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INEQUALITIES FOR DIRICHLET SERIES WITH POSITIVE TERMS

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ABSTRACT. Some fundamental inequalities for Dirichlet series with positive terms by utilising certain classical results due to Hölder, Čebyšev, Pólya-Szegö, Grüss and others are established.

1. Introduction

In the following we consider Dirichlet series of the form

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1}$$

with $s > 1$ and a_n assumed to be nonnegative for $n \geqslant 1$.

In this class of series one can find the celebrated *Zeta function* defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1 \tag{1.2}$$

and the *Dirichlet Lambda function* given by

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \left(1 - 2^{-s}\right)\zeta(s) \tag{1.3}$$

for $s > 1$.

If $\Lambda(n)$ is the *von Mangoldt function*

$$\Lambda(n) := \begin{cases} \log p, & n = p^k \quad (p \text{ prime}, k \geqslant 1) \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

then [2, p. 3]

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \quad s > 1. \tag{1.5}$$

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If $d(n)$ is the number of divisors of n, we have [2, p. 35] the following relationships with the Zeta function

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \tag{1.6}$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \tag{1.7}$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \tag{1.8}$$

and [2, p. 36]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad s > 1, \tag{1.9}$$

where $\omega(n)$ is the number of distinct prime factors of n.

Further, if $\varphi(n)$ denotes Euler's function defined by

$$\varphi(n) := n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over all prime divisors of n, then

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad s > 2. \tag{1.10}$$

For $a \in \mathbb{R}$ we define

$$\sigma_a(n) := \sum_{d|n} d^a$$

and in particular $\sigma(n) = \sigma_1(n) = \sum_{d|n} d$ is the sum of the divisors of n, then these are related to the Zeta function [2, p. 37] by

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \quad s > 1, \ s > a+1;$$

and

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s},$$

where $s > \max\{1, a+1, b+1, a+b+1\}$.

One can prove in various ways that such functions ψ defined in (1.1) are monotonic non-increasing on $(1, \infty)$ and logarithmic convex. This means that the function $\log \psi$ is convex or, alternatively

$$\psi(us_1 + vs_2) \leqslant [\psi(s_1)]^u [\psi(s_2)]^v \tag{1.11}$$

for any s_1, $s_2 > 1$ and $u, v \geqslant 0$ with $u + v = 1$.

Since, by the geometric mean – arithmetic mean inequality we have

$$[\psi(s_1)]^u [\psi(s_2)]^v \leqslant u\psi(s_1) + v\psi(s_2)$$

for $s_1, s_2 > 1$ and $u, v \geqslant 1$, $u + v = 1$, we can also state that these classes of functions ψ are also convex on $(1, \infty)$.

The main aim of this paper is to establish a number of fundamental inequalities for ψ that can be stated by utilising some classical inequalities for nonnegative real numbers such as Hölder's inequality, Čebyšev's inequality, Polyá-Szegö's reverse of Schwarz's inequality, Grüss' inequality and others.

2. Inequalities for Dirichlet series with positive terms

We consider the Dirichlet series given by (1.1). We assume that the series which defines ψ is uniformly convergent for $s > 1$. Then we have the following result.

Proposition 2.1. *Let $\alpha, \beta > 1$ with $\alpha^{-1} + \beta^{-1} = 1$. If $s, p, q \in \mathbb{R}$ are such that $s + p + q > 1$, $s + p\alpha > 1$ and $s + q\beta > 1$, then*

$$\psi(s + p + q) \leqslant [\psi(s + p\alpha)]^{\frac{1}{\alpha}} [\psi(s + q\beta)]^{\frac{1}{\beta}}. \tag{2.1}$$

Proof. We use Hölder's inequality to state that

$$\begin{aligned}
\psi(s + p + q) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \frac{1}{n^p} \cdot \frac{1}{n^q} \\
&\leqslant \left[\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \left(\frac{1}{n^p}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \left[\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \left(\frac{1}{n^q}\right)^{\beta}\right]^{\frac{1}{\beta}} \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+\alpha p}}\right)^{\frac{1}{\alpha}} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+\beta q}}\right)^{\frac{1}{\beta}} \\
&= [\psi(s + p\alpha)]^{\frac{1}{\alpha}} [\psi(s + q\beta)]^{\frac{1}{\beta}},
\end{aligned}$$

which proves the desired inequality (2.1). □

Remark 2.1. We observe that for $\alpha = \beta = 2$, one obtains from (2.1) the following inequality

$$\psi^2(s + p + q) \leqslant \psi(s + 2p)\,\psi(s + 2q), \tag{2.2}$$

provided the real numbers s, p, q satisfy the conditions $s+p+q, s+2p, s+2q > 1$. In its turn, the inequality (2.2), and in fact (2.1), is a generalisation of the following result

$$\psi^2(s + 1) \leqslant \psi(s)\,\psi(s + 2), \tag{2.3}$$

provided $s > 1$.

We remark that for $\psi = \zeta$ one obtains from (2.3) that

$$\frac{\zeta(s+1)}{\zeta(s)} \leqslant \frac{\zeta(s+2)}{\zeta(s+1)} \quad \text{for } s > 1. \tag{2.4}$$

This inequality is an improvement of a recent result due to Laforgia and Natalini [3] who proved that

$$\frac{\zeta(s+1)}{\zeta(s)} \leqslant \frac{s+1}{s} \cdot \frac{\zeta(s+2)}{\zeta(s+1)} \text{ for } s > 1.$$

Their arguments make use of an integral representation of the Zeta function and Turán-type inequalities.

It should be further noted that, if $s = 2n$, $n \in \mathbb{N}$, then (2.4) shows that

$$\zeta(2n+1) \leqslant \sqrt{\zeta(2n)\,\zeta(2n+2)},$$

demonstrating that the value of Zeta at the odd integers is bounded above by the geometric mean of its immediate even Zeta values.

We also have the following result.

Proposition 2.2. *If $a > 1$, $b, c \in \mathbb{R}$ such that $bc \geqslant (\leqslant)\, 0$ and $a+b$, $a+c$, $a+b+c > 1$, then*

$$\psi(a)\,\psi(a+b+c) \geqslant (\leqslant)\, \psi(a+b)\,\psi(a+c). \tag{2.5}$$

Proof. Consider the sequence $\alpha_n := n^b$, $n \geqslant 1$, $b \in \mathbb{R}$. It is clear that α_n is increasing if $b > 0$ and decreasing if $b < 0$. Therefore, the sequences $\frac{1}{n^b}, \frac{1}{n^c}$ are synchronous if $bc \geqslant 0$ and asynchronous when $bc < 0$.

