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ON COFINITELY δ-SEMIPERFECT MODULES

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ABSTRACT. Supplemented modules and $\oplus$-supplemented modules are useful in characterizing semiperfect modules and rings. Recently, the notion of cofinitely supplemented modules and δ-supplemented modules were introduced as generalizations of supplemented modules. In this paper, $\oplus - cof_\delta$-supplemented and cofinitely δ-semiperfect modules are defined as generalizations of $\oplus$-cofinitely supplemented modules and cofinitely semiperfect modules. Several properties of these modules are obtained.

1. Introduction

Throughout this paper, we assume that R is an associative ring with unity, M is a unital right R-module. The symbols, "$\leqslant$" will denote a submodule, "$\leqslant^{\oplus}$" a module direct summand, "$\leqslant_e$" an essential submodule and "*Rad* " the radical of a module. The texts by Anderson and Fuller [2] and Wisbauer [15] are the general references for notion of rings and modules not defined in this work.

A submodule N of M is called *small* in M, denoted by $N \ll M$, if for every submodule K of M the equality $N + K = M$ implies $K = M$. Let M be a module and N, P be submodules of M. We call P *a supplement* of N in M if $M = P + N$ and $P \cap N$ is small in P. A submodule N of M has an *ample supplement* in M if every submodule L such that $M = N + L$ contains a supplement of N in M. A module M is called *(amply, resp.) supplemented* if every submodule of M has a (an ample, resp.) supplement. Supplemented modules have been discussed by several authors (see [5], [8], [15]).

If P and M are modules, we call an epimorphism $p : P \to M$ a *small cover* in case $\operatorname{Ker}(p) \ll P$. If P is projective, then it is called *projective cover*. An R-module M is called *semiperfect* if every factor module of M has a projective cover. If R_R is semiperfect, then R is called a *semiperfect* ring.

Following Zhou [16], a submodule N of a module M is said to be a *δ-small submodule* (denoted by $N \ll_\delta M$) if, whenever $M = N + X$ with M/X singular, we have $M = X$. In [11], δ-supplemented modules are introduced as generalization of supplemented modules. Let M be a module and N, P be submodules of M. According to [11, Lemma 2.9], P is called *a δ-supplement* of N in M if $M = P + N$ and $P \cap N$ is δ-small in P. A module M is said to be a *δ-supplemented* module

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if every submodule of M has a δ-supplement in M. A submodule N of M has *δ-ample supplement* in M if every submodule L such that $M = N + L$ contains a δ-supplement of N in M. A module M is called (*amply, resp.*) *δ-supplemented* if every submodule of M has a (an ample, resp.) δ-supplement. This type modules is used to characterize δ-semiperfect and δ-perfect rings introduced and discussed in [16]. In [16], a projective module P is called *a projective δ-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $\mathrm{Ker}(f) \ll_\delta M$, and an R-module M is called *δ- semiperfect* if, for every submodule N of M, there exists a decomposition $M = A \oplus B$ such that A is a projective module with $A \leqslant N$ and $B \cap N \ll_\delta M$ (see [11]). A ring is called *δ-perfect* (or *δ-semiperfect,* resp.) if every R-module (or every simple R-module, resp.) has a projective δ-cover. For more discussion on δ-small submodules, δ-perfect and δ-semiperfect rings, we refer to [11] and [16].

A submodule N of M is called *cofinite* (in M) if M/N is a finitely generated module. A module M is called *cofinite δ-supplemented module* if every cofinite submodule of M has a δ-supplement in M.

By [3], an R-module M is called *cofinitely semiperfect* if every finitely generated factor module of M has a projective cover. Çalişici and Pancar gave some properties of semiperfect ring via cofinitely semiperfect modules. In this paper, we will use their techniques to obtain some properties of $\oplus - cof_\delta$-supplemented modules.

2. $\oplus - cof_\delta$-SUPPLEMENTED MODULES.

Definition 2.1. An R-module M is called *$\oplus - cof_\delta$-supplemented* if every cofinite submodule of M has a δ-supplement that is a direct summand of M.

