

PROOF OF THE VOLUME CONJECTURE FOR WHITEHEAD CHAINS

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ABSTRACT. We prove the volume conjecture for an infinite family of links called Whitehead chains that generalizes both the Whitehead link and the Borromean rings.

1. INTRODUCTION

The volume conjecture relates the colored Jones polynomials of a knot to the simplicial volume of its complement. More precisely, let us denote the normalized N -colored Jones polynomial of a knot K by $J_N(K)$ and let $\text{Vol}(K)$ be v_3 times the simplicial volume (Gromov norm) of $\mathbb{S}^3 - K$, where v_3 is the volume of the hyperbolic regular ideal tetrahedron. The volume conjecture can now be stated as follows:

Conjecture 1. (Volume conjecture) [7]

For any knot K we have:

$$\lim_{N \rightarrow \infty} \frac{2\pi}{N} \log \left| J_N(K) \left(e^{\frac{2\pi i}{N}} \right) \right| = \text{Vol}(K).$$

So far the conjecture has been proven for only the figure eight knot, torus knots, Whitehead doubles of certain torus knots and connected sums of these knots, [6], [4], [12].

It is well known that the volume conjecture is false for many splittable links so it is unclear how to extend the volume conjecture to links. On the other hand, the volume conjecture has been shown to hold for the Whitehead link [12] and the Borromean rings [1]. In this paper, we introduce the family of Whitehead chains generalizing both the Whitehead link and the Borromean rings and we settle the volume conjecture for this family.

The Whitehead chains are defined in terms of the tangles Belt, Clasp and Twist depicted in figure 1 on the next page.

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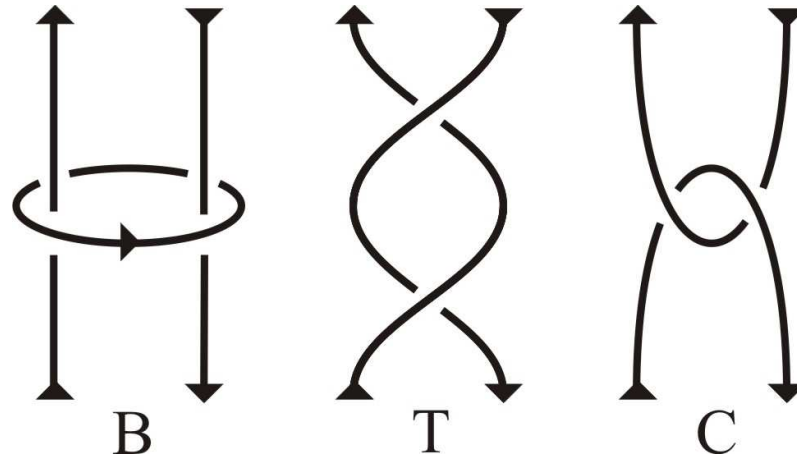


FIGURE 1. The tangles Belt, Twist and Clasp

Definition 1. Let a, b, c, d be integers such that $b \geq 1$ and $c, d \geq 0$. Define the Whitehead chain $W_{a,b,c,d}$ to be the closure of the composition of a tangles of type Twist (or a mirror images of type Twist when $a < 0$), b tangles of type Belt, c tangles of type Clasp and d mirror images of tangles of type Clasp.

The tangles Belt, Twist, Clasp and their mirror images commute, so the order of composition is immaterial. Therefore the Whitehead chains are well defined. In the notation of the previous definition the Whitehead link is $W_{0,1,1,0}$ and the Borromean rings are $W_{0,1,1,1}$.

Our main theorem is the following asymptotic expansion for the colored Jones polynomial of a Whitehead chain. In the statement below the notation $x_N \sim y_N$ means that the quotient x_N/y_N converges to 1 as N goes to infinity.

