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THE EXTREME VALUES OF LOCAL DIMENSION IN FRACTAL GEOMETRY

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Abstract. Let μ be the probability measure induced by $S = \sum_{i=0}^{\infty} q^{-i} X_i$, where $m \geqslant q \geqslant 2$ are integers and $X_0, X_1, ...$ a sequence of independent identically distributed random variables each taking integer values $0, 1, \dots, m$ with equal probability $p = 1/(m+1)$. Let $\alpha(s)$ (resp. $\alpha_*(s), \alpha^*(s)$) denote the local dimension (resp. lower, upper local dimension) of $s \in \operatorname{supp} \mu$, and let

$$\alpha^* = \sup\{\alpha^*(s) : s \in \operatorname{supp} \mu\}; \quad \alpha_* = \inf\{\alpha_*(s) : s \in \operatorname{supp} \mu\}.$$

We show that

$$\alpha_* = \frac{\log(m+1) - \log(r + \sqrt{r^2 + 4(l+1)}) + \log 2}{\log q}$$

for $rq \leqslant m < rq + r$; $r = 1, \dots, q-1$ and $l = m - rq$.

The special case of our result, $m = rq$ $(l = 0)$, was obtained earlier in [6].

1. Introduction

An *iterated function system* (IFS for short) is a finite set of contractions $\{F_1, \dots, F_m\}$ on $\mathbb{R}^d$. It is known that for any iterated function system $\{F_1, \dots, F_m\}$ there is a unique nonempty compact set E in $\mathbb{R}^d$, called the *attractor* (or the *invariant set*) of the system which is invariant under the IFS, i.e.,

$$E = \bigcup_{j=1}^{m} F_j(E).$$

Let $\{F_1, \dots, F_m\}$ be a similarity system, i.e.,

$$F_j(x) = \rho_j R_j x + b_j,$$

where $0 < \rho_j < 1, R_j$ is a $d \times d$ orthogonal matrix and b_j are vectors in $\mathbb{R}^d$, for $j = 1, \dots, m$.

Then, the invariant set E is called a *self-similar set* or a *fractal set.* If further, the similarity system $\{F_1, \dots, F_m\}$ is associated with a set of probability weights $\{p_j\}_{j=1}^m$, $0 \leqslant p_j \leqslant 1$ and $\sum_{j=1}^{m} p_j = 1$, then it will generate a unique probability

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measure μ:

$$\mu(A) = \sum_{j=1}^{m} p_j \mu(F_j^{-1}(A)) \text{ for all Borel measurable sets } A \subset E.$$

We call μ a *self-similar measure.*

When $F_1, \ldots, F_m$ are similarities with equal contraction ratio ρ, i.e., $F_j(x) = \rho(x+b_j)$ for $j = 1, \ldots, m$, the self-similar set and the self-similar measure induced by the system IFS can be seen as follows:

Let $X_0, X_1, \ldots$ be a sequence of independent identically distributed random variables each taking real values $b_1, \ldots, b_m$ with probability $p_1, \ldots, p_m$ respectively. For $\rho \in (0, 1)$, we define a random variable

$$S = \sum_{i=0}^{\infty} \rho^i X_i.$$

The probability measure μ_ρ induced by S:

$$\mu_\rho(A) = \mathrm{Prob}\{\omega : S(\omega) \in A\}$$

is called a *fractal measure.* Observe that the range of S, or the support of μ_ρ, is exactly the invariant set E under the system $\{F_1, \ldots, F_m\}$ and $\mu_\rho(A) = \sum_{j=1}^{m} p_j \mu_\rho(F_j^{-1}(A))$ for all Borel measurable sets $A \subset E$. Therefore, by uniqueness we obtain $\mu = \mu_\rho$. It is well known that μ is either singular or absolutely continuous.