Clearly, every $\oplus$-supplemented module is $\oplus - cof_\delta$-supplemented module. But in general the converse is not true.

Lemma 2.1. *Let N and L be submodules of a module M such that $N + L$ has a δ-supplement H in M and $N \cap (H + L)$ has a δ-supplement G in N. Then $H + G$ is a δ-supplement of L in M.*

Proof. Let H be a δ-supplement of $N + L$ in M and G be a δ-supplement of $N \cap (H + L)$ in N. Then $M = (N + L) + H$ such that $(N + L) \cap H \ll_\delta H$ and $N = [N \cap (H + L)] + G$ such that $(H + L) \cap G \ll_\delta G$. Since $(H + G) \cap L \leqslant H \cap (L + G) + G \cap (L + H)$, $H + K$ is a δ-supplement of L in M. □

Corollary 2.1. *Let M_1, M_2 be submodules of M such that $M = M_1 \oplus M_2$. If M_1, M_2 are $\oplus - cof_\delta$-supplemented modules, then M is also a $\oplus - cof_\delta$-supplemented module.*

Proof. Let $L \leqslant M$ such that M/L is finitely generated. Then $M = M_1 + M_2 + L$ has a δ-supplement 0 in M. We have

$$M_2/[M_2 \cap (M_1 + L)] \cong (M_1 + M_2 + L)/(M_1 + L) \cong M/(M_1 + L),$$

so that $M_2 \cap (M_1 + L)$ is a cofinite submodule of M_2. Since M_2 is $\oplus - cof_\delta$-supplemented, there exists $H \leqslant^{\oplus} M_2$ such that H is a δ-supplement of $M_2 \cap (M_1 + L)$ in M_2. By Lemma 2.1, H is a δ-supplement of $M_1 + L$ in M. Similarly, since M_2 is $\oplus - cof_\delta$-supplemented, there exists $K \leqslant^{\oplus} M_1$ such that K is a δ-supplement of $M_1 \cap (H + L)$ in M_1. Again applying Lemma 2.1, $H + K$ is a δ-supplement of L in M. Since $K \leqslant^{\oplus} M_1$ and $H \leqslant^{\oplus} M_2$, $K + H = K \oplus H$ is a direct summand of M. □

Theorem 2.1. *A direct sum $\bigoplus_{i\in I} N_i$ of $\oplus - cof_\delta$-supplemented modules N_i is a $\oplus - cof_\delta$-supplemented module.*

Proof. Let $N = \bigoplus_{i\in I} N_i$ and $L \leqslant N$ such that N/L is finitely generated. Then there exists a finitely generated submodule H of N such that $N = L + H$. There exists a finite subset I' of I such that $H \leqslant \bigoplus_{j\in I'} N_j$ and so $N = L + \bigoplus_{j\in I'} N_j$. By Corollary 2.1, $\bigoplus_{j\in I'} N_j$ is a $\oplus - cof_\delta$-supplemented module. Let $L' = \bigoplus_{j\in I'} N_j$ and so $N = L + L'$.

Note that

$$N/L = (L + L')/L \cong L'/L \cap L'$$

so that $L \cap L'$ is a cofinite submodule of L'. Since L' is $\oplus - cof_\delta$-supplemented, there exists $H \leqslant^{\oplus} L'$ such that $L' = H + L \cap L'$ and $H \cap L \ll_\delta H$. Now $N = L + L' = L + H$ and $H \cap L \ll_\delta H$. Hence H is a δ-supplement of L in N and $H \leqslant^{\oplus} N$ because $L' \leqslant^{\oplus} N$. □

From this theorem we have the following example:

Example 1. *Let $R = \mathbb{Z}$, $M_i = \mathbb{Z}(p^\infty)$ be the Prüfer p-group for all $i \in \mathbb{N}$. Then M_i are supplemented modules. Let $M = \bigoplus_{i\in\mathbb{N}} M_i$. By Theorem 2.1, M is a $\oplus - cof_\delta$-supplemented module, but M is not $\oplus$-supplemented by* [11, Example 2.14].