Main theorem. Let a, b, c, d be integers, such that $b \geq 1$ and $c, d \geq 0$.

$$J_N(W_{a,b,c,d})(e^{\frac{2\pi i}{N}}) \sim \exp\left(\frac{1}{2\pi} \{(\text{Vol}(W_{a,b,c,d}) + i\text{CS})N + D \log(N) + E\}\right).$$

- The value of CS is $\frac{-4a + c - d}{8}2\pi^2$ if $c + d = 1$ and $\frac{-4a - 7c + 7d}{8}2\pi^2$ otherwise.
- If $b = 1$ we have $D = 3\pi$ and $\exp(E)$ can be expressed explicitly as an integral depending on a, c, d , see the end of the proof of lemma 4.
- If $b \geq 2$ the expansion is only valid for odd N and $J_{2M}(W_{a,b,c,d})(e^{\frac{2\pi i}{2M}}) = 0$. For odd N we find $D = 2\pi b$ and $E = -2\pi(c + d) \log 2 + \frac{4a + 3c - 3d}{4}2\pi^2 i$.

By taking absolute values and restricting ourselves to the leading term we see that the volume conjecture holds true for $b = 1$, while it is false for $b \geq 2$. In the latter case the volume conjecture is true when we restrict to odd values of N . This phenomenon might have something to do with the fact that the complement

of the Whitehead chain is hyperbolic when $b = 1$, while the torus decomposition contains a Seifert-fibered piece when $b \geq 2$.

According to the complexified volume conjecture proposed in [8] the value of CS is equal to $2\pi^2$ times the Chern-Simons invariant. For $a = 0$ this is indeed the case because the Chern-Simons invariant is additive with respect to belted sum and its value on the Whitehead link and its mirror image is $\pm\frac{1}{8}$, [9].

In [3] the number D in the above asymptotic expansion is conjectured to be the number of prime factors of a knot. In [2] a different interpretation is given in terms of the knot complement. In the same paper it is also conjectured that the number E is determined by the Ray-Singer torsion of the complement twisted by the holonomy representation. We hope to investigate these conjectures for the Whitehead chains in a subsequent publication.

The main theorem shows that the original volume conjecture may fail even for non-splittable links, but adds credibility to the following weaker version of the volume conjecture for links:

Conjecture 2. *For any non-splittable link L we have:*

$$\limsup_{N \rightarrow \infty} \frac{2\pi}{N} \log |J_N(L)(e^{\frac{2\pi i}{N}})| = \text{Vol}(L).$$

2. PROOF OF THE MAIN THEOREM

In this section we give an overview of the proof of the main theorem, postponing the proofs of the more technical lemmas to the next section.

The first step is to obtain an expression for the colored Jones polynomials of a general Whitehead chain.

Lemma 1. *Let a, b, c, d be integers, with $b \geq 2$ and $c, d \geq 0$. We have the following formulas for the colored Jones polynomial:*

$$J_N(W_{a,1,c,d})(e^{\frac{2\pi i}{N}}) = \phi_N \sum_{n=0}^{N-1} (2n+1) \chi_{N,n}^{4a-c+d} \left(\sum_{k=0}^{N-1-n} S_{n,k} \right)^{c+d},$$

$J_N(W_{a,b,c,d})(e^{\frac{2\pi i}{N}}) = 0$ when N is even and for $N = 2M + 1$ we have:

$$J_N(W_{a,b,c,d})(e^{\frac{2\pi i}{N}}) = \phi_N N^b \chi_{N,M}^{4a-c+d} \left(\sum_{k=0}^M S_{M,k} \right)^{c+d},$$

where we define

$$\phi_N = \begin{cases} \exp\left(\frac{(N-1)(c-d)}{N}\pi i\right) & \text{if } c+d=1 \\ (-1)^{(N-1)(c-d)} & \text{otherwise,} \end{cases}$$

and we define χ and S by

$$\chi_{N,n} = \exp\left(\frac{n(n+1-N)}{2N}\pi i\right), \quad S_{n,k} = \prod_{j=1}^n \frac{2 \sin^2\left(\frac{(k+j)\pi}{N}\right)}{\sin\left(\frac{j\pi}{N}\right)}.$$

