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A CLASS OF DELTA-PLURISUBHARMONIC FUNCTIONS AND THE COMPLEX MONGE-AMPÈRE OPERATOR

NGUYEN QUANG DIEU AND DAU HOANG HUNG

Abstract. The aim of this note is to extend some fundamental properties of the complex Monge-Ampère operator for plurisubharmonic functions to the class of delta-plurisubharmonic functions. The main result is an analogue of the celebrated Bedford-Taylor comparision for plurisubharmonic functions.

1. Introduction

Let Ω be an open set in $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if u restricted to $l \cap \Omega$ is subharmonic for every complex line l. Here we allow the constant function $-\infty$ to be plurisubharmonic. Denote by $\mathcal{PSH}(\Omega)$ the cone of plurisubharmonic functions on Ω. In the study of plurisubharmonic functions, the complex Monge-Ampère operator plays a prominent role. More precisely, let $d = \partial + \overline{\partial}, d^c = i(\overline{\partial} - \partial)$, then following Bedford and Taylor in [1] and [2], the complex Monge-Ampère operator $(dd^c)^n$ can be defined on $L^\infty_{\text{loc}}(\Omega) \cap \mathcal{PSH}(\Omega)$, the cone of locally bounded plurisubharmonic functions on Ω inductively as follows

$$dd^c u \wedge T := dd^c(uT),$$

where $u \in L_{\text{loc}} \cap \mathcal{PSH}(\Omega)$ and T is a positive closed current of bidegree $(k, k), 1 \leqslant k \leqslant n-1$ on Ω. Defined in this way, $dd^c u \wedge T$ is a closed positive current of bidegree $(k+1, k+1)$. In particular, $(dd^c u)^n$ is a positive regular Borel measure on Ω. It is well known that in the class $u \in L^\infty_{\text{loc}}(\Omega) \cap \mathcal{PSH}(\Omega)$, the complex Monge-Ampère operator is well behaved under monotone convergence. From this, we can prove quasicontinuity of plurisubharmonic functions and comparison principles. The latter result is an essential tool in dealing with the generalized Dirichlet problem. See [1] and [3] for more details on this matter. Of course, these fundamental results rely heavily on the positivity of $dd^c u$ for plurisubharmonic u.

The aim of this paper is to study analogous problems for $\delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. Namely, we say that $u \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$ if locally u is the difference of two *locally bounded plurisubharmonic functions*. In particular this class contains locally bounded *plurisubharmonic* functions and $\mathcal{C}^2$ smooth functions. This class is also *strictly*

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smaller than the class $\delta\mathcal{PSH}(\Omega)$ of $\delta-$plurisubharmonic functions invented by Kiselman in [6] and later on developed in [4], [5] and [10]. Recall that $u \in \delta\mathcal{PSH}(\Omega)$ if (globally) u is the difference of two plurisubharmonic functions. An interesting result in [6] asserts that if $\delta\mathcal{PSH}(\Omega)$ contains all real valued, real analytic functions on Ω then Ω must be pseudoconvex. However, up to now, the reverse implication is still open. In recent works [5] and [10], the authors introduce Monge-Ampère norms on linear subspaces of $\delta\mathcal{PSH}(\Omega)$. Then they study these subspaces from the point of view of functional analysis.

The main result of this paper is a kind of comparison principle for functions in $\delta^*\mathcal{PSH}(\Omega)$. Roughly speaking, we replace positivity of $dd^c u$ for plurisubharmonic u by positivity of *higher* powers of $dd^c u$ for $u \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. In the course of the proof, we also establish some convergence results for the complex Monge-Ampère operator in this new class of functions. We hope that our work will be helpful in studying generalized Dirichlet problem in the class of functions which are no longer plurisubharmonic. Needless to say, in our approach, we still have to follow the original ideas on complex Monge-Ampère operator for plurisubharmonic functions given in the fundamental works [1] and [2] (see also [8], [7] and [3]).