Proposition 2.1. *Assume that M is a $\oplus - cof_\delta$-supplemented module. Then every cofinite submodule of the module $M/\delta(M)$ is a direct summand of $M/\delta(M)$.*

Proof. Let $N/\delta(M)$ be any cofinite submodule of $M/\delta(M)$. Since $(M/\delta(M))/(N/\delta(M)) \cong M/N$, we have M/N is finitely generated. Then N is a cofinite submodule of M. Since M is a $\oplus - cof_\delta$-supplemented module, there exist submodules K and K' of M such that $M = N + K = K \oplus K'$, and $N \cap K \ll_\delta K$. Since $N \cap K$ is also δ-small in M, $N \cap K \leqslant \delta(M)$. Thus $M = N + K$ and $M/\delta(M) = (N + K)/\delta(M) = N/\delta(M) \oplus [(K + \delta(M))/\delta(M)]$. Hence $N/\delta(M)$ is a direct summand of $M/\delta(M)$. □

Corollary 2.2. *Assume that M is a $\oplus - cof_\delta$-supplemented module. If $\delta(M)$ is a cofinite submodule of M, then $M/\delta(M)$ is a semisimple module.*

Let M be a module. A submodule X of M is called *fully invariant* if for every $h \in \mathrm{End}_R(M)$, $h(X) \subseteq X$. The module M is called *duo*, if every submodule of M is fully invariant.

It is well known that if $M = M_1 \oplus M_2$ is a duo module, then $A = (A \cap M_1) \oplus (A \cap M_2)$ for any submodule A of M.

Proposition 2.2. *Assume that M is a $\oplus - cof_\delta$-supplemented duo module and $N \leqslant M$. Then M/N is a $\oplus - cof_\delta$-supplemented module.*

Proof. Let $N \leqslant K \leqslant M$ with K/N cofinite submodule of M/N. Then $M/K \cong (M/N)/(K/N)$ is finitely generated. Since M is a $\oplus - cof_\delta$-supplemented module, there exist submodules L and $L^{'}$ of M such that $M = K + L = L \oplus L^{'}$, and $K \cap L$ is δ-small in L. Note that $M/N = K/N + (L + N)/N$, by modularity, $K \cap (L + N) = (K \cap L) + N$. Since $K \cap L \ll_\delta L$, we have $(K/N) \cap (L + N)/N = ((K \cap L) + N)/N \ll_\delta (L + N)/N$ by [16, Lemma 1.3 (2)]. This implies that $(L + N)/N$ is a δ-supplement of K/N in M/N. Now $N = (N \cap L) \oplus (N \cap L^{'})$ implies that

$$(L + N) \cap (L^{'} + N) \leqslant N + (L + N \cap L + N \cap L^{'}) \cap L^{'}.$$

It follows that $(L+N) \cap (L^{'}+N) \leqslant N$ and $M/N = [(L+N)/N] \oplus [(L^{'}+N)/N]$. Then $(L+N)/N$ is a direct summand of M/N. Consequently, M/N is $\oplus - cof_\delta$-supplemented. □

A module M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N+(K\cap L) = (N+K)\cap(N+L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$. A module M is said to have the *summand sum property* (SSP, for short) if the sum of any two direct summands of M is a direct summand of M. A module M has the *summand intersection property* (SIP, for short) if the intersection of two direct summands of M is again a direct summand of M.

Theorem 2.2. *Let M be a $\oplus - cof_\delta$-supplemented module and N a submodule of M.*

1. *If for every direct summand K of M, $(N + K)/N$ is a direct summand of M/N, then M/N is a $\oplus - cof_\delta$-supplemented module.*
2. *If M has the SSP, then every direct summand of M is $\oplus - cof_\delta$-supplemented.*
3. *If M is a distributive module, then M/N is a $\oplus - cof_\delta$ - supplemented module.*

Proof. (1). Any cofinite submodule of M/N has the form T/N where T is a cofinite submodule of M and $N \leqslant T$. Since M is a $\oplus - cof_\delta$-supplemented module, there exists a direct summand D of M such that $M = D \oplus D' = T + D$ and $D \cap T \ll_\delta D$ for some submodule D' of M. Now $M/N = T/N + (D + N)/N$. By hypothesis, $(D + N)/N$ is a direct summand of M/N. Note that $(T/N) \cap [(D + N)/N] = (T \cap (D + N))/N = (N + (D \cap T))/N$. Since $D \cap T \ll_\delta D$, $(N + (D \cap T))/N \ll_\delta (D + N)/N$. This implies that $(D + N)/N$ is a δ-supplement of T/N in M/N, which is a direct summand.

