0) = 0,$$
$$|uf(x,u,v) + vg(x,u,v)| \leqslant C(|u|^p + |v|^q),$$
$$|G(x,u,v)| \leqslant C(|u|^p + |v|^q),$$
for some constant $C > 0$.

H3) There are positive numbers δ, t_o, s_o such that for all $x \in \partial\Omega$
$$G(x,u,v) \leqslant 0 \text{ for } |u|^p + |v|^q \leqslant \delta$$
$$G(x,t_o,s_o) > 0.$$

H4) $\limsup\limits_{|(u,v)|\longrightarrow\infty} \dfrac{G(x,u,v)}{|u|^p + |v|^q} \leqslant 0$ uniformly with respect to $x \in \partial\Omega$.

Definition 1.1. A pair $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is called a weak solution to problem (1.4) if (u,v) satisfies:

$$\int\limits_{\Omega} (|\nabla u|^{p-2}\nabla u\nabla\varphi + |\nabla v|^{q-2}\nabla v\nabla\psi + |u|^{p-2}u\varphi + |v|^{q-2}v\psi)dx$$
$$-\lambda \int\limits_{\partial\Omega} [\varphi f(x,u,v) + \psi g(x,u,v)]d\sigma = 0$$
$$\forall\varphi,\psi \in C^\infty(\overline{\Omega}).$$

By using variational method we shall prove the following theorems.

Theorem 1.1. *Suppose that the assumptions $H1) - H2)$ are satisfied, then there exists a positive number $\underline{\lambda}$ such that for $\lambda < \underline{\lambda}$ the problem (1.4) has no positive solution.*

Theorem 1.2. *Under the assumptions $H1) - H4)$, there is a positive number $\overline{\lambda}$ such that the problem (1.4) has at least two different positive solutions $(u_1, v_1), (u_2, v_2)$ in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ for $\lambda \geqslant \overline{\lambda}$.*

The rest of the paper is organized as follows: in Section 2, we prove Theorem 1.1, and in Section 3, we prove Theorem 1.2.

2. Proof of Theorem 1.1

Firstly, we notice that the following eigenvalue problem (see [5, 7])

$$
\begin{cases} -\Delta_r u + |u|^{r-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{r-2}\dfrac{\partial u}{\partial \gamma} = \lambda |u|^{r-2}u & \text{on } \partial\Omega \\ (1 < r < +\infty) \end{cases} \tag{2.1}
$$

has the first positive eigenvalue λ_{1r} given by:

$$
\lambda_{1r} = \min_{u \in W^{1,r}(\Omega) \setminus W_0^{1,r}(\Omega)} \frac{\int\limits_\Omega (|\nabla u|^r + |u|^r)dx}{\int\limits_{\partial\Omega} |u|^r d\sigma}.
$$

Now for $2 \leqslant p, q < +\infty$ we denote

$$
\lambda_{pq} = \min\{\lambda_{1p}, \lambda_{1q}\}.
$$

Then we obtain

$$
\lambda_{pq} \leqslant \frac{\int\limits_\Omega (|\nabla u|^p + |\nabla v|^q + |u|^p + |v|^q)dx}{\int\limits_{\partial\Omega} (|u|^p + |v|^q)d\sigma}. \tag{2.2}
$$

Suppose that $(u, v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is a positive solution of problem (1.4). Multiplying the first equation of (1.4) by u and the second by v, integrating by parts and adding up, we get

$$
\begin{aligned}
\int\limits_\Omega (|\nabla u|^p + |u|^p + |\nabla v|^q + |v|^q)dx &= \int\limits_{\partial\Omega} [(|\nabla u|^{p-2}\frac{\partial u}{\partial \gamma})u + (|\nabla v|^{q-2}\frac{\partial v}{\partial \gamma})v]d\sigma \\
&= \lambda \int\limits_{\partial\Omega} (uG_u(x, u, v) + vG_v(x, u, v))d\sigma.
\end{aligned}
$$