2. Results

First we fix the notation and terminology. Throughout this paper, by Ω we always mean a bounded domain of $\mathbb{C}^n$. If $\delta > 0$ then $\Omega^\delta := \{z \in \Omega, \text{dist}\,(z, \partial\Omega) > \delta\}$. Given $u : \Omega \to \mathbb{R}$ and $h \in \mathbb{C}^n$ then we denote by u_h the translate function $u_h(z) := u(z-h)$ on Ω. The standard regularization kernels are denoted by ρ_δ. Recall that $\rho_\delta(z) = 1/\delta^{2n}\rho(z/\delta)$, where $0 \leqslant \rho \leqslant 1$ is a radial function on $\mathbb{C}^n$ with integral 1 and support in the unit ball and

$$u * \rho_\delta(z) := \int_{|t|<\delta} u_t(z)\rho_\delta(t)d\lambda_n(t),$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. If $u \in \mathcal{PSH}(\Omega), u \not\equiv -\infty$ then we denote $u^\delta := u * \rho_\delta$. It is well known that $u^\delta \in \mathcal{C}^\infty(\Omega^\delta) \cap \mathcal{PSH}(\Omega^\delta)$ and that $u^\delta \downarrow u$ as $\delta \to 0$. By β we denote the Kähler form $dd^c|z|^2$ on $\mathbb{C}^n$. Next, recall that currents of bidegree (k,k) on Ω are continuous linear functionals on the space $\mathcal{D}^{(k,k)}(\Omega)$ of differential test forms of bidegree $(n-k, n-k)$. A sequence $\{T_j\}$ of currents of bidegree (k,k) is said to converge *weakly* to T if $(T_j, \varphi) \to (T, \varphi)$ for all $\varphi \in \mathcal{D}^{(k,k)}(\Omega)$. Next T is *closed* if $dT = 0$ i.e., $(T, d\varphi) = 0$, $\forall \varphi \in \mathcal{D}^{(k,k)}(\Omega)$ and T is positive i.e., $T \geqslant 0$ if $(T, i\alpha_1 \wedge \overline{\alpha_1} \cdots i\alpha_{n-k} \wedge \overline{\alpha_{n-k}}) \geqslant 0$ for all test forms α_i of bidegree $(1,0)$. Finally a (signed) Borel measure μ in Ω is called *inner regular* if $||\mu||_K < \infty$ for every compact $K \subset \Omega$ and if for every open set $\Omega' \subset\subset \Omega$ and every sequence of compact sets $\{K_j\} \uparrow \Omega'$ we have $\mu(\Omega) = \lim_{j\to\infty} \mu(K_j)$.

It is easy to check that $\delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$ is a (real) vector space. Less evident properties are collected below.

Proposition 2.1. *Let $u, v \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. Then the following assertions hold.*

(a) $u^\delta \in \mathcal{C}^\infty(\Omega^\delta), u^\delta \to u$ *pointwise on* Ω *as* $\delta \to 0$.

(b) *If* $u \leqslant v$ *almost everywhere on* Ω *then* $u \leqslant v$ *everywhere on* Ω.

(c) *For every* $z_0 \in \Omega$, *there exists an open neighbourhood* U *of* z_0 *and* $\omega \in \mathcal{PSH}(U)$ *such that* $u+\omega$ *and* $v+\omega$ *belong to* $L^\infty(U) \cap \mathcal{PSH}(U)$.

(d) $\max(u,v) \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$.

(e) u *is quasicontinuous on* Ω *i.e., for every* $\varepsilon > 0$ *there exists an open set* $U \subset \Omega$ *such that* u *is continuous on* $\Omega \backslash U$ *and* $c(U,\Omega) < \varepsilon$, *where* $c(E,\Omega)$ *is the capacity of a Borel set* $E \subset \Omega$ *relative to* Ω *and is defined as*

$$c(E,\Omega) = \sup\left\{\int_E (dd^c u)^n : u \in \mathcal{PSH}(\Omega), 0 < u < 1\right\}.$$

Proof. (a) follows from the corresponding fact for plurisubharmonic functions and the definition of locally $\delta-$plurisubharmonic function. Notice that the convergence $u_\delta \to u$ is, in general, not monotone. Now (b) follows from (a) since in this case $u^\delta = v^\delta$ *everywhere* on Ω^δ. For (c), we choose an open neighbourhood U of z_0 and bounded plurisubharmonic functions u_1, u_2, v_1, v_2 such that $u = u_1 - u_2, v = v_1 - v_2$, on U. It is easy to check that $\omega := u_2 + v_2$ satisfies the requirement. Next, for (d), we choose ω and U as in (c). Clearly $\max(u,v) + \omega = \max(u+\omega, v+\omega) \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. Finally, (e) follows from the quasicontinuity of plurisubharmonic functions and subadditivity of capacity. ☐

The following definition of the complex Monge-Ampère is fundamental to our work.