Proof. Recall that the unnormalized N -colored Jones invariants are intertwining operators of $V_N \otimes V_N$, where V_N corresponds to the N -dimensional irreducible representation of sl_2 . Using the decomposition $V_N \otimes V_N = \bigoplus_{n=0}^{N-1} V_{2n+1}$ one can write the unnormalized colored Jones invariants of the tangles in figure 1 in the following way [12]:

$$\tilde{J}_N(t) = \bigoplus_{n=0}^{N-1} \text{Tangle}(n, t) \cdot \text{id}_{V_{2n+1}},$$

where the function $\text{Tangle}(n, t)$ depends on the tangle and $t = e^h$. If we define $[n] = \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}$ then the functions for the tangles in figure 1 are:

$$\text{T}(n, t) = t^{n(n+1)}, \quad \text{B}(n, t) = \frac{[N(2n+1)]}{[2n+1]},$$

$$\text{C}(n, t) = t^{(N^2-1)/2+N(N-1)/2} \sum_{k=0}^{N-1-n} t^{-N(n+k)} \prod_{j=1}^n \frac{(1 - t^{N-j-k})(1 - t^{j+k})}{1 - t^j}.$$

The colored Jones polynomials of the mirror-images of these tangles are obtained by replacing t by t^{-1} . The factor $t^{(N^2-1)/2}$ in the formula for $\text{C}(n, t)$ is a correction due to framing that should be included only if both strands in the clasp belong to the same component. For Whitehead chains this means that it should be included only when $c + d = 1$.

By the multiplicativity of the colored Jones invariant with respect to composition of tangles we can now calculate the colored Jones polynomial of all Whitehead chains. The general formula for the normalized version of the colored Jones polynomial of the Whitehead chains is:

$$J_N(W_{a,b,c,d})(t) = \sum_{n=0}^{N-1} \frac{[2n+1]}{[N]} \text{T}(n, t)^a \text{B}(n, t)^b \text{C}(n, t)^c \text{C}(n, t^{-1})^d.$$

The factor $\frac{[2n+1]}{[N]}$ comes from the normalization and from taking the closure of composition of the tangles.

Routine calculations now yield the above formulas. □

From now on we will use the shorthand $J_{N,a,b,c,d} = J_N(W_{a,b,c,d})(e^{\frac{2\pi i}{N}})$.

The next step in proving the main theorem is to investigate the asymptotics of the above formulas for the colored Jones polynomials as $N \rightarrow \infty$. The factors $S_{n,k}$ turn out to play a crucial role because they dominate the absolute value of the (n, k) -th term. There exists a unique maximum $\tilde{S}_N = S_{\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{4} \rfloor}$ for the $S_{n,k}$. This term dominates all others so that most of the asymptotics of $J_{N,a,b,c,d}$ can be read off from this term only. In order to make this precise we need to compare the other values of $S_{n,k}$ to the maximum value.

Let us choose a fixed number $\delta \in (\frac{1}{2}, \frac{c+d+5}{2(c+d+4)})$ once and for all, where the numbers c and d are the parameters of $W_{a,b,c,d}$. Define $n' = |n - \lfloor \frac{N}{2} \rfloor|$ and $k' = |k - \lfloor \frac{N}{4} \rfloor|$ to be the distances from the maximum. The next lemma shows how we can estimate the other values of $S_{n,k}$.

Lemma 2. *Using the above definitions of n', k' and δ we have:*

- a) *If $n' + k' < N^\delta$ then $S_{n,k} \tilde{S}_N^{-1} = \exp(-\frac{\pi}{N}(n'^2 + 2n'k' + 2k'^2)) + \mathcal{O}(N^{3\delta-2})$, as $N \rightarrow \infty$.*
- b) *There are $C, \epsilon > 0$ such that if $n' + k' \geq N^\delta$ then $S_{n,k} \tilde{S}_N^{-1} < C \exp(-\epsilon N^{2\delta-1})$*

In Lemma 3 the asymptotics of the maximum value \tilde{S}_N are expressed using the Lobachevski function $\Lambda(\theta) = -\int_0^\theta \log |2 \sin(x)| dx$.

Lemma 3. $\tilde{S}_N \sim \exp(\frac{4N}{\pi} \Lambda(\frac{\pi}{4}) - \frac{1}{2} \log 2N)$, $N \rightarrow \infty$.

The asymptotics of $J_{N,a,1,c,d}$ is reduced to that of \tilde{S}_N by the following lemma. By CS we mean the constant defined in the statement of the main theorem.