From that, by hypothesis $H2)$ we have the estimate

$$
\int\limits_\Omega (|\nabla u|^p + |u|^p + |\nabla v|^q + |v|^q)dx \leqslant \lambda C \int\limits_{\partial\Omega} (|u|^p + |v|^q)d\sigma. \tag{2.3}
$$

From (2.2), (2.3) it follows that

$$\lambda \geqslant \frac{\int\limits_{\Omega}(|\nabla u|^p + |\nabla v|^q + |u|^p + |v|^q)dx}{C\int\limits_{\partial\Omega}(|u|^p + |v|^q)d\sigma} \geqslant \frac{\lambda_{pq}}{C}.$$

Thus with $\underline{\lambda} = \dfrac{\lambda_{pq}}{C}$ and for $\lambda < \underline{\lambda}$ the problem (1.4) has no positive solution. The proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

For the proof of Theorem 1.2 we use critical point theory. Set $G(x,u,v) = 0$ for $u < 0$ or $v < 0$, hence also $f(x,u,v) = g(x,u,v) = 0$ for $u < 0$ or $v < 0$. Under hypotheses $H1) - H4)$ we consider the C^1 functional associated to the problem (1.4)

$$G_\lambda(u,v) = \int\limits_{\Omega}\left(\frac{|\nabla u|^p + |u|^p}{p} + \frac{|\nabla v|^q + |v|^q}{q}\right)dx - \lambda\int\limits_{\partial\Omega} G(x,u,v)d\sigma. \tag{3.1}$$

$$(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$$

and we have

$$\begin{aligned} &\big\langle DG_\lambda(u,v),(\varphi,\psi)\big\rangle \\ &= \int\limits_{\Omega}(|\nabla u|^{p-2}\nabla u\nabla\varphi + |\nabla v|^{q-2}\nabla v\nabla\psi)dx + \int\limits_{\Omega}(|u|^{p-2}u\varphi + |v|^{q-2}v\psi)dx \\ &\quad - \lambda\int\limits_{\partial\Omega}(f(x,u,v)\varphi + g(x,u,v)\psi)d\sigma \end{aligned} \tag{3.2}$$

for $(u,v),(\varphi,\psi) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.
It is well known that the (weak) solutions of the problem (1.4) correspond to the critical points of G_λ. To prove Theorem 1.2 we need some following facts.

Proposition 3.1. *If $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is a critical point of G_λ then $u \geqslant 0, v \geqslant 0$ in Ω.*

Proof. Let (u,v) be a critical point of G_λ. Denote

$$u^- = \min\{u,0\}, \quad v^- = \min\{v,0\}.$$

Remark that

$$\int\limits_{\partial\Omega}(u^- f(x,u,v) + v^- g(x,u,v))d\sigma = 0.$$

We have

$$\begin{aligned}
0 =& \left\langle DG_\lambda(u,v),(u^-,v^-)\right\rangle \\
=& \int\limits_\Omega (|\nabla u|^{p-2}\nabla u\nabla u^- + |\nabla v|^{q-2}\nabla v\nabla v^- + |u|^{p-2}uu^- + |v|^{q-2}vv^-)dx \\
& - \lambda\int\limits_{\partial\Omega}(u^- f(x,u,v) + v^- g(x,u,v))d\sigma \\
=& \int\limits_\Omega (|\nabla u^-|^p + |\nabla v^-|^q + |u^-|^p + |v^-|^q)dx \\
=& ||u^-||^p_{W^{1,p}(\Omega)} + ||v^-||^q_{W^{1,q}(\Omega)}.
\end{aligned}$$

Hence $||u^-||_{W^{1,p}(\Omega)} = 0, ||v^-||_{W^{1,q}(\Omega)} = 0$, it follows that $u \geqslant 0, v \geqslant 0$ in Ω. The proof is complete. □

Remark 3.1. Let (u,v) be a critical point of G_λ, then $u \geqslant 0, v \geqslant 0$ in Ω. By Harnack's inequality (see [17]), it follows that either $u > 0, v > 0$ or $u = v = 0$ in Ω. Therefore, non-trivial critical points of G_λ are positive solutions of problem (1.4).