Proposition-Definition 2.2. *Let* $1 \leqslant m \leqslant n$ *be an integer, let* $u \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$ *and let* $\{U_i\}_{i\geqslant 1}$ *be an open covering of* Ω *such that* $u = v_{i,1} - v_{i,2}$ *on* U_i *for all* $i \geqslant 1$, *where* $v_{i,1}, v_{i,2} \in L^\infty(U_i) \cap \mathcal{PSH}(U_i)$. *On each open set* U_i *we set*

$$(dd^c u)^m = \sum_{k=0}^m (-1)^{m-k} C_m^k (dd^c v_{i,1})^k \wedge (dd^c v_{i,2})^{m-k}.$$

Then $(dd^c u)^m$ *is a closed current of bidegree* (m,m) *on* Ω. *In particular the complex Monge-Ampère measure* $(dd^c u)^n$ *is a signed inner regular Borel measure. Furthermore, this definition of* $(dd^c u)^m$ *does not depend on the particular choice of the open covering* $\{U_i\}$.

Proof. We first must show that the *local* currents $(dd^c u)^n$ glue nicely on Ω. For this, it is enough to show

$$\begin{aligned}&\sum_{k=0}^m (-1)^{m-k} C_m^k (dd^c v_{i,1})^k \wedge (dd^c v_{i,2})^{m-k}\\ =&\sum_{k=0}^m (-1)^{m-k} C_m^k (dd^c v_{j,1})^k \wedge (dd^c v_{j,2})^{m-k}, \text{ on } U_i \cap U_j.\end{aligned}$$

Since $v_{i,1} - v_{i,2} = v_{j,1} - v_{j,2}$ on $U_i \cap U_j$ we infer

$$v^\delta_{i,1} - v^\delta_{i_2} = v^\delta_{j,1} - v^\delta_{j,2} \text{ on } (U_i \cap U_j)^\delta.$$

Applying the complex operator $(dd^c)^m$ (in the usual sense) to both sides we get

$$\begin{aligned}&\sum_{k=0}^{m}(-1)^{m-k}C^k_m(dd^cv^\delta_{i,1})^k \wedge (dd^cv^\delta_{i,2})^{m-k}\\ =&\sum_{k=0}^{m}(-1)^{m-k}C^k_m(dd^cv^\delta_{j,1})^k \wedge (dd^cv^\delta_{j,2})^{m-k}, \text{ on } U_i \cap U_j.\end{aligned}$$

Letting $\delta \to 0$ and applying the monotone convergence theorem of Bedford and Taylor (Theorem 7.4 in [2]) we obtain the desired equality. The same proof as above implies that $(dd^cu)^m$ is independent of the choice of the open covering. Finally, $(dd^cu)^n$ is inner regular due to the corresponding fact for locally bounded plurisubharmonic functions and to the Heine-Borel property of compact sets in $\mathbb{C}^n$. ☐

Remark 2.1. (a) By polarization, for $u_1, \dots, u_k \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega), 1 \leqslant k \leqslant n$ we can easily define the wedge product $dd^cu_1 \wedge \cdots dd^cu_k$ as a closed current of bidegree (k, k) e.g.,

$$dd^cu_1 \wedge dd^cu_2 = \frac{1}{2}\Big((dd^c(u_1 + u_2))^2 - (dd^cu_1)^2 - (dd^cu_2)^2\Big).$$

(b) If $u \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$ then $(dd^cu)^n$ does not charge pluripolar sets in Ω. This follows immediately from the corresponding fact for locally bounded plurisubharmonic functions and Proposition 2.2.

The result below is almost an immediate consequence of our basic definition.

Proposition 2.3. *Let $u, v \in \delta^*\mathcal{PSH}_{loc}(\Omega)$. Then for every $1 \leqslant m \leqslant n$ we have*

(a) $(dd^cu^\delta)^m \to (dd^cu)^m$ *weakly.*

(b) $(dd^c(u + v))^m = \sum_{k=0}^m C^k_m(dd^cu)^k \wedge (dd^cv)^{m-k}$.