Utilising Čebyšev's inequality for synchronous (asynchronous) sequences, we have

$$\begin{aligned} \psi(a)\,\psi(a+b+c) &= \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^b} \cdot \frac{1}{n^c} \\ &\geqslant (\leqslant) \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^b} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^c} \\ &= \psi(a+b)\,\psi(a+c), \end{aligned}$$

and the inequality (2.5) is proved. □

Remark 2.2. Utilising Inequality (2.5) (for $c = b$) we can state the following result

$$\psi^2(a+b) \leqslant \psi(a)\,\psi(a+2b), \tag{2.6}$$

provided the real numbers a, b are such that $a, a+b, a+2b > 1$. We also remark that the choice $b = 1$ will produce Inequality (2.3).

From a different perspective, we can state the following result as well.

Proposition 2.3. *Assume that* $m \geqslant 2$ *and* $k_1, \ldots, k_m > \frac{1}{2}$. *Then*

$$\sum_{1 \leqslant i < j \leqslant m} \psi(k_i + k_j) \leqslant \frac{m-1}{2} \sum_{j=1}^{m} \psi(2k_j). \tag{2.7}$$

Proof. By Schwarz's inequality

$$m \sum_{j=1}^{m} z_j^2 \geqslant \left(\sum_{j=1}^{m} z_j \right)^2$$

we have

$$\begin{aligned} m \sum_{j=1}^{m} \frac{1}{n^{2k_j}} \geqslant \left(\sum_{j=1}^{m} \frac{1}{n^{k_j}} \right)^2 &= \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{n^{k_i + k_j}} \\ &= \sum_{j=1}^{m} \frac{1}{n^{2k_j}} + 2 \sum_{1 \leqslant i < j \leqslant m} \frac{1}{n^{k_i + k_j}} \end{aligned} \tag{2.8}$$

giving

$$\frac{m-1}{2} \sum_{j=1}^{m} \frac{1}{n^{2k_j}} \geqslant \sum_{1 \leqslant i < j \leqslant m} \frac{1}{n^{k_i + k_j}}. \tag{2.9}$$

If we multiply (2.9) by $a_n > 0$ and sum up over $n \geqslant 1$, we get

$$\frac{m-1}{2} \sum_{j=1}^{m} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{2k_j}} \right) \geqslant \sum_{1 \leqslant i < j \leqslant m} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{k_i + k_j}} \right)$$

which gives the desired inequality (2.7). □

Remark 2.3. If $a, b, c > 1$ then from (2.7) applied for $m = 3$ we deduce the following inequality

$$\psi\left(\frac{a+b}{2}\right) + \psi\left(\frac{b+c}{2}\right) + \psi\left(\frac{c+a}{2}\right) \leqslant \psi(a) + \psi(b) + \psi(c). \tag{2.10}$$

In particular, the choice $a = x, b = x + 2, c = x + 4$ will produces the inequality

$$\psi(x+1) + \psi(x+3) \leqslant \psi(x) + \psi(x+4), \tag{2.11}$$

for each $x > 1$.

If more information about the size of k_j, $j = 1, \ldots, m$ is known, then the following reverse of (2.7) may be stated as well.

Proposition 2.4. *Assume that* $m \geqslant 2$ *and* $\frac{1}{2} < \gamma \leqslant k_1, \ldots, k_m \leqslant \Gamma < \infty$. *Then*

$$\begin{aligned} (0 \leqslant) \frac{m-1}{2} \sum_{j=1}^{m} \psi(2k_j) - \sum_{1 \leqslant i < j \leqslant m} \psi(k_i + k_j) \\ \leqslant \frac{m^2}{8} \left[\psi(2\Gamma) + \psi(2\gamma) - 2\psi(\gamma + \Gamma) \right]. \end{aligned} \tag{2.12}$$

Proof. We use the following Grüss type inequality

$$\frac{1}{m}\sum_{j=1}^{m} z_j^2 - \left(\frac{1}{m}\sum_{j=1}^{m} z_j\right)^2 \leqslant \frac{1}{4}\left(\Gamma-\gamma\right)^2,$$

provided $\gamma \leqslant z_j \leqslant \Gamma$ for each $j \in \{1,\ldots,m\}$.

Since $\gamma \leqslant k_j \leqslant \Gamma$ for $j \in \{1,\ldots,m\}$, then

$$\begin{aligned}\frac{1}{m}\sum_{j=1}^{m}\frac{1}{n^{2k_j}} - \frac{1}{m^2}\left(\sum_{j=1}^{m}\frac{1}{n^{k_j}}\right)^2 &\leqslant \frac{1}{4}\left(\frac{1}{n^{\gamma}}-\frac{1}{n^{\Gamma}}\right)^2 \\ &= \frac{1}{4}\left(\frac{1}{n^{2\gamma}}+\frac{1}{n^{2\Gamma}}-\frac{2}{n^{\gamma+\Gamma}}\right)\end{aligned}$$

for $n \geqslant 1$, which gives

$$\begin{aligned}\frac{1}{m}\sum_{j=1}^{m}\frac{1}{n^{2k_j}} - \frac{1}{m^2}\left(\sum_{j=1}^{m}\frac{1}{n^{2k_j}} + 2\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j}}\right)& \\ \leqslant \frac{1}{4}\left(\frac{1}{n^{2\gamma}}+\frac{1}{n^{2\Gamma}}-\frac{2}{n^{\gamma+\Gamma}}\right)&\end{aligned}$$

for $n \geqslant 1$. Multiplying with m^2 and rearranging the terms, we get

$$\frac{m-1}{2}\sum_{j=1}^{m}\frac{1}{n^{2k_j}} - \sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j}} \leqslant \frac{m^2}{8}\left(\frac{1}{n^{2\gamma}}+\frac{1}{n^{2\Gamma}}-\frac{2}{n^{\gamma+\Gamma}}\right) \tag{2.13}$$

for any $n \geqslant 1$.