Proposition 3.2. *G_λ is coercive and bounded from below in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$.*

Proof. By assumptions $H2$) and $H4$), for any $\lambda > 0$ there exists a constant $C_\lambda > 0$ such that

$$\lambda G(x,u,v) \leqslant \frac{\lambda_{pq}}{2}\left[\frac{|u|^p}{p} + \frac{|v|^q}{q}\right] + C_\lambda.$$

Hence

$$\begin{aligned}
G_\lambda(u,v) =& \int\limits_\Omega \left(\frac{|\nabla u|^p + |u|^p}{p} + \frac{|\nabla v|^q + |v|^q}{q}\right)dx - \lambda\int\limits_{\partial\Omega} G(x,u,v)d\sigma \\
\geqslant& \int\limits_\Omega \frac{|\nabla u|^p + |u|^p}{p}dx + \int\limits_\Omega \frac{|\nabla v|^q + |v|^q}{q}dx - \int\limits_{\partial\Omega}(\frac{\lambda_{1p}}{2p}|u|^p + \frac{\lambda_{1q}}{2q}|v|^q + C_\lambda)d\sigma \\
\geqslant& \int\limits_\Omega \frac{|\nabla u|^p + |u|^p}{p}dx + \int\limits_\Omega \frac{|\nabla v|^q + |v|^q}{q}dx - \frac{1}{2p}\frac{\int\limits_\Omega(|\nabla u|^p + |u|^p)dx}{\int\limits_{\partial\Omega}|u|^p d\sigma}\int\limits_{\partial\Omega}|u|^p d\sigma \\
& - \frac{1}{2q}\frac{\int\limits_\Omega(|\nabla v|^q + |v|^q)dx}{\int\limits_{\partial\Omega}|v|^q d\sigma}\int\limits_{\partial\Omega}|v|^q d\sigma - \int\limits_{\partial\Omega}C_\lambda d\sigma \\
\geqslant& \frac{1}{2p}\int\limits_\Omega(|\nabla u|^p + |u|^p)dx + \frac{1}{2q}\int\limits_\Omega(|\nabla v|^q + |v|^q)dx - C_\lambda\mu(\partial\Omega).
\end{aligned}$$

From this it follows that

$$G_\lambda(u,v) \geqslant \frac{1}{2p}||u||^p_{W^{1,p}(\Omega)} + \frac{1}{2q}||v||^q_{W^{1,q}(\Omega)} - C_\lambda \mu(\partial\Omega), \tag{3.3}$$

where $\mu(\partial\Omega)$ denotes the Lebesgue measure of $\partial\Omega$. So G_λ is coercive and bounded from below. □

Remark 3.2. By Proposition 3.2 and as $G_\lambda(u,v)$ is weakly lower semicontinuous, we obtain a global minimizer (u_1, v_1) of $G_\lambda(u,v)$ in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.

Proposition 3.3. *There is a positive number $\overline{\lambda}$ such that for $\lambda \geqslant \overline{\lambda}$ $\inf\limits_{(u,v)} G_\lambda(u,v) < 0$ and hence $(u_1, v_1) \neq (0,0)$.*

Proof. Take the constant functions $u_o(x) = t_o, v_o(x) = s_o$ where t_o, s_o are as in $H3$).
Then we obtain

$$\int\limits_{\partial\Omega} G(x, u_o, v_o) d\sigma = \int\limits_{\partial\Omega} G(x, t_o, s_o) d\sigma > 0,$$

hence there is a number $\overline{\lambda} > 0$ such that : for $\lambda \geqslant \overline{\lambda}$

$$G_\lambda(u_o, v_o) = \frac{1}{p}||u_o||^p_{W^{1,p}(\Omega)} + \frac{1}{q}||v_o||_{W^{1,q}(\Omega)^q} - \lambda \int\limits_{\partial\Omega} G(x, u_o, v_o) d\sigma < 0.$$