Proof. (a) Given an open set $U \subset \Omega$ and $u_1, u_2 \in L^\infty(U) \cap \mathcal{PSH}(U)$ such that $u = u_1 - u_2$, we deduce $u^\delta = u^\delta_1 - u^\delta_2$. Hence

$$(dd^cu^\delta)^m = \sum_{k=0}^{m}(-1)^kC^k_m(dd^cu^\delta_1)^{m-k} \wedge (dd^cu^\delta_2)^k.$$

It remains to apply the Bedford-Taylor monotone convergence theorem and Proposition 2.2.

(b) By (a) we get

$$(dd^cu^\delta + dd^cv^\delta)^m = (dd^c(u + v)^\delta)^m \to (dd^c(u + v))^m \text{ weakly.}$$

Notice that

$$(dd^cu^\delta + dd^cv^\delta)^m = \sum_{k=0}^{m}(-1)^kC^k_m(dd^cu^\delta)^{m-k} \wedge (dd^cv^\delta)^k.$$

We apply again the Bedford-Taylor monotone convergence theorem and Proposition 2.2 to reach the desired equality. □

Our main approximation tool for complex Monge-Ampère operator in $\delta^*\mathcal{PSH}_{\rm loc}(\Omega)$ is the following

Theorem 2.1. *Let $\{u_j\}_{j\geqslant 1}$ be a sequence in $\delta^*\mathcal{PSH}_{loc}(\Omega)$. Assume that $u_j \to u \in \delta^*\mathcal{PSH}_{loc}(\Omega)$ pointwise on Ω. Then $(dd^cu_j)^m \to (dd^cu)^m$ weakly if the following assumptions are verified.*

(a) *For every $z_0 \in \Omega$, there exists an open neighbourhood U of z_0 and a sequence $\{\omega_j\} \subset L^\infty(U) \cap \mathcal{PSH}(U)$ such that $u_j + \omega_j \in L^\infty(U) \cap \mathcal{PSH}(U)$ for all $j \geqslant 1$.*

(b) *The sequences $\{u_j+\omega_j\}$ and $\{\omega_j\}$ converge monotonically to $u+\omega \in \mathcal{PSH}(U)$ and to ω respectively (either decreasing or increasing) on U.*

Proof. Since the problem is local, it suffices to show that for every $z_0 \in \Omega$, there is an open neighbourhood U of z_0 such that $(dd^cu_j)^m \to (dd^cu)^m$ weakly on U. Choose a neighbourhood U of z_0 as in (b). By Proposition 2.2, on U, for $j \geqslant 1$ we have

(1)

$$(dd^cu_j)^m = (dd^c(u_j+\omega_j) - dd^c\omega_j)^m = \sum_{k=0}^{m}(-1)^kC_m^k dd^c(u_j+\omega_j)^k \wedge (dd^c\omega_j)^{m-k},$$

$$(2)\quad (dd^cu)^m = (dd^c(u+\omega) - dd^c\omega)^m = \sum_{k=0}^{m}(-1)^kC_m^k dd^c(u+\omega)^k \wedge (dd^c\omega)^{m-k}.$$

On the other hand, in view of the condition (a) and the monotone convergence theorems of Bedford and Taylor we obtain $\forall k \in \{0,\cdots,m\}$

$$(3)\qquad dd^c(u_j+\omega_j)^k \wedge (dd^c\omega_j)^{m-k} \to dd^c(u+\omega)^k \wedge (dd^c\omega)^{m-k}, \text{ as } j\to\infty.$$

Putting (1), (2) and (3) together we get the desired conclusion. □

Remark 2.2. (a) By a convergence theorem of Xing in [11] and the above proof, we see that the conclusion of Theorem 2.1 is still valid if the condition (a) is replaced by: $u_j \to u$ in capacity on Ω i.e.,

$$\lim_{j\to\infty} c(|u_j - u| > t, \Omega) = 0,\ \forall t > 0.$$

(b) Theorem 2.1 still holds if monotone convergence in the assumption (b) is replaced by locally uniformly convergence.

(c) It is interesting to know whether the condition (b) can be relaxed to ω_j converges to ω pointwise on U.