Lemma 4. *There is a nonzero constant $C \in \mathbb{C}$ such that:*

$$J_{N,a,1,c,d} \sim CN^{(c+d+3)/2} \tilde{S}_N^{c+d} e^{\frac{iCS}{2\pi}}, \quad N \rightarrow \infty.$$

There is a similar lemma for the case $b \geq 2$. We confine ourselves to the odd-colored Jones polynomials, since the even ones are 0 in $e^{\frac{2\pi i}{N}}$. In lemma 5 we reduce the asymptotics of the odd colored Jones polynomial to those of the maximal term \tilde{S}_N :

Lemma 5. *For $b \geq 2$ and N odd we have: $J_{N,a,b,c,d} \sim$*

$$\exp\left(- (c+d) \log \sqrt{2} + (4a + 3c - 3d)\pi i/4\right) N^{(c+d+2b)/2} \tilde{S}_N^{c+d} e^{\frac{iCS}{2\pi}}.$$

Postponing the proofs of these lemmas to the next subsection we can now prove the main theorem.

Proof. (of the main theorem)

Using an explicit decomposition of the complement into ideal octahedra it can be shown that $\text{Vol}(W_{a,b,c,d}) = 8(c+d)\Lambda(\frac{\pi}{4})$, see [10]. Let us first suppose that $b = 1$. According to Lemma 4 there is a constant C' such that we have:

$$J_{N,a,1,c,d} \sim \exp\left(\log(\tilde{S}_N^{c+d}) + \frac{c+d+3}{2} \log(N) + Ni \frac{CS}{2\pi} + C'\right).$$

Using Lemma 3 we get:

$$\sim \exp\frac{1}{2\pi} ((\text{Vol}(W_{a,1,c,d}) + iCS)N + 3\pi \log(N) + E), \quad N \rightarrow \infty.$$

The case $b \geq 2$ follows in the same way by combining Lemma 3 and Lemma 5. \square

3. PROOF OF THE LEMMAS

In this section we prove the more technical lemmas 2,3,4 and 5.

Proof. (of lemma 2) The proof of this lemma hinges on the following key estimate of $S_{n,k}$ in terms of the Lobachevski function $\Lambda(x)$ that was proved in [12]. Define $f(x, y) = -2\Lambda(x + y) + 2\Lambda(y) + \Lambda(x)$. For integers $0 \leq n, k, n + k < N$ we have the uniform estimate

$$\log S_{n,k} = \frac{N}{\pi} f\left(\frac{n\pi}{N}, \frac{k\pi}{N}\right) + \mathcal{O}(\log N), \quad N \rightarrow \infty.$$

Before we can apply this result we first need to show that inside the triangle $0 < x, y, x + y < \pi$ the function f has a unique critical point $(\frac{\pi}{2}, \frac{\pi}{4})$ and reaches its maximum there, which equals $4\Lambda(\frac{\pi}{4})$. Moreover the Taylor expansion of f around the critical point is:

$$f\left(\frac{\pi}{2} + x, \frac{\pi}{4} + y\right) = f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) - (x^2 + 2xy + 2y^2) + \mathcal{O}(|x|^3 + |y|^3).$$

Lemma 2 part a) and b) are direct consequences of these facts once we note that the difference between $S_{\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{4} \rfloor}$ and the actual critical value $f(\frac{\pi}{2}, \frac{\pi}{4})$ becomes negligibly small as N grows.

To find the critical points of f in $0 < x, y, x + y < \pi$ we use the fundamental theorem of calculus to differentiate Λ and find the system of equations: $2\sin^2(x + y) = \sin(x)$ and $\sin(x + y) = \sin(y)$. For $0 < x, y, x + y < \pi$ this has the unique solution $(x, y) = (\frac{\pi}{2}, \frac{\pi}{4})$. To determine the nature of the critical point we differentiate again and this will be left to the reader.

The value of f at its critical point is $f(\frac{\pi}{2}, \frac{\pi}{4}) = -2\Lambda(\frac{3}{4}\pi) + 2\Lambda(\frac{\pi}{4}) + \Lambda(\frac{\pi}{2}) = 4\Lambda(\frac{\pi}{4})$ because $\Lambda(x) = -\Lambda(\pi - x)$. □

The next lemma is an expanded version of a result proven in [12].