From this it follows that
$\inf\limits_{(u,v)} G_\lambda(u,v) \leqslant G_\lambda(u_o, v_o) < 0$ for $\lambda \geqslant \overline{\lambda}$. So $G_\lambda(u_1, v_1) < 0$ with $\lambda \geqslant \overline{\lambda}$, hence $(u_1, v_1) \neq 0$. Proposition 3.3 is proved. □

Proposition 3.4. *The origin $(0,0)$ is a strict local minimizer of G_λ in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.*

Proof. Let $\Gamma = \{x \in \partial\Omega : |u(x)|^p + |v(x)|^q > \delta\}, \delta$ be as in $H3$).
So $G(x, u(x), v(x)) \leqslant 0$ for $x \in \partial\Omega \setminus \Gamma$, hence $-\lambda \int\limits_{\partial\Omega\setminus\Gamma} G(x,u,v) d\sigma \geqslant 0$ with $\lambda \geqslant \overline{\lambda} > 0$.
Therefore, for $\lambda \geqslant \overline{\lambda} > 0$,

$$\begin{aligned} G_\lambda(u,v) &= \frac{1}{p}||u||^p_{W^{1,p}(\Omega)} + \frac{1}{q}||v||^q_{W^{1,q}(\Omega)} - \lambda \int\limits_{\partial\Omega\setminus\Gamma} G(x,u,v) d\sigma - \lambda \int\limits_{\Gamma} G(x,u,v) d\sigma \\ &\geqslant \frac{1}{p}||u||^p_{W^{1,p}(\Omega)} + \frac{1}{q}||v||^q_{W^{1,q}(\Omega)} - \lambda \int\limits_{\Gamma} G(x,u,v) d\sigma. \end{aligned}$$

By H2), Holder's inequality and Sobolev trace embedding theorem, we have

$$\begin{aligned} \int\limits_{\Gamma} G(x,u,v) d\sigma &\leqslant \int\limits_{\Gamma} C(|u|^p + |v|^q) d\sigma \\ &\leqslant C(||u||^p_{W^{1,p}(\Omega)} \mu(\Gamma)^{1-\frac{p}{s}} + ||v||^q_{W^{1,q}(\Omega)} \mu(\Gamma)^{1-\frac{q}{r}}), \end{aligned}$$

where

$$\begin{cases} s = \dfrac{(n-1)p}{n-p} \text{ if } p < n \quad \text{and} \quad s > p \text{ if } p \geqslant n \\ r = \dfrac{(n-1)q}{n-q} \text{ if } q < n \quad \text{and} \quad r > q \text{ if } q \geqslant n. \end{cases} \tag{3.4}$$

So, in order to finish the proof, it suffices to show that $\mu(\Gamma) \longrightarrow 0$ as $||u||^p_{W^{1,p}}(\Omega) \longrightarrow 0$ and $||v||^q_{W^{1,q}(\Omega)} \longrightarrow 0$.
We recall that

$$\frac{\int\limits_{\Omega}(|\nabla u|^p + |u|^p + |\nabla v|^q + |v|^q)dx}{\int\limits_{\partial\Omega}(|u|^p + |v|^q)d\sigma} \geqslant \lambda_{pq} = min\,(\lambda_{1p}, \lambda_{1q}) > 0.$$

Then

$$||u||^p_{W^{1,p}(\Omega)} + ||v||^q_{W^{1,q}(\Omega)} \geqslant \lambda_{pq}\int\limits_{\partial\Omega}(|u|^p + |v|^q)d\sigma \geqslant \lambda_{pq}\int\limits_{\Gamma}(|u|^p + |v|^q)d\sigma$$

$$\geqslant \lambda_{pq}\int\limits_{\Gamma}\delta d\sigma = \lambda_{pq}\delta\mu(\Gamma).$$

Now $\mu(\Gamma) \longrightarrow 0$ when $\|u\|^p_{W^{1,p}(\Omega)} + \|v\|^q_{W^{1,q}(\Omega)} \longrightarrow 0$.
Hence $G_\lambda(u,v) > G_\lambda(0,0)$ when $\|u\|^p_{W^{1,p}(\Omega)} \longrightarrow 0$, $\|v\|^q_{W^{1,q}(\Omega)} \longrightarrow 0$.
This completes the proof. □