Corollary 2.1. *Let $u, v \in \delta^*\mathcal{PSH}_{\rm loc}(\Omega)$. For $\delta \geqslant 0$ set $u(\delta, z) := \max\{u(z) + \delta, v(z)\}$. Then $(dd^cu(\delta,\cdot))^n \to (dd^cu(0,\cdot))^n$ as $\delta \to 0$.*

Proof. By (a) and (d) of Proposition 2.1 we have $u(\delta,\cdot)\in\delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$ and $u(\delta,\cdot)\downarrow u(0,\cdot)$ as $\delta\to 0$. Furthermore, by the proof of Proposition 2.1 (d) and Theorem 2.1 we get the desired convergence. $\square$

Before formulating our comparison principle for a subclass of $\delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$, it is convenient to recall the following classical version due to Bedford and Taylor in [2].

Theorem 2.2. *Let $u,v\in L^\infty(\Omega)\cap\mathcal{PSH}(\Omega)$ be such that*

$$\liminf_{z\to\partial\Omega}(u(z)-v(z))\geqslant 0.$$

Then

$$\int_{\{u<v\}}(dd^cu)^n\geqslant\int_{\{u<v\}}(dd^cv)^n.$$

The main result of the paper is the following

Theorem 2.3. *Let $u,v\in\delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. Assume that the following conditions are satisfied.*

(a) $\liminf\limits_{z\to\Omega}(u(z)-v(z))\geqslant 0.$

(b) *u and v are locally a difference of two continuous plurisubharmonic functions.*

(c) *For every open subset $\Omega'\subset\subset\Omega$ there exists $h>0$ such that $h<\text{dist}\,(\partial\Omega',\partial\Omega)$ and that for all $h_1,\cdots,h_n\in\mathbb{C}^n$ satisfying $|h_1|<h,\cdots,|h_n|<h$, on Ω' we have*

$$dd^cu_{h_1}\wedge dd^cu_{h_2}\wedge\cdots\wedge dd^cu_{h_n}\geqslant 0, dd^cv_{h_1}\wedge dd^cv_{h_2}\wedge\cdots\wedge dd^cv_{h_n}\geqslant 0,$$
$$dd^c(u+v)_{h_1}\wedge dd^c(u+v)_{h_2}\wedge\cdots\wedge dd^c(u+v)_{h_n}\geqslant 0,$$
$$dd^c(u+v)_{h_1}\wedge dd^c(u+v)_{h_2}\wedge\cdots\wedge dd^c(u+v)_{h_{n-1}}\geqslant 0.$$

Then

$$\int_{\{u<v\}}(dd^cu)^n\geqslant\int_{\{u<v\}}(dd^cv)^n.$$

Remark 2.3. The assumptions (b) and (c) are satisfied in case $u,v\in\mathcal{C}^\infty(\Omega)$ such that

$$(dd^cu)^n=\varphi_1d\lambda_n, (dd^cv)^n=\varphi_2d\lambda_n,$$
$$(dd^c(u+v))^n=\varphi_3d\lambda_n, (dd^c(u+v))^{n-1}-\varphi_4\beta^{n-1}\geqslant 0,$$

where $\varphi_1,\cdots,\varphi_4$ are positive continuous functions on Ω. For instance, in case $n=4$ we may take

$$u(z_1,z_2,z_3,z_4)=-|z_1|^2-|z_2|^2+2|z_3|^2+2|z_4|^2,$$
$$v(z_1,z_2,z_3,z_4)=2|z_1|^2+|z_2|^2-|z_3|^2-|z_4|^2.$$

The proof requires the following technical lemma.

Lemma 2.1. *Let $u, v \in \delta^*\mathcal{PSH}_{\text{loc}}(\Omega)$. Assume that u and v satisfy the assumptions (b) and (c) of Theorem 2.3. Then for every open set $\Omega' \subset\subset \Omega$, there exists $0 < h < \text{dist }(\partial\Omega', \partial\Omega)$ such that for all $\delta \in (0, h)$ we have*

(a) $(dd^c u^\delta)^n \geqslant 0, (dd^c v^\delta)^n \geqslant 0, (dd^c(u+v)^\delta)^n \geqslant 0, (dd^c(u+v)^\delta)^{n-1} \geqslant 0$ *on* Ω'.

(b) $(dd^c \max(u^\delta, v^\delta))^n \geqslant 0,$ *on* Ω'.

(c) $(dd^c \max(u, v))^n \geqslant 0$ *on* Ω.