Finally, if we multiply (2.13) by $a_n \geqslant 0$ and sum up over $n \geqslant 1$, we get the desired inequality (2.12). □

Remark 2.4. If $R > a,b,c > r > 1$ then from (2.12) applied for $m = 3$ we deduce the following result

$$\begin{aligned}0 &\leqslant \psi(a)+\psi(b)+\psi(c)-\psi\left(\frac{a+b}{2}\right)-\psi\left(\frac{b+c}{2}\right)-\psi\left(\frac{c+a}{2}\right) \\ &\leqslant \frac{9}{4}\cdot\left[\frac{\psi(r)+\psi(R)}{2}-\psi\left(\frac{r+R}{2}\right)\right].\end{aligned} \tag{2.14}$$

We have the following result as well.

Proposition 2.5. *Assume that* $m \geqslant 1$ *and* $\frac{1}{2} < \gamma \leqslant k_1,\ldots,k_m \leqslant \Gamma < \infty$. *Then*

$$\sum_{j=1}^{m}\left[\psi(k_j+\gamma)+\psi(k_j+\Gamma)\right] \geqslant \sum_{j=1}^{m}\psi(2k_j) + m\psi(\gamma+\Gamma). \tag{2.15}$$

Proof. We have

$$\left(\frac{1}{n^{\gamma}}-\frac{1}{n^{k_j}}\right)\left(\frac{1}{n^{k_j}}-\frac{1}{n^{\Gamma}}\right) \geqslant 0$$

for each $j \in \{1, \dots, m\}$ and $n \geqslant 1$. This is clearly equivalent to

$$\frac{1}{n^{\gamma+k_j}} + \frac{1}{n^{\Gamma+k_j}} \geqslant \frac{1}{n^{2k_j}} + \frac{1}{n^{\gamma+\Gamma}}$$

for $j \in \{1, \dots, m\}$ and $n \geqslant 1$.

Summing up over j from 1 to m, we get

$$\sum_{j=1}^{m} \frac{1}{n^{\gamma+k_j}} + \sum_{j=1}^{m} \frac{1}{n^{\Gamma+k_j}} \geqslant \sum_{j=1}^{m} \frac{1}{n^{2k_j}} + \frac{m}{n^{\gamma+\Gamma}} \tag{2.16}$$

for each $n \geqslant 1$.

Multiplying (2.16) with $a_n \geqslant 0$ and summing up over $n \geqslant 1$, we deduce the desired inequality (2.15). □

We have the following result as well.

Proposition 2.6. *Assume that $m \geqslant 1$ and $\frac{1}{2} < \gamma \leqslant k_1, \dots, k_m \leqslant \Gamma < \infty$. Then*

(2.17)

$$\begin{aligned}\left(m - \frac{1}{2}\right) \sum_{j=1}^{m} \psi\left(2k_j\right) \leqslant & \frac{1}{2} \sum_{j=1}^{m} \left[\frac{\psi\left(2k_j - \gamma + \Gamma\right) + \psi\left(2k_j - \Gamma + \gamma\right)}{2}\right] \\ & + \sum_{1 \leqslant i < j \leqslant m} \left[\frac{\psi\left(k_i + k_j - \Gamma + \gamma\right) + \psi\left(k_i + k_j - \gamma + \Gamma\right)}{2}\right] \\ & + \sum_{1 \leqslant i < j \leqslant m} \psi\left(k_i + k_j\right).\end{aligned}$$

Proof. We apply Polyá-Szegö's inequality

$$(1 \leqslant) \frac{m \sum_{j=1}^{m} z_j^2}{\left(\sum_{j=1}^{m} z_j\right)^2} \leqslant \frac{(\Gamma + \gamma)^2}{4\gamma\Gamma}, \tag{2.18}$$

provided $\gamma \leqslant z_j \leqslant \Gamma$, $j \in \{1, \dots, m\}$.

Observe that

$$\frac{1}{n^{\Gamma}} \leqslant \frac{1}{n^{k_j}} \leqslant \frac{1}{n^{\gamma}}, \quad j = 1, \dots, m.$$

Hence, by (2.18) we have

$$\begin{aligned} m \sum_{j=1}^{m} \frac{1}{n^{2k_j}} \leqslant & \frac{\left(\frac{1}{n^{\gamma}} + \frac{1}{n^{\Gamma}}\right)^2}{4 \frac{1}{n^{\gamma}} \cdot \frac{1}{n^{\Gamma}}} \left(\sum_{j=1}^{m} \frac{1}{n^{k_j}}\right)^2 \\ = & \frac{1}{4}\left(n^{\Gamma-\gamma} + n^{\gamma-\Gamma} + 2\right) \left[\sum_{j=1}^{m} \frac{1}{n^{2k_j}} + 2 \sum_{1 \leqslant i < j \leqslant m} \frac{1}{n^{k_i + k_j}}\right] \end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{4}\left[\sum_{j=1}^{m}\frac{1}{n^{2k_j-\Gamma+\gamma}}+\sum_{j=1}^{m}\frac{1}{n^{2k_j-\gamma+\Gamma}}+2\sum_{j=1}^{m}\frac{1}{n^{2k_j}}\right]\\
&+\frac{1}{2}\left[\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j-\Gamma+\gamma}}+\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j-\gamma+\Gamma}}+2\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j}}\right],
\end{aligned}
$$

which is clearly equivalent to

$$
\begin{aligned}
\left(m-\frac{1}{2}\right)\sum_{j=1}^{m}\frac{1}{n^{2k_j}}\leqslant&\frac{1}{4}\left[\sum_{j=1}^{m}\frac{1}{n^{2k_j-\Gamma+\gamma}}+\sum_{j=1}^{m}\frac{1}{n^{2k_j-\gamma+\Gamma}}\right]\\
&+\frac{1}{2}\left[\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j-\Gamma+\gamma}}+\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j-\gamma+\Gamma}}\right]\\
&+\sum_{1\leqslant i<j\leqslant m}\frac{1}{n^{k_i+k_j}}
\end{aligned}
\tag{2.19}
$$

for any $n \geqslant 1$.

Multiplying (2.19) by $a_n \geqslant 0$ and summing up over n, we deduce the desired result (2.17). □

3. Representations as double sums

Consider the sequences

$$
I_k^{\pm}(p,s) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p \pm m^p)^2}{n^s m^s}a_n a_m, \quad k \geqslant 1 \tag{3.1}
$$

where $a_n \geqslant 0$, $n \geqslant 1$ and $s, p \in \mathbb{R}$. We have the following representation.