If μ is singular, the degree of singularity of μ can be analyzed on a pointwise basis by studying its local dimensions. Recall that for $s \in \mathrm{supp}\ \mu$, the *lower local dimension* $\alpha_*(s)$ of μ at s is defined by

$$\alpha_*(s) = \liminf_{h \to 0^+} \frac{\log \mu(B(s,h))}{\log h},$$

where $B(s, h)$ is the closed interval $[s - h, s + h]$. Similarly, the *upper local dimension* $\alpha^*(s)$ is defined by using the upper limit. If $\alpha^*(s)$ and $\alpha_*(s)$ coincide, the common value is called the *local dimension* of μ at s and is denoted by $\alpha(s)$.

We are interested in identifying the extreme values of local dimension of fractal measures. Observe that the least upper bound of local dimensions is not difficult to identify, however the problem of calculating the greatest lower bound is harder. Our aim in this paper is to establish the greatest lower bound α_* in the case $rq \leqslant m < rq + r,\ r = 1, \ldots, q - 1$. We prove that under the notation

$$\alpha^* = \sup\{\alpha^*(s) : s \in \mathrm{supp}\ \mu\} \text{ and } \overline{\alpha} = \sup\{\alpha(s) : s \in \mathrm{supp}\ \mu\};$$
$$\alpha_* = \inf\{\alpha_*(s) : s \in \mathrm{supp}\ \mu\} \text{ and } \underline{\alpha} = \inf\{\alpha(s) : s \in \mathrm{supp}\ \mu\},$$

we have

Main Theorem. *Let $m \geqslant q \geqslant 2$ be integers. Then*

$$\underline{\alpha} = \alpha_* = \frac{\log(m+1) - \log(r + \sqrt{r^2 + 4(l+1)}) + \log 2}{\log q}$$

for $rq \leqslant m < rq + r$, $r = 1, \dots, q-1$, and $l = m - rq$.

For $m = rq$ $(l = 0)$, the result was established in [6].

The proof of our Main Theorem is divided into two steps. In Section 2 we introduce some notations and recall some primary results which will be used throughout the paper. The Main Theorem will be proved in Section 3.

2. Notations and primary results

Let $\mathbb{N}$ denote the set of all nonnegative integers. For $m \in \mathbb{N}$ we denote

$$\mathbb{D}_m = \{0, 1, \dots, m\} \text{ and } \mathbb{D}_m^n = \{0, 1, \dots, m\}^n, \text{ where } n \leqslant \infty.$$

For $q \geqslant 2$, denote

$$S = \sum_{k=0}^{\infty} q^{-k} X_k \quad \text{and} \quad S_n = \sum_{k=0}^{n} q^{-k} X_k.$$

Then for $x = (x_0, x_1, \dots) \in \mathbb{D}_m^\infty$, we have

$$S(x) = \sum_{k=0}^{\infty} q^{-k} x_k \text{ and } S_n(x) = \sum_{k=0}^{n} q^{-k} x_k.$$

Let μ_n and μ denote the probability measures induced by S_n and S respectively. For $x = (x_0, x_1, ...) \in \mathbb{D}_m^\infty$, let $x(n) = (x_0, x_1, ..., x_n) \in \mathbb{D}_m^{n+1}$ and

$$F_n(x) = S_n^{-1}(s_n) \subset \mathbb{D}_m^{n+1}, \text{ where } s_n = S_n(x(n)). \tag{2.1}$$

For $s_{n+1} \in \operatorname{supp} \mu_{n+1}$ we denote

$$\sigma(s_{n+1}) = \max\{t_n \in \operatorname{supp} \mu_n : t_n \leqslant s_{n+1}\}. \tag{2.2}$$

The following fact, established in [6], will be the first point of our calculation.