Proof. (a) We will use an idea from the proof of Theorem 3.6 of [9]. Choose $h > 0$ satisfying the assumption (c) of Theorem 2.3. Fix $\delta \in (0, h)$, it is enough to show $(dd^c u^\delta)^n \geqslant 0$ on Ω' since the proof of the other statements are similar. Fix $z_0 \in \Omega$, then there exist a neighbourhood U of z_0 and $\omega \in \mathcal{C}(U) \cap \mathcal{PSH}(U)$ such that $u + \omega \in \mathcal{C}(U) \cap \mathcal{PSH}(U)$. Since u and ω are continuous on U, by the definitions of Riemannian integral and convolution, we can find sequences $\{u_j\}_{j\geqslant 1}$ and $\{\omega_j\}_{j\geqslant 1}$ of continuous functions on U^δ converging locally uniformly to u^δ and ω^δ respectively such that

$$u_j = \lambda_{1,j} u_{\delta_{1,j}} + \cdots \lambda_{a_j,j} u_{\delta_{a_j,j}},$$

$$\omega_j = \lambda_{1,j} \omega_{\delta_{1,j}} + \cdots \lambda_{a_j,j} \omega_{\delta_{a_j,j}},$$

where $\lambda_{i,j}$ and $\delta_{i,j}$ are non negative numbers satisfying

$$\lambda_{1,j} + \cdots + \lambda_{a_j,j} = 1, 0 < \delta_{i,j} < \delta, \ \forall 1 \leqslant i \leqslant a_j.$$

Then $u_j + \omega_j \in \mathcal{C}(U^\delta) \cap \mathcal{PSH}(U^\delta)$. Moreover, $u_j + \omega_j$ converges to $u + \omega$ locally uniformly on U^δ. Thus by Theorem 2.1 we must have $(dd^c u_j)^n \to (dd^c u)^n$ weakly on U^δ. On the other hand, by the assumption (c) and Proposition 2.3 (b) we deduce that $(dd^c u_j)^n \geqslant 0$ on U^δ for all $j \geqslant 1$. It follows that $(dd^c u^\delta)^n \geqslant 0$ on U^δ.
(b) By part (a) we have $(dd^c u^\delta)^n$ and $(dd^c v^\delta)^n$ are positive regular Borel measures on Ω. Thus it suffices to check that for every compact $K \subset \{u^\delta = v^\delta\}$ we have

$$(dd^c \max(u^\delta, v^\delta))^n(K) \geqslant 0. \tag{4}$$

To this end, we use an idea of Zeriahi in [12] (which also goes back to earlier work of Stoll). For $j \geqslant 1$ we set

$$u_{j,\delta} = \frac{1}{j} \log(e^{ju^\delta} + e^{jv^\delta}).$$

It is easy to check that $u_{j,\delta} \downarrow \max(u^\delta, v^\delta)$ on Ω. For a point $z_0 \in \Omega$, by Proposition 2.1 (c), we can find an open neighbourhood U of z_0 and $\omega \in L^\infty(U) \cap \mathcal{PSH}(U)$ such that $u + \omega, v + \omega \in L^\infty(U) \cap \mathcal{PSH}(U)$. Then we get

$$u_{j,\delta} + \omega^\delta = \frac{1}{j} \log(e^{j(u^\delta + \omega^\delta)} + e^{j(v^\delta + \omega^\delta)}) \in L^\infty(U^\delta) \cap \mathcal{PSH}(U^\delta).$$

Thus by Theorem 2.1 we obtain

$$(dd^c u_{j,\delta})^n \to (dd^c \max(u^\delta, v^\delta))^n \text{ weakly}.$$

An easy computation shows that

$$(dd^c u_{j,\delta})^n$$
$$= \Big(\frac{e^{ju^\delta}}{e^{ju^\delta}+e^{jv^\delta}} dd^c u^\delta + \frac{e^{jv^\delta}}{e^{ju^\delta}+e^{jv^\delta}} dd^c v^\delta + \frac{e^{j(u^\delta+v^\delta)}}{(e^{ju^\delta}+e^{jv^\delta})^2} d(u^\delta - v^\delta)\wedge d^c(u^\delta - v^\delta)\Big)^n.$$