Proof. (of Lemma 3) If we define $s_n = -\sum_{j=1}^n \log |2\sin(j\pi/N)|$ then we can write $\log \tilde{S}_N = -2s_{\lfloor N/4 \rfloor + \lfloor N/2 \rfloor} + 2s_{\lfloor N/4 \rfloor} + s_{\lfloor N/2 \rfloor}$. It was shown in [12] that for $0 < n < \frac{5}{6}N$ we have $s_n = \frac{N}{\pi} \Lambda(\frac{n\pi}{N}) - \frac{1}{2} \log n + \mathcal{O}(1)$, as $N \rightarrow \infty$. To prove the lemma we need to expand a little further. Assuming that $r = \lim_{N \rightarrow \infty} n/N$ exists, we show that

$$s_n = \frac{N}{\pi} \Lambda\left(\frac{n\pi}{N}\right) - \frac{1}{2} \log \frac{n \sin(r\pi)}{r\pi} + \mathcal{O}(N^{-1}), \quad N \rightarrow \infty.$$

By applying the above expansion for s_n three times we find the desired expansion for \tilde{S}_N .

To prove the expansion for s_n we reason as follows: for $x \in [0, \frac{5}{6}\pi] - \{0, u\}$ we have

$$\frac{\sin(x - u)}{x - u} \frac{x}{\sin x} = 1 + \frac{-x \cos x + \sin x}{x \sin x} u + \mathcal{O}(u^2), \quad u \rightarrow 0.$$

Now taking the logarithm and expanding $\log(1 + f)$ around $f = 0$ we find:

$$\log \left| \frac{\sin(x - u)}{\sin x} \right| = \log \left| \frac{x - u}{x} \right| + \frac{-x \cos x + \sin x}{x \sin x} u + \mathcal{O}(u^2).$$

Following [12] p.7 we set $x = j\pi/N$ and find:

$$s_n - \frac{N}{\pi} \Lambda\left(\frac{n\pi}{N}\right) = \sum_{j=1}^n \frac{N}{\pi} \int_0^{\frac{\pi}{N}} \log \frac{\sin(j\pi/N - u)}{\sin j\pi/N} du =$$

$$-\frac{1}{2} \log n - \frac{1}{2} \log 2\pi + \sum_{j=1}^n \frac{N}{\pi} \int_0^{\frac{\pi}{N}} \frac{-(j\pi/N) \cos(j\pi/N) + \sin j\pi/N}{j\pi/N \sin j\pi/N} u du + \mathcal{O}(N^{-1}).$$

The term $-\frac{1}{2} \log 2\pi$ is the constant contribution of the Stirling series used in [12] p.7. After integration with respect to u we can write the above sum as a Riemann sum. A computation then shows that the limit of this sum equals $\frac{1}{2}(\log r\pi - \log \sin r\pi)$. \square

Although the proof of lemma 4 below is quite long the main idea is simple: Use lemma 2 to estimate the value of the colored Jones polynomial in terms of the maximum value \tilde{S}_N .

Proof. (of Lemma 4) For convenience we will assume throughout the proof that $c + d \geq 2$, so that the value of CS is $\frac{-4a - 7c + 7d}{8} 2\pi^2$. The proof in the case $c + d = 1$ is completely analogous. According to lemma 1 the formula for the N -colored Jones polynomial of the one-belted Whitehead chain is

$$J_{N,a,1,c,d} = \phi_N \sum_{n=0}^{N-1} (2n+1) \chi_{N,n}^{4a-c+d} \left(\sum_{k=0}^{N-1-n} S_{n,k} \right)^{c+d}.$$

Define the quotient $Q_N = J_{N,a,1,c,d} N^{-(c+d+3)/2} \tilde{S}_N^{-(c+d)} e^{\frac{4a+7c-7d}{8} N\pi i}$. We aim to show that $Q_\infty = \lim_{N \rightarrow \infty} Q_N = C$ for some nonzero complex constant.