Proposition 3.5. *G_λ satisfies the Palais-Smale condition in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$.*

Proof. Let $\{(u_m, v_m)\}_{m=1}^{+\infty}$ be a Palais-Smale sequence of G_λ in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$. We have then $|G_\lambda(u_m, v_m)| \leqslant K$, for any m, $DG_\lambda(u_m, v_m) \longrightarrow 0$ as $m \longrightarrow +\infty$. Due to Proposition 3.2, G_λ is coercive and bounded, and from (3.3) we have

$$G_\lambda(u_m, v_m) \geqslant \frac{1}{2p}\|u_m\|^p_{W^{1,p}(\Omega)} + \frac{1}{2q}\|v_m\|^q_{W^{1,q}(\Omega)} - C_\lambda\mu(\partial\Omega).$$

Hence (u_m, v_m) is a bounded sequence in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$. Thus, there exists a subsequence $\{(u_{m_j}, v_{m_j})\}_{j=1}^{\infty}$ of $\{(u_m, v_m)\}_{m=1}^{\infty}$ which converges weakly to (u_o, v_o) in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. We shall prove that $\{(u_{m_j}, v_{m_j})\}$ converges strongly to (u_o, v_o) in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.

Firsly, by Rellich-Kondrachov theorem (see[1], p.144), the embedding $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$ into $L^p(\Omega) \times L^q(\Omega)$ is continuous and compact. Therefore the sequence $\{(u_{m_j}, v_{m_j})\}_j$ converges strongly to (u_o, v_o) in $L^p(\Omega) \times L^q(\Omega)$. This implies that the sequence $\{(u_{m_j}, v_{m_j})\}_j$ is bounded in $L^p(\Omega) \times L^q(\Omega)$, hence the sequence

$$\{|u_{m_j}|^{p-2}u_{m_j}, |v_{m_j}|^{q-2}v_{m_j}\}_j$$

is bounded in $L^{p^{'}}(\Omega) \times L^{q^{'}}(\Omega)$, where $p^{'} = \dfrac{p}{p-1}$, $\quad q^{'} = \dfrac{q}{q-1}$ so that

$$\lim_{j \longrightarrow +\infty} \int_{\Omega} (|u_{m_j}|^{p-2} u_{m_j}(u_{m_j} - u_o) + |v_{m_j}|^{q-2} v_{m_j}(v_{m_j} - v_o))dx = 0. \tag{3.5}$$

On the other hand, from hypothesis $H2)$ it follows that $f(x, u_{m_j}, v_{m_j})$ is bounded in $L^{p^{'}}$ and $g(x, u_{m_j}, v_{m_j})$ is bounded in $L^{q^{'}}$, hence

$$\lim_{j \longrightarrow +\infty} \int_{\partial\Omega} [(u_{m_j} - m_o).f(x, u_{m_j}, v_{m_j}) + (v_{m_j} - v_o)g(x, u_{m_j}, v_{m_j})]d\sigma = 0. \tag{3.6}$$

Besides, we have

$$\lim_{j \longrightarrow +\infty} \big\langle DG_\lambda(u_{m_j}, v_{m_j}), (u_{m_j} - u_o, v_{m_j} - v_o)\big\rangle = 0. \tag{3.7}$$

By applying the equality (3.2) we have

$$\begin{aligned}
&\int_{\Omega} (|\nabla u|^{p-2} \nabla u \nabla \varphi + |\nabla v|^{q-2} \nabla v \nabla \psi)dx \\
&= \big\langle DG_\lambda(u, v), (\varphi, \psi)\big\rangle - \int_{\Omega} (|u|^{p-2} u\varphi + |v|^{q-2} v\psi)dx \\
&\quad + \lambda \int_{\partial\Omega} [\varphi f(x, u, v) + \psi g(x, u, v)]d\sigma
\end{aligned}$$