Proposition 3.1. *If $s > 1$ and $p \in \mathbb{R}$ such that $s - 1 > 2p$ and $s - 1 > p$, then*

$$
I^{\pm}(p,s) := \lim_{k\to\infty} I_k^{\pm}(p,s) = \psi(s-2p)\,\psi(s) \pm [\psi(s-p)]^2\,(\geqslant 0)\,. \tag{3.2}
$$

Proof. We observe that

$$
\begin{aligned}
I_k^{\pm}(p,s) =&\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\left(\frac{n^{2p}\pm 2n^p m^p + m^{2p}}{n^s m^s}\right)a_n a_m\\
=&\frac{1}{2}\left[\sum_{n=1}^{k}\frac{a_n}{n^{s-2p}}\sum_{m=1}^{k}\frac{a_m}{m^s}\pm 2\sum_{n=1}^{k}\frac{a_n}{n^{s-p}}\sum_{m=1}^{k}\frac{a_m}{m^{s-p}}\right.\\
&\left.+\sum_{n=1}^{k}\frac{a_n}{n^s}\sum_{m=1}^{k}\frac{a_m}{m^{s-2p}}\right].
\end{aligned}
$$

Since, for $s>1$, $s-1>2p$, $s-1>p$,

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^{s-2p}}=\psi\left(s-2p\right),\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^{s-p}}=\psi\left(s-p\right),\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^{s}}=\psi\left(s\right),$$

$\lim_{k\to\infty} I_k^{\pm}\left(p,s\right)$ exists and the relation (3.2) is proved. □

Remark 3.1. We observe that for $s>1$ and $p=-1$, one has

$$\psi\left(s+2\right)\psi\left(s\right)-\left[\psi\left(s+1\right)\right]^2=\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{\left(n-m\right)^2}{n^{s+2}m^{s+2}}a_na_m\geqslant 0. \tag{3.3}$$

We have the following result.

Proposition 3.2. *Let $\alpha,\beta>1$ with $\alpha^{-1}+\beta^{-1}=1$. If $s,p,q,r\in\mathbb{R}$ are such that $s+q+r>1$, $s+q+r-1>2p$, $s+q+r-1>p$ and $s+\alpha q>1$, $s+\alpha q-1>2p$, $s+\alpha q-1>p$, $s+\beta r>1$, $s+\beta r-1>2p$, $s+\beta r-1>p$, then*

$$I^{\pm}\left(p,s+q+r\right)\leqslant\left[I^{\pm}\left(p,s+\alpha q\right)\right]^{\frac{1}{\alpha}}\left[I^{\pm}\left(p,s+\beta r\right)\right]^{\frac{1}{\beta}}. \tag{3.4}$$

Proof. Using the representation (3.1), (3.2) and Hölder's inequality for double sums, we have

$$\begin{aligned}
I^{\pm}\left(p,s+q+r\right)&=\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{\left(n^p\pm m^p\right)^2}{n^{s+q+r}m^{s+q+r}}a_na_m\\
&=\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{1}{n^q\cdot m^q}\cdot\frac{1}{n^r\cdot m^r}\cdot\frac{\left(n^p\pm m^p\right)^2}{n^s\cdot m^s}a_na_m\\
&\leqslant\left[\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{\left(n^p\pm m^p\right)^2}{n^s\cdot m^s}a_na_m\left(\frac{1}{n^q\cdot m^q}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\\
&\quad\times\left[\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{\left(n^p\pm m^p\right)^2}{n^s\cdot m^s}a_na_m\left(\frac{1}{n^r\cdot m^r}\right)^{\beta}\right]^{\frac{1}{\beta}}\\
&=\left[I^{\pm}\left(p,s+\alpha q\right)\right]^{\frac{1}{\alpha}}\left[I^{\pm}\left(p,s+\beta r\right)\right]^{\frac{1}{\beta}}
\end{aligned}$$

and Inequality (3.4) follows. □

Remark 3.2. In particular, if we define

$$I\left(s\right):=\psi\left(s+2\right)\psi\left(s\right)-\left[\psi\left(s+1\right)\right]^2\quad\text{for}\quad s>1, \tag{3.5}$$

then we have

$$I\left(s+q+r\right)\leqslant\left[I\left(s+\alpha q\right)\right]^{\frac{1}{\alpha}}\left[I\left(s+\beta r\right)\right]^{\frac{1}{\beta}}, \tag{3.6}$$

where $\alpha,\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $s,q,r\in\mathbb{R}$ with $s+q+r$, $s+\alpha q$ and $s+\beta r>1$.

We have the following log-convexity property.

Proposition 3.3. *Let $p \in \mathbb{R}$ and $s_0 := \max\{1, p+1, 2p+1\}$. Then the function $s \mapsto I_k^{\pm}(p,s)$ is log-convex on the interval $(s_0, +\infty)$.*

Proof. Let $s_1, s_2 \in (s_0, +\infty)$. Then for $\alpha, \beta > 0$, $\alpha + \beta = 1$ by Hölder's inequality for double sums we have

$$\begin{aligned} I_k^{\pm}(p, \alpha s_1 + \beta s_2) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p \pm m^p)^2}{n^{\alpha s_1+\beta s_2} m^{\alpha s_1 + \beta s_2}} a_n a_m \\ &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p \pm m^p)^2 a_n a_m}{(nm)^{\alpha s_1}(nm)^{\beta s_2}} \\ &\leqslant \left[\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p \pm m^p)^2 a_n a_m}{\left[(nm)^{\alpha s_1}\right]^{1/\alpha}}\right]^{\alpha} \\ &\qquad \times \left[\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p \pm m^p)^2 a_n a_m}{\left[(nm)^{\beta s_2}\right]^{1/\beta}}\right]^{\beta} \\ &= \left[I_k^{\pm}(p, s_1)\right]^{\alpha}\left[I_k^{\pm}(p, s_2)\right]^{\beta} \end{aligned}$$

for any $k \geqslant 1$.

Taking the limit over $k \to \infty$, and using the representation (3.2) we deduce the desired result. □

Corollary 3.1. *The function $I(s) := \psi(s+2)\psi(s) - [\psi(s+1)]^2$ is log-convex on $(1, \infty)$.*

For given $s, p \in \mathbb{R}$ and $k \in \mathbb{N}$, $k \geqslant 1$, we consider the sequence

$$\Delta_k(s,p) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n - a_m)\left(\frac{1}{m^s} - \frac{1}{n^s}\right)\frac{1}{n^p m^p},$$

where a_n is also a sequence of real numbers.