Step 1. By expanding the $c + d$ -th power of the sum we obtain a multi-sum over all N -tuples of natural numbers $n, \mathbf{k} < N$ such that $n + k_j < N$. Define Central to be the set of all such tuples that satisfy $n + \max k_j < N^\delta$ and define Far to be the set of tuples such that $n + \max k_j \geq N^\delta$. We can then rewrite Q_N as follows:

$$Q_N = \phi_N e^{\frac{4a+7c-7d}{8} N\pi i} N^{-(c+d+3)/2} \sum_{\text{Central}} (2n+1) \chi_{N,n}^{4a-c+d} S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}$$

$$+ \phi_N e^{\frac{4a+7c-7d}{8} N\pi i} N^{-(c+d+3)/2} \sum_{\text{Far}} (2n+1) \chi_{N,n}^{4a-c+d} S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}.$$

The Far sum converges to zero absolutely, because each tuple n, \mathbf{k} in the sum contains a k_m such that $n + k_m \geq N^\delta$. According to lemma 2b this means that $S_{n,k_m} \tilde{S}_N^{-1} = \mathcal{O}(\exp(-\epsilon N^{2\delta-1}))$. We can estimate all other factors $S_{n,k_j} \tilde{S}_N^{-1}$ by a constant and conclude that the sum of absolute values of the Far sum converges to zero since $\delta > 1/2$. So far we have shown that

$$Q_\infty = \lim_{N \rightarrow \infty} \phi_N e^{\frac{4a+7c-7d}{8} N\pi i} N^{-\frac{c+d+3}{2}} \sum_{\text{Central}} (2n+1) \chi_{N,n}^{4a-c+d} S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}.$$

Step 2. In the next step we replace the factor $(2n + 1)$ in the expression for Q_∞ by a factor N . This is done by showing that the difference converges to zero. The absolute value of the difference is less than

$$N^{-(c+d+3)/2} \sum_{\text{Central}} |2n + 1 - N| S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}.$$

For central tuples n, \mathbf{k} we have $n' < N^\delta$ and hence $2n + 1 = N + \mathcal{O}(N^\delta)$. The number of terms in the sum is of order $\mathcal{O}(N^{(c+d+1)\delta})$ and the product $S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}$ is $\mathcal{O}(1)$ by lemma 2a. Therefore the absolute value of the difference is at most of order $\mathcal{O}(N^{-(c+d+3)/2+(c+d+2)\delta})$. Since $\delta < \frac{c+d+5}{2(c+d+4)} < \frac{c+d+3}{2(c+d+2)}$ the difference converges to zero and we have:

$$Q_\infty = \lim_{N \rightarrow \infty} \phi_N e^{\frac{4a+7c-7d}{8} N \pi i} N^{-(c+d+1)/2} \sum_{\text{Central}} \chi_{N,n}^{4a-c+d} S_{n,k_1} \cdots S_{n,k_{c+d}} \tilde{S}_N^{-(c+d)}.$$

Step 3. Define the $(c + d + 1)$ -variable Gaussian function

$$g_N(x, \mathbf{y}) = \exp -\frac{\pi}{N} \left((c + d - (4a + c - d)i) \frac{x^2}{2} + 2 \sum_{j=1}^{c+d} (y_j + \frac{x}{2})^2 \right)$$

and let $\psi = \exp(\frac{4a+3c-3d}{4} \pi i)$. We will show that

$$Q_\infty = \lim_{N \rightarrow \infty} \psi N^{-(c+d+1)/2} \sum_{\text{Central}} g_N(n', \mathbf{k}').$$

Starting with the sum from step 2 we use lemma 2 to replace the factors $S_{n,k_j} \tilde{S}^{-1}$ by exponentials and error terms:

$$\lim_{N \rightarrow \infty} \phi_N e^{\frac{4a+7c-7d}{8} N \pi i} N^{-(c+d+1)/2} \sum_{\text{Central}} \chi_{N,n}^{4a-c+d} \prod_{j=1}^{c+d} \left(\exp(-\frac{\pi}{N} (n'^2 + 2n'k'_j + 2k'^2_j)) + \mathcal{O}(N^{3\delta-2}) \right).$$

To show that the error terms can be removed we estimate the contribution of their absolute values. There are $\mathcal{O}(N^{(c+d+1)\delta})$ terms in the sum so their contribution is of order $\mathcal{O}(N^{(c+d+1)\delta+3\delta-2-(c+d+1)/2})$. This converges to zero because $\delta < \frac{c+d+5}{2(c+d+4)}$.