Proposition 2.1 ([6]). *Let $s_n(0) < s_n(1) < \dots < s_n(k_n)$ denote the set of all distinct values of supp μ_n. Then we have:*

(1) *The distance between any two consecutive points in supp μ_n is q^{-n}.*

(2) *supp $\mu_n \subset$ supp μ_{n+1} for every $n \in \mathbb{N}$ and supp $\mu = \overline{\cup_{n=0}^{\infty} \text{supp } \mu_n}$.*

(3) *The set supp μ_n consists of $k_n + 1 = \dfrac{m(q^{n+1}-1)}{q-1} + 1$ points ranging from 0 to $\dfrac{m(q^{n+1}-1)}{q^n(q-1)}$.*

In notation of Proposition 2.1, $s_n(k_n)$ is the largest value in supp μ_n. Note that if $s_{n+1} \leqslant s_n(k_n) + (q-1)q^{-(n+1)}$, then by Proposition 2.1 (1) we have

$$s_{n+1} = \sigma(s_{n+1}) + kq^{-(n+1)} \text{ for some } k \in \{0, 1, \dots, q-1\} \tag{2.3}$$

and if $s_{n+1} \geqslant s_n(k_n) + q^{-n}$, then

$$s_{n+1} = \sigma(s_{n+1}) + kq^{-(n+1)} = s_n(k_n) + kq^{-(n+1)} \quad \text{for some} \quad k \in \{q, \dots, m\}. \tag{2.4}$$

In the case (2.4), we claim that

$$\# S_{n+1}^{-1}(s_{n+1}) \leqslant q - 1. \tag{2.5}$$

To prove (2.5) we will show that $y(n+1) = (y_0, \dots, y_{n+1}) \in S_{n+1}^{-1}(s_{n+1})$ if and only if

$$y(n+1) = (y_0, \dots, y_{n+1}) = (m, \dots, m, m-t, k+tq) \tag{2.6}$$

for some integer $t = 0, \dots, [(m-k)/q]$, where $[a]$ denotes the largest integer not exceeding a.

In fact, if $y(n+1) = (m, \dots, m, m-t, k+tq)$, then $\sum\limits_{i=0}^{n+1} q^{-i} y_i = s_{n+1}$ and since $k + tq \leqslant m$, we get $0 \leqslant t \leqslant [(m-k)/q]$. Therefore, $y(n+1) \in S_{n+1}^{-1}(s_{n+1})$.

Conversely, if $y(n+1) = (y_0, \dots, y_{n+1}) \in S_{n+1}^{-1}(s_{n+1})$, and $y_j = m - i$, $1 \leqslant i \leqslant m$ for some $j \in \{0, 1, \dots, n-1\}$, then

$$\begin{aligned} s_{n+1} = \sum_{i=0}^{n+1} q^{-i} y_i &\leqslant s_n(k_n) + y_{n+1} q^{-(n+1)} - iq^{-j} \\ &= s_n(k_n) + (y_{n+1} - iq^{n+1-j}) q^{-(n+1)}. \end{aligned}$$

Therefore, $y_(n+1) > iq^{n+1-j}$. Since $j \leqslant n-1$, we have $y_{n+1} \geqslant iq^2 \geqslant q^2$. Hence, $y_{n+1} \notin D_m$, a contradiction.

From (2.6) it follows that

$$y(n+1) = (y_0, \dots, y_{n+1}) \in S_{n+1}^{-1}(s_{n+1}) \;\Rightarrow\; y_0 = \dots = y_{n-1} = m.$$

Consequently,

$$S_{n+1}^{-1}(s_{n+1}) = \{(m, \dots, m, m-t, k+tq) : \; t \in \{0, \dots, [\frac{m-k}{q}]\}\}.$$

Since $[(m-k)/q] \leqslant q - 2$ for $k \in \{q, \dots, m\}$, we have

$$\# S_{n+1}^{-1}(s_{n+1}) \leqslant q - 1.$$

Therefore, (2.5) is proved.

By claim (2.5), we need only consider the case

$$s_{n+1} = \sigma(s_{n+1}) + kq^{-(n+1)} \quad \text{for some} \quad k \in \{0, \dots, q-1\}.$$

We need the following lemma.