The following representation result may be stated:

Proposition 3.4. *If $a_n \geqslant 0$, $n \in \mathbb{N}$, $n \geqslant 1$ and $p > 1$, $s \in \mathbb{R}$ such that $s + p > 1$, then we have the representation*

$$\lim_{k\to\infty}\Delta_k(s,p) = \psi(p)\zeta(s+p) - \zeta(p)\psi(s+p), \tag{3.7}$$

where ζ is the Zeta function, i.e.,

$$\zeta(p) := \sum_{n=1}^{\infty}\frac{1}{n^p}, \quad p > 1.$$

Proof. Observe that, by Korkine's identity

$$\sum_{i=1}^{m}p_i\sum_{i=1}^{m}p_i a_i b_i - \sum_{i=1}^{m}p_i a_i\sum_{i=1}^{m}p_i b_i = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}p_i p_j(a_i - a_j)(b_i - b_j),$$

we have

$$\begin{aligned}
&\sum_{n=1}^{k}\frac{1}{n^p}\sum_{n=1}^{k}\frac{1}{n^p}\cdot a_n\cdot\frac{1}{n^s}-\sum_{n=1}^{k}\frac{1}{n^p}\cdot a_n\cdot\sum_{n=1}^{k}\frac{1}{n^p}\cdot\frac{1}{n^s}\\
&=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{n^p m^p}\left(a_n-a_m\right)\left(\frac{1}{n^s}-\frac{1}{m^s}\right)\\
&=-\Delta_k\left(s,p\right)
\end{aligned}$$

for each $k \geqslant 1$ and p, s as above.

Since

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{n^p}=\zeta\left(p\right)\quad\text{and}\quad\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^p}=\psi\left(p\right),$$

$\lim_{k\to\infty}\Delta_k\left(p,s\right)$ exists and the identity (3.7) holds true. □

Corollary 3.2. *If the sequence* $(a_n)_{n\in\mathbb{N}}$ *is decreasing (increasing) then*

$$\zeta\left(s+p\right)\psi\left(p\right)\leqslant\left(\geqslant\right)\zeta\left(p\right)\psi\left(s+p\right)\tag{3.8}$$

for $p > 1$ *and* $s \in \mathbb{R}$ *such that* $s + p > 1$.

The following result concerning some bounds for the quantity

$$\zeta\left(s+p\right)\psi\left(p\right)-\zeta\left(p\right)\psi\left(s+p\right)$$

in the case when the sequences $(a_n)_{n\in\mathbb{N}}$ satisfy some Lipschitz type conditions may be stated as well.

Proposition 3.5. *Assume that for* $(a_n)_{n\in\mathbb{N}}$ *there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *such that*

$$\gamma\leqslant\frac{a_n-a_m}{n-m}\leqslant\Gamma\tag{3.9}$$

for any $n, m \in \mathbb{N}$, $n \neq m$. *Then for* $p > 2$ *and* $s \in \mathbb{R}$ *such that,* $p + s > 2$

$$\begin{aligned}
&\gamma\left[\zeta\left(p-1\right)\zeta\left(p+s\right)-\zeta\left(p\right)\zeta\left(p+s-1\right)\right]\\
&\leqslant\zeta\left(s+p\right)\psi\left(p\right)-\zeta\left(p\right)\psi\left(s+p\right)\\
&\leqslant\Gamma\left[\zeta\left(p-1\right)\zeta\left(p+s\right)-\zeta\left(p\right)\zeta\left(p+s-1\right)\right].
\end{aligned}\tag{3.10}$$

Proof. With the assumption (3.9) we have

$$\begin{aligned}
&\frac{1}{2}\gamma\sum_{n=1}^{k}\sum_{m=1}^{k}\left(n-m\right)\left(\frac{1}{m^s}-\frac{1}{n^s}\right)\frac{1}{n^p m^p}\\
&\qquad\leqslant\Delta_k\left(p,s\right)\leqslant\frac{1}{2}\Gamma\sum_{n=1}^{k}\sum_{m=1}^{k}\left(n-m\right)\left(\frac{1}{m^s}-\frac{1}{n^s}\right)\frac{1}{n^p m^p}
\end{aligned}\tag{3.11}$$

for each $k \in \mathbb{N}$, $k \geqslant 1$.

Further, utilising Korkine's identity produces

$$
\begin{aligned}
I_k &:= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}(n-m)\left(\frac{1}{m^s}-\frac{1}{n^s}\right)\frac{1}{n^p m^p} \\
&= \sum_{n=1}^{k}\frac{n}{n^p}\cdot\sum_{n=1}^{k}\frac{1}{n^s}\cdot\frac{1}{n^p}-\sum_{n=1}^{k}\frac{1}{n^p}\sum_{n=1}^{k}\frac{1}{n^p}\cdot n\cdot\frac{1}{n^s} \\
&= \sum_{n=1}^{k}\frac{1}{n^{p-1}}\sum_{n=1}^{k}\frac{1}{n^{p+s}}-\sum_{n=1}^{k}\frac{1}{n^p}\sum_{n=1}^{k}\frac{1}{n^{p+s-1}}
\end{aligned}
$$

for each $k\in\mathbb{N}$, $k\geqslant 1$ and so, for $p>2$, $s\in\mathbb{R}$ with $p+s, p+s-1>1$, we have

$$\lim_{k\to\infty} I_k=\zeta(p-1)\,\zeta(p+s)-\zeta(p)\,\zeta(p+s-1).$$

Taking the limit in (3.11) we deduce the desired inequality (3.10). ◻

The following simple result also holds.