Next we look at the phase factors ϕ_N and $\chi_{N,n}$. We have $\lim_{N \rightarrow \infty} \phi_N = e^{(c-d)\pi i}$. Furthermore:

$$\chi_{N,n} e^{\frac{N \pi i}{8}} = \exp\left(\frac{n'^2}{2N} - \frac{N}{8} + \frac{n}{2N} + \frac{N}{8}\right) \pi i = \exp\left(\frac{n'^2}{2N} \pi i\right) \exp\left(\frac{n}{2N} \pi i\right).$$

Since $n = \frac{N}{2} + \mathcal{O}(N^\delta)$ we have $\lim_{N \rightarrow \infty} \frac{n}{2N} = \frac{1}{4}$. If we apply this to the above sum we get the desired expression.

Step 4. It seems natural to replace the previous sum by an integral, this is done in step 4. We will show that

$$Q_\infty = \lim_{N \rightarrow \infty} \psi N^{-(c+d+1)/2} \int_{|x| + \max |y_j| < N^\delta - 1} g_N(|x|, |\mathbf{y}|) dx d\mathbf{y}.$$

The strategy is to estimate the absolute value of the difference between sum and integral. First we write the sum from step 3 as follows:

$$Q_\infty = \lim_{N \rightarrow \infty} \psi N^{-(c+d+1)/2} \sum_{\text{Central}} \int_{B(n, \mathbf{k})} g_N(n', \mathbf{k}') dx d\mathbf{y},$$

where $B(n, \mathbf{k}) = [n - \frac{N}{2} - \frac{1}{2}, n - \frac{N}{2} + \frac{1}{2}] \times \prod_{j=1}^{c+d} [k_j - \frac{N}{4} - \frac{1}{2}, k_j - \frac{N}{4} + \frac{1}{2}]$. The absolute value of the difference between the sum and the proposed integral above is:

$$N^{-(c+d+1)/2} \left| \sum_{\text{central}} \int_{B(n, \mathbf{k})} g_N(n', \mathbf{k}') dx d\mathbf{y} - \int_{|x| + \max |y_j| < N^\delta - 1} g_N(|x|, |\mathbf{y}|) dx d\mathbf{y} \right|.$$

The union of the disjoint blocks $B(n, \mathbf{k})$ as n, \mathbf{k} runs through Central, covers the entire integration domain $|x| + \max |y_j| < N^\delta - 1$ so we can subtract the integrals. However some blocks continue over the boundary of the domain of integration, resulting in a slight error. The terms n, \mathbf{k} such that the corresponding blocks $B(n, \mathbf{k})$ intersect the complement of the domain will be called Border terms. We can estimate the above quantity as

$$\begin{aligned} &< N^{-(c+d+1)/2} \sum_{\text{Central}} \int_{B(n, \mathbf{k})} |g_N(n', \mathbf{k}') - g_N(|x|, |\mathbf{y}|)| dx d\mathbf{y} \\ &+ N^{-(c+d+1)/2} \sum_{\text{Border}} |g_N(n', \mathbf{k}')|. \end{aligned}$$

Both sums will be shown to converge to zero, we start with the second one. Since the number of terms on the Border of the integration domain is $\mathcal{O}(N^{(c+d)\delta})$ and $g_N(n', \mathbf{k}') = \mathcal{O}(1)$ the second sum is of order $\mathcal{O}(N^{-(c+d+1)/2 + (c+d)\delta})$. This implies that the sum converges to zero because $\delta < \frac{c+d+5}{2(c+d+4)} < \frac{c+d+1}{2(c+d)}$.

For the first sum we need to estimate the integrands:

$$\begin{aligned} &|g_N(n', \mathbf{k}') - g_N(|x|, |\mathbf{y}|)| = |g_N(n', \mathbf{k}')| |1 - \\ &\exp \frac{-\pi}{N} \left((c+d - (4a+c-d)i) \frac{|x|^2 - n'^2}{2} + 2 \sum_{j=1}^{c+d} (|y_j| + \frac{|x|}{2})^2 - (k'_j + \frac{n'}{2})^2 \right) | \end{aligned}$$