Notice that $(d(u^\delta - v^\delta)\wedge d^c(u^\delta - v^\delta))^2 = 0$, so we have

$$(dd^c u_{j,\delta})^n = \frac{1}{(e^{ju^\delta}+e^{jv^\delta})^n}\Big((e^{ju^\delta} dd^c u^\delta + e^{jv^\delta} dd^c v^\delta)^n +$$
$$+ \frac{e^{j(u^\delta+v^\delta)}}{e^{ju^\delta}+e^{jv^\delta}} (e^{ju^\delta} dd^c u^\delta + e^{jv^\delta} dd^c v^\delta)^{n-1} d(u^\delta - v^\delta)\wedge d^c(u^\delta - v^\delta)\Big). \tag{5}$$

Since $u^\delta = v^\delta$ on K and since, by part (a)

$$(dd^c u^\delta + dd^c v^\delta)^n = (dd^c(u+v)^\delta)^n \geqslant 0,$$
$$(dd^c u^\delta + dd^c v^\delta)^{n-1} = (dd^c(u+v)^\delta)^{n-1} \geqslant 0,$$

on Ω', we infer $(dd^c u_{j,\delta})^n(K) \geqslant 0$. Combining this with (5) one obtains (4).
(c) We do the same trick as in (b). Notice that $\max(u_\delta, v_\delta)$ converges locally uniformly to $\max(u,v)$ on Ω. Now given $z_0 \in U$, there exist a neighbourhood U of z_0 and $\omega \in \mathcal{C}(U)\cap\mathcal{PSH}(U)$ such that $u+\omega, v+\omega \in \mathcal{C}(U)\cap\mathcal{PSH}(U)$. It follows that $\max(u^\delta, v^\delta)+\omega^\delta$ converges locally uniformly to $\max(u,v)+\omega \in \mathcal{C}(U)\cap\mathcal{PSH}(U)$ as $\delta \to 0$. Thus by the remark following Theorem 2.1 we get

$$(dd^c \max(u^\delta, v^\delta)) \to (dd^c \max(u,v)) \text{ weakly as } \delta \to 0.$$

Putting this with the result obtained in (b) we get $(dd^c \max(u,v))^n \geqslant 0$. The lemma is completely proven. □

Proof of Theorem 2.3. We consider two cases.
Case 1. $u, v \in \mathcal{C}^\infty(\Omega)$ and $E := \{u < v\} \subset\subset \Omega$ has a smooth boundary. Then we set

$$v_k := \max(v, u + 1/k),\ \forall k \geqslant 1.$$

Then $(dd^c v_k)^n \to (dd^c v)^n$ on E by Corollary 2.1. On the other hand, since $v_k = u + 1/k$ on a neighbourhood of ∂E, by Stokes' theorem we get

$$\int_E (dd^c v_k)^n = \int_{\partial E} d^c v_k \wedge (dd^c v_k)^{n-1} = \int_{\partial E} d^c u \wedge (dd^c u)^{n-1} = \int_E (dd^c u)^n.$$

Fix $\varepsilon > 0$. Then by Lemma 2.1 (c) and by inner regularity of $(dd^c v)^n$, we can choose a compact subset $K \subset E$ such that $(dd^c v_k)^n \geqslant 0$ on $E\backslash K$ and that

$$||(dd^c v)^n||_{E\backslash K} < \varepsilon. \tag{6}$$

Pick $\varphi \in \mathcal{C}^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighbourhood of K but $\varphi = 0$ on $\Omega\backslash E$. Thus we have

$$\int_K (dd^c v)^n - \varepsilon \leqslant \int \varphi (dd^c v)^n = \lim_{k\to\infty} \int \varphi (dd^c v_k)^n \leqslant \liminf_{k\to\infty} \int_E (dd^c v_k)^n. \tag{7}$$