(2). Let N_1 be a direct summand of M. Then $M = N_1 \oplus N'$ for some $N' \leqslant M$. We want to show that M/N' is $\oplus - cof_\delta$-supplemented. In fact, assume that L is a direct summand of M. Since M has the SSP, $L + N^{'}$ is a direct summand of M. Let $M = (L+N')\oplus K$ for some $K \leqslant M$. Then $M/N' = (L+N')/N' \oplus (K+N')/N'$. Therefore M/N' is a $\oplus - cof_\delta$-supplemented module by (1).

(3). Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/N = [(D + N)/N] + [(D' + N)/N]$ and $N = N + (D \cap D') = (N + D) \cap (N + D')$ by distributivity of M. This implies that $M/N = [(D + N)/N] \oplus [(D' + N)/N]$. By (1), M/N is a $\oplus - cof_\delta$-supplemented module. □

Lemma 2.2 ([12], Corollary 18)**.** *Let M be a duo module. Then M has the SIP and the SSP.*

As a result of Theorems 2.2 and Lemma 2.2, we obtain the following result:

Corollary 2.3. *Let M be a $\oplus - cof_\delta$-supplemented duo module. Then every direct summand of M is $\oplus - cof_\delta$-supplemented.*

A module M is called $\delta-$*small* if it can be embedded as a δ-small submodule of some module. It is clear that:

1. Every small module is a δ-small module.
2. Any nonzero nonsingular injective semisimple module is a δ-small module, but not a small module.

Proposition 2.3. *M is a $\delta-$small module if and only if M is $\delta-$small in $E(M)$.*

Proof. Suppose M is a δ-small submodule of a module N. Then M is δ-small in $E(N)$ by [16, Lemma 2.1]. Since $E(M)$ is a direct summand of $E(N)$, M is a δ-small in $E(M)$ by [16, Lemma 1.5]. The converse is clear. □

Let M, N be R-modules. We denote

$$\overline{\delta(M)} = \bigcap\{\mathrm{Ker}(g) : g \in \mathrm{Hom}(M, N), N \ll_\delta E(N)\}.$$

Clearly, in case $\overline{\delta(M)} = M$, the class

$$\bigcap\{\mathrm{Ker}(g) : g \in \mathrm{Hom}(M, N), N \ll_\delta E(N)\}$$

is closed under homomorphic images.

Lemma 2.3.

1. *Let M be a module with $\overline{\delta(M)} = M$. If N is a δ-small module with $N \leqslant M$, then $N \ll_\delta M$.*
2. *Let $B \leqslant A \leqslant M$. If A is a direct summand of M and $A/B \ll_\delta M/B$ then $A = B$.*

Proof. (1). Let $M = N + K$ with M/K singular. Since $N/(N \cap K)$ is a homomorphic image of N, it is a δ-small module. Since $N/(N \cap K)$ is a homomorphic image of M, we have $\overline{\delta(N/(N \cap K))} = N/(N \cap K)$. Hence $N \cap K = N$ and so

$K = M$.
(2). Let $B \leqslant A \leqslant M$ and $M = A \oplus A'$ for some submodule A' of M. Then $M/B = A/B + (A' + B)/B$ and $(M/B)/((A' + B)/B) \cong M/(A' + B)$. Since $A/B \ll_\delta M/B$, we have two cases:
Case (i): Assume that $A' + B \leqslant_e M$. Then $M = A' + B$. By modularity, we have $A = A \cap M = A \cap (A' + B) = B + (A \cap A') = B$.
Case (ii): Assume that $A' + B$ is not essential in M. Then there exits a submodule X of M such that $(A' + B) \oplus X \leqslant_e M$. This implies that $M = (A' + B) \oplus X$, $A = A \cap (B + A^{'} + X) = B \cap (A + A^{'} + X) = B \cap M = B$. □