for $(u, v), (\varphi, \psi) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. With $u = u_{m_j}$, $v = v_{m_j}$, $\varphi = u_{m_j} - u_o$, $\psi = v_{m_j} - v_o$, we get

$$\begin{aligned}
&\int_{\Omega} (|\nabla u_{m_j}|^{p-2} \nabla u_{m_j} \nabla(u_{m_j} - u_o) + |\nabla v_{m_j}|^{q-2} \nabla v_{m_j} \nabla(v_{m_j} - v_o))dx \\
&= \big\langle DG_\lambda(u_{m_j}, v_{m_j}), (u_{m_j} - u_o, v_{m_j} - v_o)\big\rangle \\
&\quad - \int_{\Omega} (|u_{m_j}|^{p-2} u_{m_j}(u_{m_j} - u_o) + |v_{m_j}|^{q-2} v_{m_j}(v_{m_j} - v_o))dx \\
&\quad + \lambda \int_{\partial\Omega} [(u_{m_j} - u_o) f(x, u_{m_j}, v_{m_j}) + (v_{m_j} - v_o) g(x, u_{m_j}, v_{m_j})]d\sigma.
\end{aligned}$$

Letting $j \longrightarrow +\infty$ from (3.5), (3.6), (3.7) we obtain that

$$\int_{\Omega} (|\nabla u_{m_j}|^{p-2} \nabla u_{m_j} \nabla(u_{m_j} - u_o) + |\nabla v_{m_j}|^{q-2} \nabla v_{m_j} \nabla(v_{m_j} - v_o))dx = 0. \tag{3.8}$$

Using a similar approach we get

$$\int_{\Omega} (|\nabla u_o|^{p-2} \nabla u_o \nabla(u_{m_j} - u_o) + |\nabla v_o|^{q-2} \nabla v_o \nabla(v_{m_j} - v_o))dx = 0. \tag{3.9}$$

Remark that for $r \geqslant 2$, there exists a positive contant C_r such that

$$(|s|^{r-2}s - |\overline{s}|^{r-2}\overline{s})(s-\overline{s}) \geqslant C_r|s-\overline{s}|^r \tag{3.10}$$

for any $s, \overline{s} \in R^n$ (Proposition 2, [21]).
Applying (3.10) with $s = \nabla u_{m_j}(\nabla v_{m_j})$, $\overline{s} = \nabla u_o(\nabla v_o)$ we obtain the estimate

$$\begin{aligned} &\int_\Omega (|\nabla u_{m_j}|^{p-2}\nabla u_{m_j} - |\nabla u_o|^{p-2}\nabla u_o)(\nabla u_{m_j} - \nabla u_o)dx \\ &\quad + \int_\Omega (|\nabla v_{m_j}|^{q-2}\nabla v_{m_j} - |\nabla v_o|^{q-2}\nabla v_o)(\nabla v_{m_j} - \nabla v_o)dx \\ &\geqslant C_p||\nabla u_{m_j} - \nabla u_o||^p_{L^p(\Omega)} + C_q||\nabla v_{m_j} - \nabla v_o||^q_{L^q(\Omega)}. \end{aligned} \tag{3.11}$$

Letting $j \longrightarrow \infty$, using (3.8), (3.9), from (3.11), we get

$$\lim_{j\longrightarrow\infty} ||u_{m_j} - u_o||_{W^{1,p}(\Omega)} = 0,$$

$$\lim_{j\longrightarrow\infty} ||v_{m_j} - v_o||_{W^{1,q}(\Omega)} = 0.$$

Besides, $(u_{m_j}, v_{m_j}) \longrightarrow (u_o, v_o)$ in $L^p(\Omega) \times L^q(\Omega)$ that the sequence $\{(u_{m_j}, v_{m_j})\}_j$ converges strongly to (u_o, v_o) in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. The proof of Proposition 3.5 is complete. □

Now we are in position to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.5 and Proposition 3.4, G_λ satisfies the Palais-Smale condition in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, the origin $(0,0)$ is a strict local minimizer of G_λ and $G_\lambda(0,0) = 0$. Moreover, from Proposition 3.2 and Remark 3.2, G_λ has a global minimizer $(u_1, v_1) \neq (0,0)$, $G_\lambda(u_1, v_1) < 0$. Now applying the mountain-pass theorem (Theorem 10.3 [18]), there exists a critical point $(u_2, v_2) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ of G_λ which is not of minimizer type. Thus $(u_2, v_2) \neq (u_1, v_1)$. Theorem 1.2 is proved. □

References

[1] R. A. Adams, *Sobolev Spaces*, Academice Press, New York - San Francisco - London, 1975.

