ACTA MATHEMATICA VIETNAMICA
Volume 33, Number 1, 2008, pp. 53-64

LIPSCHITZ ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS ON HARDY TYPE SPACES

LIU LANZHE

Abstract. In this paper, the boundedness for some multilinear operators generated by Littlewood-Paley operators and Lipschitz functions on Hardy and Herz-Hardy spaces is obtained.

1. Introduction and results

In this paper, we will consider a class of multilinear operators related to Littlewood-Paley operators, whose definitions are as follows.

Let m be a positive integer and A be a function on R^n. We denote

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

Fix $\delta > 0$, $\varepsilon > 0$ and $\mu > 1$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leqslant C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

Denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Littlewood-Paley operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

Received January 5, 2007.
1991 *Mathematics Subject Classification.* 42B20, 42B25.
Key words and phrases. Littlewood-Paley operator; Multilinear operator; Hardy space; Herz space; Lipschitz space.

and

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. The variants of g_ψ^A, S_ψ^A and g_μ^A are defined by

$$\tilde{g}_\psi^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$\tilde{S}_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

and

$$\tilde{g}_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z)f(z)dz.$$

Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [14]).

Note that for $m = 0$ and $\delta = 0$, g_ψ^A, S_ψ^A and g_μ^A are just the commutator of Littlewood-Paley operators (see [1], [9]), while for $m > 0$ they are nontrivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$ (see [3-6]). In [2] and [15], the authors obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions. The main purpose of this paper is to discuss the boundedness properties of the multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces.

Let us introduce some more definitions (see [7], [11], [12], [13]). Throughout this paper, $M(f)$ denotes the Hardy-Littlewood maximal function of f, Q denotes a cube in R^n with sides parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leqslant 1)$ has the atomic decomposition characterization (see [14]). The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} |f(x+h) - f(x)|/|h|^\beta < \infty,$$

where $\beta > 0$ (see [13]).

Definition 1.1. Let $0 < p$, $q < \infty$, $\alpha \in R$, $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $C_k = B_k \setminus B_{k-1}, k \in Z$. Denote $\chi_k = \chi_{C_k}$ for $k \in Z$ and $\chi_0 = \chi_{B_0}$, where χ_E is the characteristic function of a set E.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_0||_{L^q}^p\right]^{1/p}.$$

Definition 1.2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 1.3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type) if

1) $\mathrm{Supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geqslant 1$),
2) $||a||_{L^q} \leqslant |B(0, r)|^{-\alpha/n}$,
3) $\int a(x)x^\gamma dx = 0$ for $|\gamma| \leqslant [\alpha - n(1 - 1/q)]$.

Lemma 1.1 (see [12])**.** *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geqslant n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(or||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Now we can state our results as follows.

Theorem 1.1. *Let $0 < \beta \leqslant 1$, $0 \leqslant \delta < n-\beta$, $\max(n/(n+\beta), n/(n+\varepsilon)) < p \leqslant 1$ and $1/p - 1/q = (\delta + \beta)/n$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$, then g_ψ^A, S_ψ^A and g_μ^A are all bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Theorem 1.2. *Let $0 < \beta < \min(1, \varepsilon)$, $0 \leqslant \delta < n - \beta$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$, then $\tilde{g}_\psi^A$, $\tilde{S}_\psi^A$ and $\tilde{g}_\mu^A$ are all bounded from $H^{n/(n+\beta)}(R^n)$ to $L^{n/(n-\delta)}(R^n)$.*

Theorem 1.3. *Let $0 < \beta < \min(1, \varepsilon)$, $0 < \delta < n - \beta$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$, then g_ψ^A, S_ψ^A and g_μ^A are all bounded from $H^{n/(n+\beta)}(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.*

Theorem 1.4. *Let $0 < \beta \leqslant 1$, $0 < \delta < n - \beta$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (\delta+\beta)/n$ and $n(1-1/q_1) \leqslant \alpha < \min(n(1-1/q_1)+\beta, n(1-1/q_1)+\varepsilon)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$, then g_ψ^A, S_ψ^A and g_μ^A are all bounded from $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ to $\dot{K}_{q_2}^{\alpha,p}(R^n)$.*

Remark 1. Theorem 1.4 also holds for the nonhomogeneous Herz type Hardy space.

2. Some lemmas

For proving the above results we need the following two lemmas.