Lemma 2.1 ([6]). *Let σ be defined by (2.2) and $k \in \{0, ..., q-1\}$ be defined by (2.3). Then we have*

$$\#S_{n+1}^{-1}(s_{n+1}) = \sum_{j=0}^{[(m-k)/q]} \#S_n^{-1}(\sigma(s_{n+1}) - jq^{-n}),$$

where $\#S_n^{-1}(\sigma(s_{n+1}) - jq^{-n}) = 0$ if $\sigma(s_{n+1}) - jq^{-n} < 0$.

The following proposition establishes a relation between $\alpha(s)$ and $\#F_n(x)$ which will be useful for calculating the local dimension $\alpha(s)$.

Proposition 2.2 ([6]). *For $x = (x_0, x_1, \dots) \in \mathbb{D}_m^\infty$, let $s = \sum_{i=0}^{\infty} q^{-k}x_k$. Then*

$$\alpha(s) = \frac{\log(m+1)}{\log q} - \lim_{n\to\infty} \frac{\log \#F_n(x)}{n \log q},$$

provided that the limit exists. Otherwise

$$\alpha^*(s) = \frac{\log(m+1)}{\log q} - \lim_{n\to\infty} \inf \frac{\log \#F_n(x)}{n \log q},$$

$$\alpha_*(s) = \frac{\log(m+1)}{\log q} - \lim_{n\to\infty} \sup \frac{\log \#F_n(x)}{n \log q}.$$

By Proposition 2.2, to calculate the local dimension $\alpha(s)$ we need to identify the number $\#F_n(x)$. Following [6] we say that $x = (x_0, x_1, \dots) \in \mathbb{D}_m^\infty$ is a *maximal sequence* (respectively, *minimal sequence*) if

$$\#F_n(y) \leqslant \#F_n(x) \quad (\text{respectively}, \#F_n(y) \geqslant \#F_n(x))$$

for every $y = (y_0, y_1, \dots) \in \mathbb{D}_m^\infty$, and for every $n \in \mathbb{N}$.

From Proposition 2.2, we get

Corollary 2.1. *If $x \in \mathbb{D}_m^\infty$ is a maximal sequence (respectively, a minimal sequence), then $\alpha_* = \alpha_*(s)$ (respectively, $\alpha^* = \alpha^*(s)$), where $s = S(x)$.*

Observe that both $x = (0, 0, \dots)$ and $x = (m, m, \dots)$ are minimal sequences since $\#F_n(x) = 1$ for every $n \in \mathbb{N}$. So the least upper bound of local dimension is easy to identify. The problem of calculating the greatest lower bound will be established in the next section.

3. The greatest lower bounds of local dimensions

Theorem 3.1. *Let $m \geqslant q \geqslant 2$ be integers. Then*

$$\underline{\alpha} = \alpha_* = \frac{\log(m+1) - \log(r + \sqrt{r^2 + 4(l+1)}) + \log 2}{\log q}$$

for $rq \leqslant m < rq + r$, $r = 1, \dots, q-1$, and $l = m - rq$.

As in [6], the key point for our proof of Theorem 3.1 is to establish a Fibonacci recurrence formula for $\#F_n(x)$:

Lemma 3.1. *For $rq \leqslant m < rq + r$, $r = 1, \dots, q-1$, $l = m - rq$ and $x = (0, m, 0, m, \dots)$, $\#F_{n+1}(x)$ is given by the Fibonacci recurrence formula*

$$\#F_{n+1}(x) = r\#F_n(x) + (l+1)\#F_{n-1}(x) \text{ for every } n \in \mathbb{N},$$

where $\#F_0(x) = 1$, $\#F_1(x) = r + 1$.