Proposition 3.6. *Let $a_n\geqslant 0$, $n\in\mathbb{N}$, $n\geqslant 1$ and $s>1$.*

(i) *If a_n is increasing and*

$$M:=\sup_{\substack{k\in\mathbb{N}\\ k\geqslant 1}}\left\{\frac{1}{k}\sum_{n=1}^{k}a_n\right\},$$

then

$$\psi(s)\leqslant M\cdot\zeta(s). \tag{3.12}$$

(ii) *If a_n is decreasing and*

$$m:=\inf_{\substack{k\in\mathbb{N}\\ k\geqslant 1}}\left\{\frac{1}{k}\sum_{n=1}^{k}a_n\right\}$$

then

$$\psi(s)\geqslant m\cdot\zeta(s). \tag{3.13}$$

Proof. Utilising Korkine's identity we have for each $k\geqslant 1$ that

$$k\sum_{n=1}^{k}\frac{a_n}{n^s}-\sum_{n=1}^{k}a_n\sum_{n=1}^{k}\frac{1}{n^s}=\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)\left(\frac{1}{n^s}-\frac{1}{m^s}\right). \tag{3.14}$$

(i) If a_n is increasing, then by (3.14) we deduce that

$$\sum_{n=1}^{k}\frac{a_n}{n^s}\leqslant\left(\frac{1}{k}\sum_{n=1}^{k}a_n\right)\sum_{n=1}^{k}\frac{1}{n^s}\leqslant M\sum_{n=1}^{k}\frac{1}{n^s}. \tag{3.15}$$

Taking the limit over $k\to\infty$ in (3.15) we deduce (3.12).

(ii) is treated similarly and we omit the details.

◻

4. Inequalities in terms of the first and second derivatives

We consider the sequence

$$S_k(s) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(\ln n - \ln m)^2}{n^s m^s} a_n a_m, \quad s > 1, \tag{4.1}$$

where $k \in \mathbb{N}$, $k \geqslant 1$. The following representation holds.

Proposition 4.1. *Consider the Dirichlet series $\psi(s) := \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ with $a_n \geqslant 0$ and assume it to be uniformly convergent on $(1,\infty)$. Then*

$$S(s) := \lim_{k\to\infty} S_k(s) = \psi''(s)\,\psi(s) - \left[\psi'(s)\right]^2 (\geqslant 0), \tag{4.2}$$

for $s \in (1,\infty)$.

Proof. It is obvious that

$$\psi'(s) = -\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n$$

and

$$\psi''(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot (\ln n)^2$$

for $s > 1$.

Now, observe that for $k \geqslant 1$

$$\begin{aligned} S_k(s) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\left[\frac{(\ln n)^2 + (\ln m)^2 - 2\ln n\cdot \ln m}{n^s m^s}\right] a_n a_m \\ &= \sum_{n=1}^{k}\frac{a_n}{n^s}\cdot(\ln n)^2 \sum_{n=1}^{k}\frac{a_n}{n^s} - \left(\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n\right)^2, \end{aligned}$$

and since

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot(\ln n)^2 = \psi''(s) \quad \text{and} \quad \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n = \psi'(s),$$

(4.2) holds true. □

We have the following result concerning the convexity property of $S(s)$.

Proposition 4.2. *The function $S(s) = \psi''(s)\,\psi(s) - [\psi'(s)]^2$ is log-convex on $(1,\infty)$.*

The proof follows by making use of the representation (4.2) and utilising the Hölder inequality for double sums. The details are omitted.

Theorem 4.1. *The inequality*

$$(0 \leqslant) \psi''(s)\psi(s) - [\psi'(s)]^2 \leqslant \psi(s-1)\psi(s+1) - [\psi(s)]^2, \tag{4.3}$$

holds true for any $s > 2$.

Proof. We use the following inequality between the geometric mean and the logarithmic mean of two positive numbers a, b, $a \neq b$,

$$\frac{b-a}{\ln b - \ln a} > \sqrt{ab},$$

to state that

$$\frac{\ln n - \ln m}{n-m} \leqslant \frac{1}{\sqrt{nm}} \quad \text{for } n, m \geqslant 1, \ n \neq m.$$

This obviously implies that

$$(\ln n - \ln m)^2 \leqslant \frac{(n-m)^2}{nm}$$

for each $n, m \geqslant 1$ and then from (4.1)

$$\begin{aligned} S_k(s) &\leqslant \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{n^{s+1}m^{s+1}}a_n a_m \\ &= \sum_{n=1}^{k}\frac{1}{n^{s-1}}a_n \cdot \sum_{n=1}^{k}\frac{a_n}{n^{s+1}} - \left(\sum_{n=1}^{k}\frac{a_n}{n^s}\right)^2, \end{aligned} \tag{4.4}$$

for each $k \in \mathbb{N}$, $k \geqslant 1$.

Since

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^s} = \psi(s)$$

for $s > 1$, by (4.4) we deduce the desired inequality (4.3). □

In [4], F. Topsøe obtained among other results, the following inequality for the logarithmic function

$$|\ln x| \leqslant \frac{1}{2}\left|x - \frac{1}{x}\right| \quad \text{for } x > 0. \tag{4.5}$$

We can state the following result based on (4.5).

Theorem 4.2. *We have the inequality*

$$(0 \leqslant) \psi''(s)\psi(s) - [\psi'(s)]^2 \leqslant \frac{1}{2}\left[\psi(s+2)\psi(s-2) - [\psi(s)]^2\right], \tag{4.6}$$

for any $s > 3$.

Proof. Making use of (4.5), we have

$$(\ln n - \ln m)^2 \leqslant \frac{1}{2}\left(\frac{n}{m} - \frac{m}{n}\right)^2 \quad \text{for } \; n, m \in \mathbb{N},\; n \neq m; n, m \geqslant 1$$

which together with (4.1) give

$$\begin{aligned} S_k(s) &\leqslant \frac{1}{4}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{n^4 - 2n^2m^2 + m^4}{n^{s+2}m^{s+2}}a_n a_m \\ &= \frac{1}{2}\left[\sum_{n=1}^{k}\frac{a_n}{n^{s-2}}\sum_{n=1}^{k}\frac{a_n}{n^{s+2}} - \left(\sum_{n=1}^{k}\frac{a_n}{n^s}\right)^2\right]. \end{aligned}$$

This implies the desired inequality (4.6). □

Remark 4.1. From (4.3) and (4.6), a computer comparison of the bounds

$$B_1(s) := \psi(s-1)\psi(s+1) - [\psi(s)]^2, \quad s > 2$$

and

$$B_2(s) := \frac{1}{2}\left[\psi(s+2)\psi(s-2) - [\psi(s)]^2\right], \quad s > 3$$

for $s > 3$ and $\psi = \zeta$ (Zeta function) shows that

$$B_2(s) \leqslant B_1(s) \text{ for all } s > 3.$$

However, we do not have an analytic proof for this inequality.