M is said to satisfy (D3) if M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Theorem 2.3. *Let M be a module.*

1. *Assume that M is a $\oplus - cof_\delta$-supplemented module satisfying $(D3)$. Then every cofinite direct summand of M is $\oplus - cof_\delta$-supplemented.*
2. *Assume that M satisfies $(D3)$. Let K and N be cofinite direct summands of M such that $\overline{\delta(M/(N \cap K))} = M/(N \cap K)$. If M/N is a $\oplus - cof_\delta$-supplemented module then $(N + K)/N$ is a direct summand of M/N.*
3. *Assume that M satisfies $(D3)$ with $\overline{\delta(M)} = M$. If M is a $\oplus - cof_\delta$-supplemented module then M has the SSP on cofinite direct summands.*

Proof. (1). Let N be a cofinite direct summand of M. Then $M = N \oplus N'$ for some submodule N of M. Let K be a cofinite submodule of N. Then K is a cofinite submodule of M. Since M is $\oplus - cof_\delta$-supplemented, there exist submodules L, L' of M such that $M = K + L = L \oplus L'$ and $K \cap L \ll_\delta L$. This implies that $N = K + (N \cap L)$. By (D_3), $N \cap L$ is a direct summand of M and so is a direct summand of N. By [16, Lemma 1.3], we have $K \cap (N \cap L) = K \cap L \ll_\delta N \cap L$.

(2). Since $(K + N)/N$ is a cofinite submodule of M/N and M/N is a $\oplus - cof_\delta$-supplemented module, there exist submodules N_1, N_2 such that $M/N = N_1/N \oplus N_2/N = (K + N)/N + N_2/N$ and $[(K + N)/N] \cap (N_2/N) = (N + (K \cap N_2))/N \ll_\delta N_2/N$. This implies that $N = N_1 \cap N_2$ and $M = N_1 + N_2 = K + N_2$. Note that $(N + (K \cap N_2))/N$ is a δ-small module by definition. We consider the monomorphism $f : ((K \cap N_2) + N_1)/N_1 \to ((K \cap N_2) + N)/N$ defined by $f(x + N_1) = x + N$ for all $x \in K \cap N_2$. Thus $((K \cap N_2) + N_1)/N_1$ is a δ-small module. Then $((K \cap N_2) + N_1)/N_1 \cong (K \cap N_2)/(K \cap N)$ is a δ-small module. Hence $(K \cap N_2)/(K \cap N) \ll_\delta M/(N \cap K)$ by Lemma 2.3(1). Since N_2 is a direct summand of M and M satisfies (D3), $(K \cap N_2)$ is a direct summand of M. We have $K \cap N_2 = K \cap N$. Hence $(N + K)/N$ is a direct summand of M/N.

(3). Let N and K be cofinite direct summands of M. Then

$$\overline{\delta(M/(N \cap K))} = M/(N \cap K).$$

By (1), M/N is a $\oplus - cof_\delta$-supplemented module, then $(N + K)/N$ is a cofinite direct summand of M/N by (2). Clearly $N + K$ is a direct summand of M. □

Clearly, $SIP \Rightarrow (D3)$. On the other hand, by [10, Lemma 2.6], every module satisfying (D3) with the SSP has the SIP.

Lemma 2.4. *Assume that M satisfies $(D3)$. If M has the SSP on cofinite direct summands then M has the SIP on cofinite direct summands.*

Proof. Assume that M satisfies (D3) and M has the SSP on cofinite direct summands of M. Let N and K be cofinite direct summands of M. Then M/N and M/K are finitely generated and so $M/(N+K)$ is also finitely generated. Since M has the SSP on cofinite direct summands of M, then $N+K$ is also a direct summand of M. Let $M = (N+K) \oplus L$ for some submodule L of M. Note that $M/(N+L)$ and $M/(K+L)$ are finitely generated. Hence $N+L$ and $K+L$ are cofinite direct summands of M because M has the SSP. Since $M = (N+L)+(K+L)$ and M satisfies (D3), then $(N+L)\cap(K+L)$ is a direct summand of M. Let $M = [(N+L)\cap(K+L)] \oplus X$ for some submodule X of M. Since $M/(N\cap K)$ is finitely generated and $N\cap(K+L) \leqslant N\cap K$, then $M = (N\cap K)\oplus L \oplus X$. □