Combining (6) and (7) we get the desired conclusion. Observe that the proof of this case only requires the positivity of $(dd^c v_k)^n$ on a *fixed* neighbourhood of ∂E

in Ω for all k large enough.
Case 2. General case. Fix $\varepsilon > 0$. Then $\{u < v - 2\varepsilon\}$ is open and relatively compact in Ω. By Sard's theorem, we can choose sequences $\delta_k \downarrow 0$ such that the open set $E_k := \{u^{\delta_k} < v^{\delta_k} - \varepsilon\}$ is relatively compact in Ω and has smooth boundary. For ease of notation, we put $u_k := u^{\delta_k}, v_k := v^{\delta_k}$. By Proposition 2.3 and Lemma 2.1, $(dd^c u_k)^n$ and $(dd^c v_k)^n$ are positive regular Borel measures converging weakly to $(dd^c u)^n$ and $(dd^c v)^n$ respectively. By Dini's theorem we also have u_k and v_k converge locally uniformly to u and v. So there exists an open set Ω' such that

$$\{u < v - 2\varepsilon\} \cup_{k \geqslant 1} E_k \subset\subset \Omega' \subset\subset \Omega.$$

Now we claim that for all k large enough

$$\int_{E_k} (dd^c v_k)^n \leqslant \int_{E_k} (dd^c u_k)^n.$$

For this, by the proof given in the first case, it suffices to check that if $t > 0$ then $(dd^c \max(v_k - \varepsilon, u_k + t))^n \geqslant 0$ on Ω' for k sufficiently large. However, this assertion follows from Lemma 2.1 applied to $u + t$ and $v - \varepsilon$. Thus we have

$$\int_{\{u<v-2\varepsilon\}} (dd^c v)^n \leqslant \liminf_{k\to\infty} \int_{E_k} (dd^c v_k)^n \leqslant \liminf_{k\to\infty} \int_{E_k} (dd^c u_k)^n \leqslant \int_{\{u<v\}} (dd^c u)^n.$$

By letting $\varepsilon \to 0$ we get the desired conclusion. $\square$

Remark 2.4. It is tempting to generalize Theorem 2.3 to the case where u, v are *not* necessarily continuous. Although quasicontinuity of u and v still holds (see Proposition 2.1 (e)), it is not clear to us how to prove Lemma 2.1, especially when no convergence of Riemannian sums is available for integral of discontinuous functions.

Corollary 2.2. *Let $u, v \in \mathcal{C}^\infty(\Omega)$. Then $u \geqslant v$ on Ω if the following conditions are satisfied.*

(a) $\liminf_{z\to\partial\Omega} (u(z) - v(z)) \geqslant 0$.

(b) $(dd^c u)^n \leqslant (dd^c v)^n$.

(c) *u and v satisfy the conditions given in the remark following Theorem 2.3.*

(d) *There exists $\varphi \in L^\infty(\Omega) \cap \mathcal{C}^2(\Omega)$ such that $(dd^c v)^{n-1} \wedge dd^c\varphi = \varphi^* d\lambda_n$, where φ^* is a positive continuous function on Ω.*

Proof. Choose M so large that $\varphi(z) - M < 0$ on Ω. Assume that $E := \{u < v\} \neq \emptyset$. Then by the assumption (a), we can find $\varepsilon > 0$ such that $E_\varepsilon := \{u < v + \varepsilon(\varphi - M)\}$ is a non empty, relatively compact, open subset of Ω. Observe that for $\varepsilon > 0$ small enough, the functions u and $v + \varepsilon(\varphi - M)$ satisfy the conditions (b) and (c) of Theorem 2.3. Thus by the Theorem and the assumption (b) we obtain for all $\varepsilon > 0$ sufficiently small

$$\int_{E_\varepsilon} (dd^c v + \varepsilon dd^c \varphi)^n \leqslant \int_{E_\varepsilon} (dd^c u)^n \leqslant \int_{E_\varepsilon} (dd^c v)^n.$$

This implies

$$\int_{E_\varepsilon} \Big[(dd^c v)^{n-1} \wedge dd^c \varphi + \varepsilon((dd^c v)^{n-2} \wedge (dd^c \varphi)^2 + \cdots + \varepsilon^{n-1}(dd^c \varphi)^n)\Big] \leqslant 0.$$

Letting $\varepsilon \to 0$ and using (d) we arrive at

$$0 \leqslant \int_E (dd^c v)^{n-1} \wedge dd^c \varphi = \int_E \varphi^* d\lambda_n \leqslant 0,$$

which is clearly absurd. The proof is thereby concluded. □

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Department of Mathematics
Hanoi University of Education
Cau Giay, Tu Liem, Hanoi, Vietnam

E-mail address: dieu_vn@yahoo.com

Department of Mathematics,
Vinh University
Vinh, Vietnam

E-mail address: dauhoanghung@vnn.vn