The last expression is of order $\mathcal{O}(N^{\delta-1})$ because $|g_N(n', \mathbf{k}')| = \mathcal{O}(1)$ and $|x|^2 - n'^2 = (|x| + n')(|x| - n')$ and $(|y_j| + \frac{|x|}{2})^2 - (k'_j + \frac{n'}{2})^2 = (|y_j| + \frac{|x|}{2} + k'_j + \frac{n'}{2})(|y_j| + \frac{|x|}{2} - k'_j - \frac{n'}{2})$. We integrate over $B(n, \mathbf{k})$ so $|n' - |x|| < 1$ and $|k'_j - |y_j|| < 1$ and all terms are central so $|x| + n' < N^\delta$ and $k'_j + |y_j| < N^\delta$. Therefore the second sum is of order $\mathcal{O}(N^{-(c+d+1)/2 + \delta - 1 + (c+d+1)\delta})$ and thus converges to zero, because $\delta < \frac{c+d+5}{2(c+d+4)} < \frac{c+d+3}{2(c+d+2)}$.

Step 5. We make the substitution $x = \sqrt{N}w$ and $y_j = \sqrt{N}z_j$ with Jacobian $N^{(c+d+1)/2}$. This gives:

$$Q_\infty = \lim_{M \rightarrow \infty} \psi \int_{w+\max|z_j| < \frac{(N^\delta-1)}{\sqrt{N}}} \exp \left(-\pi \left((c+d - (4a+c-d)i) \frac{|w|^2}{2} + 2 \sum_{j=1}^{c+d} (|z_j| + \frac{|x|}{2})^2 \right) \right) dw dz.$$

Now $\delta > \frac{1}{2}$ and the integrand is rapidly decreasing so the limit exists and is equal to:

$$\psi \int_{\mathbb{R}^{c+d+1}} \exp \left(-\pi \left((c+d - (4a+c-d)i) \frac{|w|^2}{2} + 2 \sum_{j=1}^{c+d} (|z_j| + \frac{|w|}{2})^2 \right) \right) dw dz$$

Step 6. In this final step we need to show that $Q_\infty \neq 0$. We know that $\psi \neq 0$, but what about the complicated Gaussian integral above? We can write it as an iterated integral and get rid of the absolute value signs by integrating 2^{c+d+1} times over the positive hyper-quadrant $w, z_j > 0$:

$$Q_\infty = \psi 2^{c+d+1} \int_0^\infty \exp \left(-\pi (c+d - (4a+c-d)i) \frac{w^2}{2} \right) dw \cdot \prod_{j=1}^{c+d} \int_0^\infty \exp(-2\pi(z_j + \frac{w}{2})^2) dz_j dw.$$

Using the substitutions $y_j = \sqrt{2\pi}(z_j + \frac{w}{2})$ we get the integral below. We only want to check that the integral is nonzero so all constants in front of the integral and the Jacobian are written as C .

$$Q_\infty = C \int_0^\infty \exp(-\pi(c+d - (4a+c-d)i) \frac{w^2}{2}) \cdot \prod_{j=1}^{c+d} \int_{\frac{\sqrt{\pi}w}{\sqrt{2}}}^\infty \exp(-y_j^2) dy_j dw.$$

If we define the complementary error function [11] by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-y^2) dy,$$

then we can write the integral more concisely as:

$$Q_\infty = C \int_0^\infty \exp(-\pi(c+d - (4a+c-d)i) \frac{w^2}{2}) \operatorname{erfc}^{c+d} \left(\frac{\sqrt{\pi}w}{\sqrt{2}} \right) dw = C \int_0^\infty \exp(-\pi(c+d) \frac{w^2}{2}) \operatorname{erfc}^{c+d} \left(\frac{\sqrt{\pi}w}{\sqrt{2}} \right) \cdot \left(\cos\left(\frac{4a+c-d}{2} \pi w^2\right) + i \sin\left(\frac{4a+c-d}{2} \pi w^2\right) \right) dw.$$

After the substitution $x = \sqrt{\frac{4a+c-d}{2}}w$ we note that the imaginary part of integral is nonzero, because both the exponential function and the complementary error

function are bounded and decrease monotonically and the decrease is quite rapid. Therefore the integral over the part where the sine is positive dominates the integral over its complement, showing that the integral is a positive scalar multiple of C . Note that the constant E mentioned in the main theorem equals $\log Q_\infty$. \square

The same proof also works for lemma 5 but in this case it is much easier because the value of n is fixed at the maximum. The Gaussian integral at the end of step 5 is now a standard Gaussian integral so that the constant in front of lemma 5 can be computed easily.

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