Proposition 2.4. (1) *Assume that M satisfies $(D3)$ with $\overline{\delta(M)} = M$. If M is a $\oplus - cof_{\delta}$-supplemented module then M has the SIP on cofinite direct summands.*
(2) *Assume that M is a $\oplus - cof_{\delta}$-supplemented module with $\overline{\delta(M)} = M$. Then M satisfies $(D3)$ if and only if M has the SIP on cofinite direct summands.*

Proof. (1). It follows from Lemma 2.4 and Theorem 2.3.
(2). It is clear from definition of (D3) and (1). □

3. Cofinitely δ-semiperfect modules

Definition 3.1. An R-module M is called *cofinitely δ-semiperfect* if every finitely generated factor module of M has a projective δ-cover.

Clearly, δ-semiperfect modules and cofinitely semiperfect modules are cofinitely δ-semiperfect. It is well-known that the δ-semiperfect module is not semiperfect. Thus a cofinitely δ-semiperfect module is not cofinitely semiperfect in general, see [16, Example 4.1].

Proposition 3.1. *Let M be a module and U a fully invariant submodule of M. If M is a cofinitely δ-semiperfect module, then M/U is a cofinitely δ-semiperfect module. If, moreover, U is a cofinite direct summand of M, then U is also a cofinitely δ-semiperfect module.*

Proof. Suppose that M is cofinitely δ-semiperfect and L/U is a cofinite submodule of M/U. Thus $M/L \cong (M/U)/(L/U)$ is a finitely generated module and hence L is a cofinite submodule of M. Since M is a cofinitely δ-semiperfect module, there exist submodules N and N' of M such that $M = N \oplus N'$, $M = N + L$ and $N \cap L \ll_{\delta} N$. It is easy to see that $(N+U)/U$ is a δ-supplement of L/U in M/U and $U = (U\cap N)\oplus(N\cap N')$. Thus we have $(N+U)\cap(N'+U) = U$ and $((N+U)/U)\oplus((N'+U)/U)) = M/U$ and hence $(N+U)/U$ is a direct summand of M/U. So M/U is a cofinitely δ-semiperfect module.

Now suppose that U is a cofinite direct summand of M. Then there exists a finitely generated submodule U' of M such that $M = U \oplus U'$. Let V be a cofinite submodule of U. Note that $M/V = (U \oplus U')/V \cong U/V \oplus U'$ is finitely generated so that V is a cofinite submodule of M. Since M is a cofinitely δ-semiperfect module, there exist submodules K and K' of M such that $M = K \oplus K'$, $M = V + K$ and $V \cap K \ll_\delta K$. Thus $U = V + (U \cap K)$. But $U = (U \cap K) \oplus (U \cap K')$ and hence $U \cap K$ is a direct summand of U. Moreover, $V \cap (U \cap K) = V \cap K \ll_\delta K$. Then $V \cap (U \cap K) \ll_\delta U \cap K$ by [16, Lemma 1.3]. Therefore $U \cap K$ is a δ-supplement of V in U and it is a direct summand of U. Thus U is a cofinitely δ-semiperfect module. □

Theorem 3.1. *Let M be a projective module. Then M is cofinitely δ-semiperfect if and only if M is $\oplus - cof_\delta$-supplemented.*

Proof. (⇒) Let N be a cofinite submodule of M. Then M/N is finitely generated and so, by assumption, M/N has a projective δ-cover. Then by [16, Lemma 2.4], there are $M_1, M_2 \leqslant M$ such that $M = M_1 \oplus M_2$ with $M_1 \leqslant N$ and $M_2 \cap N \ll_\delta M$. Hence by [16, Lemma 1.3], $M_2 \cap N \ll_\delta M_2$ or M_2 is a δ-supplement of N in M.