Proof. First, observe that $\#F_0(x) = 1$ and $\#F_1(x) = r+1$. To prove the Fibonacci recurrence formula, without loss of generality we may assume that n is odd. Then

$$x(n+1) = (x_0, x_1, \dots, x_{n+1}) = (0, m, \dots, 0, m, 0).$$

So we have $s_{n+1} = s_n + 0q^{-(n+1)} = \sigma(s_{n+1})$, where $\sigma(s_{n+1})$ is defined by (2.2). Since $rq \leqslant m < rq + r$, by Lemma 2.1, we have

$$\#F_{n+1}(x) = \#S_{n+1}^{-1}(s_{n+1}) = \sum_{j=0}^{r} \#S_n^{-1}(s_n - jq^{-n}).$$

Since

$$s_n - jq^{-n} = s_{n-1} + (m-j)q^{-n} = s_{n-1} + (rq + l - j)q^{-n},$$

we have

$$\sigma(s_n - jq^{-n}) = \begin{cases} s_{n-1} + rq^{-(n-1)} & \text{if } 0 \leqslant j \leqslant l \\ \\ s_{n-1} + (r-1)q^{-(n-1)} & \text{if } j > l. \end{cases}$$

Hence,

$$s_n - jq^{-n} = \begin{cases} \sigma(s_n - jq^{-n}) + (l-j)q^{-n} & \text{if } 0 \leqslant j \leqslant l \\ \\ \sigma(s_n - jq^{-n}) + (q - j + l)q^{-n} & \text{if } j > l. \end{cases}$$

Observe that for $j = 0, 1, \dots r$,

$$[\frac{m - (l-j)}{q}] = [\frac{rq + j}{q}] = r \text{ and } [\frac{m - (q + l - j)}{q}] = [\frac{rq - q + j}{q}] = r - 1.$$

Therefore,

$$\#S_n^{-1}(s_n - jq^{-n}) = \begin{cases} \sum_{i=0}^{r} \#S_{n-1}^{-1}(s_{n-1} + (r-i)q^{-(n-1)}) & \text{if } 0 \leqslant j \leqslant l \\ \\ \sum_{i=0}^{r-1} \#S_{n-1}^{-1}(s_{n-1} + (r-1-i)q^{-(n-1)}) & \text{if } j > l \end{cases}$$

$$= \begin{cases} \sum_{i=0}^{r} \#S_{n-1}^{-1}(s_{n-1} + iq^{-(n-1)}) & \text{if } 0 \leqslant j \leqslant l \\ \\ \sum_{i=0}^{r-1} \#S_{n-1}^{-1}(s_{n-1} + iq^{-(n-1)}) & \text{if } j > l. \end{cases}$$

Thus,

$$\#S_n^{-1}(s_n - jq^{-n}) = \begin{cases} \#F_n(x) & \text{if } 0 \leqslant j \leqslant l \\ \\ \#F_n(x) - \#S_{n-1}^{-1}(s_{n-1} + rq^{-(n-1)}) & \text{if } j > l. \end{cases}$$

Consequently,

$$\begin{aligned} \#F_{n+1}(x) &= (l+1)\#F_n(x) + (r-l)[\#F_n(x) - \#S_{n-1}^{-1}(s_{n-1} + rq^{-(n-1)})] \\ &= (r+1)\#F_n(x) - (r-l)\#S_{n-1}^{-1}(s_{n-1} + rq^{-(n-1)}) \end{aligned} \tag{3.1}$$

Since

$$s_{n-1} + rq^{-(n-1)} = s_{n-2} + rq^{-(n-1)}, \ \sigma(s_{n-1} + rq^{-(n-1)}) = s_{n-2};$$
$$m = rq + l < rq + r, \ [\frac{m-r}{q}] = r - 1,$$

we get

$$\begin{aligned} \#S_{n-1}^{-1}(s_{n-1} + rq^{-(n-1)}) &= \sum_{i=0}^{[\frac{m-r}{q}]} \#S_{n-2}^{-1}(s_{n-2} - iq^{-(n-2)}) \\ &= \sum_{i=0}^{r-1} \#S_{n-2}^{-1}(s_{n-2} - iq^{-(n-2)}). \end{aligned}$$