[2] A. Ambrossetti and A. Rabinowitz, Dual variational methods in critical points theory and applications, *J. Funct. Anal.* **4** (1973), 349–381.

[3] A. Ambrossetti, K. C. Chay and I. Ekeland, *Nonlinear functional analysis and Applications to differential equations*, Procedings of the second school ICTP, Trieste, Italy, 1997.

[4] A. Anane, Etude des valeurs propres et de la resonnance pour l'operateur p-laplacien, PhD thesis, Universite Libre de Bruxelles, *C. R. Acad. Sci. Paris Ser. I Math.* **305** (16) (1987), 725-728.

[5] I. Babuska and J. Osborn, *Eigenvalue Problems*, Handbook of Number. Anal., Vol.II. North-Holland, 1991.

[6] L. Boccacdo and D. G. de Figueirrdo, Some remarks on a system of quasilinear elliptic equations, *Nonlinear Anal.* **9** (3) (2002), 309–323.

[7] E. di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Academice Press, New York - San Francisco - London* **7** (8) (1983), 827–850.

[8] J. Fernandez Bonder and J. D. Rossi, Existence results for the p-laplacian with nonlinear boundary conditions, *J. Math. Anal. Appl.* **263** (2001), 195–223.
[9] J. Fernandez Bonder and J. D. Rossi, A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding, *Publ. Math.* **46** (2002), 221–235.
[10] J. Fernandez Bonder, Multiple positive solutions for quasilinear elliptic problems with Sign-Changing nonlinearities, *Abstr. Appl. Anal.* **2004** (12) (2004), 1047–1056.
[11] P. Lindqvit, On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1) (1990), 157–164, Addendum: *Proc. Amer. Math. Soc.* **116** (2) (1992), 583–584.
[12] S. Martinez and J. D. Rossi, Isolation and simplicity for the first eigenvalue of the p-laplacian with a nonlinear boundary condition, *Abstr. Appl. Anal.* **7** (2002) (5), 287–293.
[13] C. Maya and R. Shivaji, Multiple positive solutions for a class of semilinear elliptic boundary value problems, *Nonlinear Anal.* **38** (4) (1999), Ser. A: Theory Methods, 497–504.
[14] M. Giaguinta, Giuseppe Modica Jiri Soucek, *Cartesian currents in the calculus of variation*, 1998.
[15] K. Perera, Multiple positive solutions for a class of quasilinear elliptic boundary value problems, *Electron. J. Differential Equation* **2003** (7) (2003), 1–5.
[16] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.
[17] N. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967), 721–747.
[18] M. Struwe, *Variational Methods, Application to Nonlinear Partial Differential Equation and Hamiltonian Systems*, 2000.
[19] P. Tolksdorf, On the Dirichlet problem for quasilinear equation in domain with conical boundary points, *Comm. Pure Appl. Math.* **8** (1983), 773–817.
[20] J. L. Vazquez, A strong maximun priciple for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (3) (1984), 191–202.
[21] X. Wu, K.-K. Tan, On existence and multiplicity of solutions of Neumann boundary elliptic equations, *Nonlinear Anal.* **65** (2006), 1334–1347.

Departement of Informatics
Hanoi University of Civil Engineering
55 Giai Phong, Hanoi Vietnam

E-mail address: quoctrung032007@yahoo.com

Department of Mathematics
Hanoi University of Science
334 Nguyen Trai, Hanoi, Vietnam

E-mail address: hq-toan@yahoo.com