(⇐) Let M/N be a finitely generated factor module of M. Then N is cofinite. Since M is $\oplus - cof_\delta$-supplemented, there exist submodules K and K' of M such that $M = N + K$, $N \cap K \ll_\delta K$, and $M = K \oplus K'$. Clearly, K is projective. For the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\sigma : M \to M/N$, $\mathrm{Ker}\sigma i = N \cap K \ll_\delta K$. □

Corollary 3.1. *Let M be a projective module. Then the following conditions are equivalent:*

(1) *M is cofinitely δ-semiperfect.*
(2) *M is $\oplus - cof_\delta$-supplemented.*
(3) *For each cofinite submodule N of M, there is a decomposition $M = K \oplus K'$ such that $K \leqslant N$ and $K' \cap N \ll_\delta K'$.*

Proof. (1) ⇔ (2). By Theorem 3.1.
(2) ⇒ (3). Let N be a cofinite submodule of M. By hypothesis, there exist submodules K and K' of M such that $M = N + K'$, $K' \cap N \ll_\delta K'$ and $M = K \oplus K'$. Since M is projective, there exists a submodule $K" \leqslant N$ such that $M = K" \oplus K'$ by [15, 4.14].
(3) ⇒ (2) is clear. □

Theorem 3.2. *Let M be a projective module with $\delta(M) \ll_\delta M$. Then the following conditions are equivalent:*

1. *M is a cofinitely δ-semiperfect module.*
2. *For every cofinite submodule N of M, M/N has a projective $\delta-$cover.*
3. *Every cofinite submodule N of M can be written as $N = A \oplus S$ with $A \leqslant_e M$ and $S \ll_\delta M$.*
4. *M is a $\oplus - cof_\delta$-supplemented module.*

5. *Every cofinite submodule of the module $M/\delta(M)$ is a direct summand of $M/\delta(M)$ and each cofinite direct summand of $M/\delta(M)$ lifts to a direct summand of M.*

Proof. By Corollary 3.1. □

Proposition 3.2. *Every homomorphic image of a cofinitely δ-semiperfect module is cofinitely δ-semiperfect.*

Proof. Let $f : M \to N$ be a homomorphism and M be a cofinitely δ-semiperfect module. Let $f(M)/U$ be a finitely generated factor module of $f(M)$. Consider the epimorphism $\psi : M \to f(M)/U$, defined by $m \mapsto f(m) + U$. Since M is cofinitely δ-semiperfect, by the natural isomorphism $M/f^{-1}(U) \cong f(M)/U$, we have $f(M)/U$ has a projective δ-cover. Hence $f(M)$ is cofinitely δ-semiperfect. □

Corollary 3.2. *Every factor module of a cofinitely δ-semiperfect module is cofinitely δ-semiperfect.*

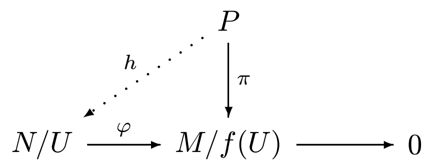
A module N is called a δ-*small cover* of a module M if there exists an epimorphism $f : N \to M$ with $\mathrm{Ker} f \ll_\delta N$.

Proposition 3.3. *Every δ-small cover of a cofinitely δ-semiperfect module is cofinitely δ-semiperfect.*

Proof. Let N be a δ-small cover of a module M and $f : N \to M$ be an epimorphism with $\mathrm{Ker} f \ll_\delta N$. For a finitely generated factor module N/U of N, the homomorphism $\varphi : N/U \to M/f(U)$, defined by $n + U \mapsto f(n) + f(U)$ is epic. We have $\mathrm{Ker}\varphi = (U + \mathrm{Ker} f)/U$. Let $L/U \leqslant N/U$ such that $(U + \mathrm{Ker} f)/U + L/U = N/U$ and $(N/U)/(L/U)$ is singular. Then $L + \mathrm{Ker} f = N$ and $N/L \cong (N/U)/(L/U)$ is singular. This implies $L = N$ since $\mathrm{Ker} f \ll_\delta N$. Hence $\mathrm{Ker}\varphi \ll_\delta N/U$. Note that

$$M/f(U) = \varphi(N/U) \cong (N/U)/((U + \mathrm{Ker} f)/U)$$

so that $M/f(U)$ is finitely generated. Because M is cofinitely δ-semiperfect, $M/f(U)$ has a projective δ-cover $\pi : P \to M/f(U)$. Since P is projective, there is a homomorphism $h : P \to N/U$ such that the diagram



is commutative; i.e., we have $\pi = \varphi h$. Then $N/U = h(P) + \mathrm{Ker}\varphi$.