Therefore, from the identies $s_{n-1} = s_{n-2} = \sigma(s_{n-1})$, we obtain

$$\#F_{n-1}(x) = \sum_{i=0}^{r} \#S_{n-2}^{-1}(s_{n-2} - iq^{-(n-2)}).$$

It follows that

$$\#S_{n-1}^{-1}(s_{n-1} + rq^{-(n-1)}) = \#F_{n-1}(x) - \#S_{n-2}^{-1}(s_{n-2} - rq^{-(n-2)}).$$

Therefore,

$$\#F_{n+1}(x) = (r+1)\#F_n(x) - (r-l)\#F_{n-1}(x) + (r-l)\#S_{n-2}^{-1}(s_{n-2} - rq^{-(n-2)}).$$

As in (3.1), we have

$$\#F_n(x) = (r+1)\#F_{n-1}(x) - (r-l)\#S_{n-2}^{-1}(s_{n-2} - rq^{-(n-2)}).$$

Consequently,

$$\#F_{n+1}(x) = r\#F_n(x) + (l+1)\#F_{n-1}(x).$$

The lemma is proved. $\square$

Now by solving the difference equations of Fibonacci recurrence formula

$$\#F_{n+1}(x) = r\#F_n(x) + (l+1)\#F_{n-1}(x) \ \text{ with } \ \#F_0(x) = 1, \ \#F_1(x) = r + 1,$$

we obtain

Corollary 3.1. *For* $x = (0, m, 0, m, \dots)$, $rq \leqslant m < rq + r, 1 \leqslant r \leqslant q-1$, *then for every* $n \in \mathbb{N}$, *we have*

$$\#F_n(x) = a^+\Big(\frac{r+\sqrt{r^2+4(l+1)}}{2}\Big)^n - a^-\Big(\frac{r-\sqrt{r^2+4(l+1)}}{2}\Big)^n,$$

where $a^\pm = \dfrac{r+2 \pm \sqrt{r^2+4(l+1)}}{2\sqrt{r^2+4(l+1)}}$ *and* $l = m - rq$.

The following lemma is the final step in the proof of our mail result.

Lemma 3.2. *Under the assumption of Lemma 3.1,* $x = (0, m, 0, m, \dots)$ *is a maximal sequence.*

Proof. Observe that if $t_{n+1} = \sigma(t_{n+1}) + kq^{n+1}$, $k = 0, 1, \dots, q-1$, where σ is defined by (2.2), then we have

$$\#S^{-1}_{n+1}(t_{n+1}) = \sum_{j=0}^{r} \#S_n^{-1}(\sigma(t_{n+1}) - jq^{-n}) \ \text{ if } \ 0 \leqslant k \leqslant l, \tag{3.2}$$

$$\#S^{-1}_{n+1}(t_{n+1}) = \sum_{j=0}^{r-1} \#S_n^{-1}(\sigma(t_{n+1}) - jq^{-n}) \text{ if } k > l. \tag{3.3}$$

Since

$$[(m-k)/q] = \begin{cases} r & \text{if } 0 \leqslant k \leqslant l \\ r-1 & \text{if } k > l. \end{cases}$$

the value of $\#S^{-1}_{n+1}(t_{n+1})$ in (3.2) is greater than the one in (3.3). This implies that if $(y_0, y_1, \dots)$ is a maximal sequence, then $y_j \equiv k \pmod q$ where $0 \leqslant k \leqslant l$, for all $j = 0, 1, \dots$. Let

$$A_n = \max\{\ \#S_n^{-1}(\sum_{j=0}^{n} q^{-j}y_j) : y_n \equiv k \pmod q,\ 0 \leqslant k \leqslant l\};$$

$$a_n = \max\{\ \#S_n^{-1}(\sum_{j=0}^{n} q^{-j}y_j) : y_n \equiv k \pmod q,\ l < k \leqslant q-1\}.$$

Then by (3.2) and (3.3) we have $a_n < A_n$. Consequently, the maximality of $x = (0, m, 0, m, \dots)$ will be established if $\#F_n(x) = \#S_n^{-1}(s_n) = A_n$ for every $n \in \mathbb{N}$.