We have the following result as well.

Theorem 4.3. *The inequality*

$$(0 \leqslant)\, \psi(s+2)\psi(s) - [\psi(s+1)]^2 \leqslant \psi''(s)\psi(s) - \left[\psi'(s)\right]^2 \tag{4.7}$$

holds true for any $s > 1$.

Proof. We use the following elementary inequality for the logarithmic mean

$$\frac{b-a}{\ln b - \ln a} \leqslant \frac{a+b}{2}, \qquad a, b > 0 \quad (a \neq b)$$

which implies

$$\frac{\ln n - \ln m}{n - m} \geqslant \frac{2}{n+m} \quad \text{for } \; n, m \in \mathbb{N},\; n \neq m; n, m \geqslant 1.$$

This obviously implies

$$(\ln n - \ln m)^2 \geqslant \frac{4(n-m)^2}{(n+m)^2} \quad \text{for any } \; n, m \in \mathbb{N},\; n, m \geqslant 1.$$

Consequently, with the above notation, we have from (4.1)

$$S_k(s) \geqslant 2\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{(n+m)^2}\cdot\frac{1}{n^s m^s}a_n a_m \tag{4.8}$$
$$= 2\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{\left(\frac{1}{n}+\frac{1}{m}\right)^2}\cdot\frac{1}{n^{s+2}m^{s+2}}a_n a_m$$
$$\geqslant \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{n^{s+2}m^{s+2}}\cdot a_n a_m$$
$$=: L_k(s),$$

where we have used the fact that $\frac{1}{n}+\frac{1}{m}\leqslant 2$ for $n,m\geqslant 1$.

Observe that

$$L_k(s) = \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{n^2-2nm+m^2}{n^{s+2}m^{s+2}}a_n a_m \tag{4.9}$$
$$= \sum_{n=1}^{k}\frac{a_n}{n^{s+2}}\sum_{n=1}^{k}\frac{a_n}{n^s} - \left(\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\right)^2$$
$$= M_k(s).$$

Then, using (4.8) and (4.9) we deduce

$$S_k(s) \geqslant M_k(s) \quad \text{for} \quad k\geqslant 1 \text{ and } s>1. \tag{4.10}$$

Further, since

$$\lim_{k\to\infty} S_k(s) = \psi''(s)\psi(s) - \left[\psi'(s)\right]^2$$

and

$$\lim_{k\to\infty} M_k(s) = \psi(s+2)\psi(s) - \left[\psi(s+1)\right]^2$$

uniformly for $s>1$, (4.7) follows from (4.10). □

Remark 4.2. Theorem 4.3 provides a lower bound for $\psi''(s)\psi(s) - [\psi'(s)]^2$ whereas Theorems 4.1 and 4.2 give upper bounds.

5. Other inequalities for the first derivative

In this section we establish some bounds for the quantity

$$Q(s) := \frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)}, \quad s>1 \tag{5.1}$$

provided ψ is defined by the Dirichlet series

$$\psi(s) := \sum_{n=1}^{\infty}\frac{a_n}{n^s}, \quad s>1 \tag{5.2}$$

and ζ is the Zeta function.

We observe that if $(a_n)_{n\in\mathbb{N}}$ is nonnegative and monotonic nondecreasing (nonincreasing) then (see [1])

$$\frac{\zeta'(s)}{\zeta(s)} \geqslant (\leqslant) \frac{\psi'(s)}{\psi(s)} \quad \text{for} \quad s>1. \tag{5.3}$$

We have the following result as well.

Theorem 5.1. *If $(a_n)_{n\in\mathbb{N}}$ is nonnegative and nondecreasing, then we have the reverse inequality*

$$(0\leqslant)\frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)} \leqslant \frac{\psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right) - \psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right)}{\zeta(s)\psi(s)}, \tag{5.4}$$

for any $s > \frac{3}{2}$.

Proof. Consider the sequence

$$Q_k(s) := \frac{\sum_{n=1}^k \frac{a_n \ln n}{n^s} \cdot \sum_{n=1}^k \frac{1}{n^s} - \sum_{n=1}^k \frac{a_n}{n^s} \cdot \sum_{n=1}^k \frac{\ln n}{n^s}}{\zeta(s)\psi(s)}$$

for $k \geqslant 1$. We observe that for $s > 1$ the sequence $Q_n(s)$ is uniformly convergent and

$$\lim_{n\to\infty} Q_n(s) = Q(s) = \frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)}, \quad s>1.$$

Utilising Korkine's identity, we also have

$$Q_k(s) = \frac{1}{2}\cdot\frac{\sum_{n=1}^k\sum_{m=1}^k (a_n-a_m)(\ln n - \ln m)\frac{1}{n^s m^s}}{\sum_{n=1}^k \frac{1}{n^s}\cdot\sum_{n=1}^k \frac{a_n}{n^s}} \tag{5.5}$$

for $k\geqslant 1$, $s>1$.

Utilising the fact that (a_n) is monotonic nondecreasing and the elementary inequality

$$\frac{\ln n - \ln m}{n-m} \leqslant \frac{1}{\sqrt{nm}}, \quad n,m\geqslant 1,\ n\neq m,$$

we get

$$\begin{aligned} Q_k(s) &\leqslant \frac{1}{2}\cdot\frac{\sum_{n=1}^k\sum_{m=1}^k (a_n-a_m)(n-m)\frac{1}{n^{s+\frac{1}{2}}m^{s+\frac{1}{2}}}}{\sum_{n=1}^k \frac{1}{n^s}\cdot\sum_{n=1}^k \frac{a_n}{n^s}} \\ &= \frac{\sum_{n=1}^k \frac{a_n\cdot n}{n^{s+\frac{1}{2}}}\cdot\sum_{n=1}^k \frac{1}{n^{s+\frac{1}{2}}} - \sum_{n=1}^k \frac{a_n}{n^{s+\frac{1}{2}}}\cdot\sum_{n=1}^k \frac{n}{n^{s+\frac{1}{2}}}}{\sum_{n=1}^k \frac{1}{n^s}\cdot\sum_{n=1}^k \frac{a_n}{n^s}} \\ &=: V_k(s), \quad s>1. \end{aligned} \tag{5.6}$$

Since

$$\lim_{k\to\infty} V_k(s) = \frac{\psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right) - \psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right)}{\zeta(s)\psi(s)}$$

for $s>\frac{3}{2}$, we deduce by (5.6) the desired result (5.4). ∎

The following upper bound for $Q(s)$, $s>1$, can be established as well.