Since $\mathrm{Ker}\varphi \ll_\delta N/U$, there exists a semi-simple projective submodule Y of $\mathrm{Ker}\varphi$ such that $N/U = h(P) \oplus Y$. Let $\phi : P \oplus Y \longrightarrow N/U$, defined by $\phi(p, y) = h(p) + y$. Then ϕ is an epimorphism and $\mathrm{Ker}\phi = \mathrm{Ker} h \oplus 0$. Because $\mathrm{Ker} h \leqslant$

Ker$\pi \ll_\delta P$, Ker$h \ll_\delta P$. This implies Ker$h \oplus 0 \ll_\delta P \oplus Y$. Thus $P \oplus Y$ is a projective δ-cover of N/U. □

Corollary 3.3. *If $N \ll_\delta M$ and M/N is cofinitely δ-semiperfect, then M is cofinitely δ-semiperfect.*

Corollary 3.4. *Let $\pi : P \longrightarrow M$ be a projective δ-cover of M. Then the following conditions are equivalent:*

(1) *M is cofinitely δ-semiperfect.*
(2) *P is cofinitely δ-semiperfect*
(3) *P is cofinitely δ-supplemented.*

Proof. (1) ⇔ (2) By Proposition 3.3 and Proposition 3.2.
(2) ⇔ (3) By Theorem 3.1. □

Theorem 3.3. *A direct sum $\bigoplus_{i\in I} P_i$ of projective modules P_i is a cofinitely δ-semiperfect module if and only if every summand P_i is cofinitely δ-semiperfect.*

Proof. (⇒). Let $P_i (i \in I)$ be a collection of projective R-modules and $P = \bigoplus_{i\in I} P_i$ be a cofinitely δ-semiperfect module. Since $P_j \cong P/(\bigoplus_{i\in I\setminus\{j\}} P_i)$ for all $j \in I$, by Corollary 3.2, every P_i is cofinitely δ-semiperfect.

(⇐). Since every P_i is projective and cofinitely δ-semiperfect, by Theorem 3.1, every P_i is $\oplus - cof_\delta$-supplemented and so P is $\oplus - cof_\delta$-supplemented by Theorem 2.1. Thus P is cofinitely δ-semiperfect by Theorem 3.1. □

Let M and N be R-modules. N is said to be *(finitely) M-generated* if there is an epimorphism $f : M^{(\Lambda)} \longrightarrow N$ for some (finite) index set Λ.

Lemma 3.1. *Let M be a projective module. If M is δ-semiperfect then every M-generated module is cofinitely δ-semiperfect. The converse holds if M is finitely generated.*

Proof. If M is δ-semiperfect, then M is cofinitely δ-semiperfect by [16, Lemma 2.4]. By Theorems 3.1 and 3.3, for every index set Λ, $M^{(\Lambda)}$ is cofinitely δ-semiperfect. If M is a finitely generated and cofinitely δ-semiperfect module, then it is δ-semiperfect. □

Theorem 3.4. *For a ring R, the following conditions are equivalent:*

(1) *R is δ-semiperfect.*
(2) *Every free R-module is cofinitely δ-semiperfect.*
(3) *Every finitely generated free R-module is δ-semiperfect.*

Proof. (1) ⇒ (2). Assume that R is δ-semiperfect, R is cofinitely δ-semiperfect by Lemma 3.1. Thus every free R-module is cofinitely δ-semiperfect by Theorem 3.3.

(2) ⇒ (3) is clear.

(3) $\Rightarrow$ (1). By hypothesis, R is cofinitely δ-semiperfect. Thus we have (1) by Lemma 3.1. $\square$

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