We prove the following more general result

$$\#F_n(x) = \#S_n^{-1}(s_n) = \#S_n^{-1}(s_n + (-1)^n kq^{-n}) = A_n \text{ for } \ k = 0, \dots, l;$$

$$\#S_n^{-1}(s_n + (-1)^n kq^{-n}) = a_n \text{ for } \ k = l+1, \dots, r \ \text{ and } \ n \in \mathbb{N}^*.$$

We prove the result by induction. First, if $x(1) = (0, m)$, $\#F_1(x) = \#S_1^{-1}(s_1) = r + 1 = A_1$ and $s_1 - kq^{-1} = s_0 + (rq + l - k)q^{-1}$, then

$$\#S_1^{-1}(s_1 - kq^{-1}) = \begin{cases} r = a_1 & \text{if } k > l \\ \\ r + 1 = A_1 & \text{if } 0 \leqslant k \leqslant l. \end{cases}$$

Suppose that the statement has been verified up to n. Without loss of generality, assume that n is even, i.e.,

$$\#F_n(x) = \#S_n^{-1}(s_n) = \#S_n^{-1}(s_n + kq^{-n}) = A_n \text{ if } \ k = 0, \dots, l$$

and

$$\#S_n^{-1}(s_n + kq^{-n}) = a_n \text{ if } \ l < k \leqslant r.$$

Since

$$s_{n+1} - kq^{-(n+1)} = s_n + (m - k)q^{-(n+1)} = s_n + (rq + l - k)q^{-(n+1)},$$

using the argument in the proof of the Lemma 3.1, we get

$$\#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}) = \begin{cases} \sum_{i=0}^{r} \#S_n^{-1}(s_n + iq^{-n}) & \text{if } 0 \leqslant k \leqslant l \\ \\ \sum_{i=0}^{r-1} \#S_n^{-1}(s_n + iq^{-n}) & \text{if } k > l. \end{cases}$$

Therefore,

$$\#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}) = \begin{cases} (l + 1)A_n + (r - l)a_n & \text{if } 0 \leqslant k \leqslant l \\ \\ (l + 1)A_n + (r - l - 1)a_n & \text{if } k > l. \end{cases}$$

Observe that for $r + 1$ consecutive points in supp μ_{n+1} there exist at most $l + 1$ points for which the final digit in its series representation equals $k \pmod q$, for $0 \leqslant k \leqslant l$. Since $r + 1 \leqslant q$, each of the sums (3.2) and (3.3) contains $l + 1$ terms equal to A_n and the remaining terms are less than or equals a_n. Consequently, if $t_{n+1} = \sum_{j=0}^{n+1} q^{-j} y_j$ with $\ y_{n+1} \equiv k \pmod q$, then by (3.3)

$$\begin{aligned} \#S_{n+1}^{-1}(t_{n+1}) &= \sum_{j=0}^{r-1} \#S_n^{-1}(\sigma(s_n) - jq^{-n}) \\ &\leqslant (l + 1)A_n + (r - 1 - l)a_n \\ &= \#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}) \ \text{ if } k > l, \end{aligned}$$

and if $0 \leqslant k \leqslant l$, then by (3.2)

$$\begin{aligned} \#S_{n+1}^{-1}(t_{n+1}) &= \sum_{j=0}^{r} \#S_n^{-1}(\sigma(s_n) - jq^{-n}) \\ &\leqslant (l + 1)A_n + (r - l)a_n \\ &= \#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}). \end{aligned}$$

It follows that

$$\#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}) = A_{n+1} \text{ for } 0 \leqslant k \leqslant l;$$
$$\#S_{n+1}^{-1}(s_{n+1} - kq^{-(n+1)}) = a_{n+1} \text{ for } k > l.$$

The lemma is proved. □

Finally, we observe that Theorem 3.1 follows from Proposition 2.2, Corollary 3.1 and Lemma 3.2.

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