Theorem 5.2. *With the assumptions of Theorem 5.1, we have*

$$(0\leqslant)\,\frac{\zeta'(s)}{\zeta(s)}-\frac{\psi'(s)}{\psi(s)}\leqslant\frac{1}{2}\cdot\left[\frac{\psi(s-1)\zeta(s+1)-\psi(s+1)\zeta(s-1)}{\zeta(s)\psi(s)}\right] \tag{5.7}$$

for any $s>2$.

Proof. From Inequality (4.9) we have

$$\frac{\ln n-\ln m}{n-m}\leqslant\frac{n+m}{2nm},\quad \text{for any } n,m\geqslant 1,\ n\neq m.$$

Therefore (5.5) implies that

$$\begin{aligned}Q_k(s)&\leqslant\frac{1}{4}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\frac{n+m}{n^{s+1}m^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}} \\ &=\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\frac{a_n\cdot n^2}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{1}{n^{s+1}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{n^2}{n^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}} \\ &=: W_k(s),\quad s>1.\end{aligned} \tag{5.8}$$

Since

$$\lim_{k\to\infty}W_k(s)=\frac{1}{2}\cdot\frac{\psi(s-1)\zeta(s+1)-\psi(s+1)\zeta(s-1)}{\zeta(s)\psi(s)}$$

for $s>1$, inequality (5.8) implies the desired result (5.7). ∎

Finally, we have the following refinement of the inequality (5.3).

Theorem 5.3. *With the assumptions of Theorem 5.1, we have the inequality*

$$0\leqslant\frac{\zeta(s+1)}{\zeta(s)}-\frac{\psi(s+1)}{\psi(s)}\leqslant\frac{\zeta'(s)}{\zeta(s)}-\frac{\psi'(s)}{\psi(s)}, \tag{5.9}$$

for $s>1$.

Proof. Utilising the inequality

$$\frac{\ln n-\ln m}{n-m}\leqslant\frac{2}{n+m},\quad \text{for } n,m\in\mathbb{N},\ n\neq m,\ n,m\geqslant 1,$$

we have

$$\begin{aligned}Q_k(s)&\geqslant\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\cdot\frac{2}{n+m}\cdot\frac{1}{n^sm^s}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}} \\ &\geqslant\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\cdot\frac{1}{n^{s+1}m^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}} \\ &=Z_k(s)\end{aligned} \tag{5.10}$$

since for $n,m>1$,

$$\frac{2}{n+m}=\frac{2}{nm\left(\frac{1}{n}+\frac{1}{m}\right)}\geqslant\frac{1}{nm}.$$

Observe that

$$\begin{aligned}Z_k(s) &= \frac{\sum_{n=1}^{k} \frac{a_n \cdot n}{n^{s+1}} \cdot \sum_{n=1}^{k} \frac{1}{n^{s+1}} - \sum_{n=1}^{k} \frac{a_n}{n^{s+1}} \cdot \sum_{n=1}^{k} \frac{n}{n^{s+1}}}{\sum_{n=1}^{k} \frac{1}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^s}} \\ &= \frac{\sum_{n=1}^{k} \frac{a_n}{n^{s}} \cdot \sum_{n=1}^{k} \frac{1}{n^{s+1}} - \sum_{n=1}^{k} \frac{a_n}{n^{s+1}} \cdot \sum_{n=1}^{k} \frac{n}{n^{s+1}}}{\sum_{n=1}^{k} \frac{1}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^s}}\end{aligned}$$

for $k \geqslant 1$, and

$$\begin{aligned}\lim_{k\to\infty} Z_k(s) &= \frac{\zeta(s+1)\,\psi(s) - \psi(s+1)\,\zeta(s)}{\psi(s)\,\zeta(s)} \\ &= \frac{\zeta(s+1)}{\zeta(s)} - \frac{\psi(s+1)}{\psi(s)}.\end{aligned}$$

Hence by (5.10) we deduce the desired result (5.9). □

Remark 5.1. Inequalities (5.4), (5.7) and (5.9) are obviously equivalent to

$$\begin{aligned}&(0 \leqslant)\zeta'(s)\,\psi(s) - \psi'(s)\,\zeta(s) \\ &\leqslant \psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right) - \psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right), \quad s > \frac{3}{2}\end{aligned} \tag{5.11}$$

$$\begin{aligned}&(0 \leqslant)\zeta'(s)\,\psi(s) - \psi'(s)\,\zeta(s) \\ &\leqslant \frac{1}{2}\left[\psi(s-1)\,\zeta(s+1) - \psi(s+1)\,\zeta(s-1)\right], \quad s > 2\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}&(0 \leqslant)\zeta(s+1)\,\psi(s) - \psi(s+1)\,\zeta(s) \\ &\leqslant \zeta'(s)\,\psi(s) - \psi'(s)\,\zeta(s), \quad s > 1,\end{aligned} \tag{5.13}$$

respectively.

Now, consider $\psi(s) := \sum_{n=1}^{\infty} \frac{\ln n}{n^s}$, $s > 1$. We observe that this Dirichlet series satisfies the assumptions of Theorem 5.1. Further, $\psi(s) = -\zeta'(s)$, $s > 1$. Therefore, by (5.11), (5.12) and (5.13) we have the inequalities

$$\begin{aligned}&(0 \leqslant)\zeta''(s)\,\zeta(s) - \left[\zeta'(s)\right]^2 \\ &\leqslant \zeta'\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right) - \zeta'\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right), \quad s > \frac{3}{2}\end{aligned} \tag{5.14}$$

$$\begin{aligned}&(0 \leqslant)\zeta''(s)\,\zeta(s) - \left[\zeta'(s)\right]^2 \\ &\leqslant \frac{1}{2}\left[\zeta'(s+1)\,\zeta(s-1) - \zeta'(s-1)\,\zeta(s+1)\right], \quad s > 2\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}&(0 \leqslant)\zeta'(s+1)\,\zeta(s) - \zeta(s+1)\,\zeta'(s) \\ &\leqslant \zeta''(s)\,\zeta(s) - \left[\zeta'(s)\right]^2, \quad s > 2,\end{aligned} \tag{5.16}$$

respectively.

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