Lemma 2.1 (see [5])**.** *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leqslant C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2.2. *Let $0 < \beta \leqslant 1$, $1 < p < n/(\delta+\beta)$, $1/q = 1/p - (\delta+\beta)/n$ and $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then g_ψ^A, S_ψ^A and g_μ^A are all bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof. By the Minkowski inequality and the condition on ψ,

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta}.$$

For g_ψ^A we have

$$\begin{aligned} g_\psi^A(f)(x) &\leqslant \int_{R^n} \frac{|f(y)||R_{m+1}(A;x,y)|}{|x-y|^m} \left(\int_0^\infty |\psi_t(x-y)|^2 \frac{dt}{t}\right)^{1/2} dy \\ &\leqslant C\int_{R^n} \frac{|f(y)||R_{m+1}(A;x,y)|}{|x-y|^m} \left(\int_0^\infty \frac{t^{-2n+2\delta}}{(1+|x-y|/t)^{2(n+1-\delta)}} \frac{dt}{t}\right)^{1/2} dy \\ &\leqslant C\int_{R^n} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n-\delta}} |f(y)|dy. \end{aligned}$$

For S_ψ^A, noting that $2t+|y-z| \geqslant 2t+|x-z|-|x-y| \geqslant t+|x-z|$ when $|x-y| \leqslant t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$\begin{aligned} & S_\psi^A(f)(x) \\ \leqslant\ & C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\leqslant t} \frac{2^{2n+2-2\delta}\cdot t^{1-n}}{(2t+|y-z|)^{2n+2-2\delta}} dydt\right)^{1/2} dz \\ \leqslant\ & C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2} dz \\ \leqslant\ & C\int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz. \end{aligned}$$

For g_μ^A, noting that

$$\begin{aligned} t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{dy}{(t+|y-z|)^{2n+2-2\delta}} &\leqslant CM\left(\frac{1}{(t+|x-z|)^{2n+2-2\delta}}\right) \\ &\leqslant C\frac{1}{(t+|x-z|)^{2n+2-2\delta}}, \end{aligned}$$

we obtain

$$\begin{aligned}
& g_\mu^A(f)(x) \\
\leqslant\ & C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \\
& \times\left(\int_0^\infty\int_{R^n}\frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dydt}{t^{1+n}}\right)^{1/2}dz \\
\leqslant\ & C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \\
& \times\left[\int_0^\infty\left(t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)tdt\right]^{1/2}dz \\
\leqslant\ & C\int_{R^n}\frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2}dz \\
\leqslant\ & C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}dz.
\end{aligned}$$

Thus, the lemma follows from [2]. □

3. Proofs of the theorems

Proof of Theorem 1.1. For simplicity, we denote $T^A = g_\psi^A$ or S_ψ^A or g_μ^A. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||T^A(a)||_{L^q} \leqslant C.$$

Let a be a H^p-atom, that is a is supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \leqslant |Q|^{-1/p}$ and $\int a(x)x^\gamma dx = 0$ for $|\gamma| \leqslant [n(1/p-1)]$. We write

$$\int_{R^n}[T^A(a)(x)]^q dx = \left(\int_{|x-x_0|\leqslant 2r} + \int_{|x-x_0|>2r}\right)[T^A(a)(x)]^q dx = I + II.$$

For I, taking $1 < p_1 < n/(\delta+\beta)$ and q_1 such that $1/p_1 - 1/q_1 = (\delta+\beta)/n$, by Holder's inequality and the (L^{p_1}, L^{q_1})-boundedness of T^A (see Lemma 2.2), we see that

$$I \leqslant C||T^A(a)||_{L^{q_1}}^q|2Q|^{1-q/q_1} \leqslant C||a||_{L^{p_1}}^q|Q|^{1-q/q_1} \leqslant C.$$

To obtain the estimate of II, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{Q} = 5\sqrt{n}Q$ and

$$\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha.$$

Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}(y) = D^\alpha A(y) - (D^\alpha A)_Q$.

For g_ψ^A, by the vanishing moment of a, we write

$$\begin{aligned} F_t^A(a)(x) &= \int_{R^n}\left[\frac{\psi_t(x-y)R_m(\tilde{A};x,y)}{|x-y|^m}-\frac{\psi_t(x-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]a(y)dy \\ &\quad -\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{\psi_t(x-y)(x-y)^\alpha D^\alpha\tilde{A}(y)}{|x-y|^m}a(y)dy. \end{aligned}$$

By Lemma 2.1 and the inequality

$$|b(x)-b_Q|\leqslant\frac{1}{|Q|}\int_Q||b||_{Lip_\beta}|x-y|^\beta dy\leqslant||b||_{Lip_\beta}(|x-x_0|+r)^\beta,$$

we get

$$|R_m(\tilde{A};x,y)|\leqslant\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}(|x-y|+r)^{m+\beta}.$$

By the formula (see [5])

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x_0,y)(x-x_0)^\eta,$$

noting that $|x-y|\sim|x-x_0|$ for $y\in Q$ and $x\in R^n\setminus Q$, similarly as in the proof of Lemma 2.2, we obtain

$$\begin{aligned} g_\psi^A(a)(x) &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\int_{R^n}[\frac{|y-x_0|}{|x-x_0|^{n+1-\delta-\beta}}+\frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\delta-\beta}} \\ &\quad +\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|-\delta}}+\frac{|y-x_0|^\beta}{|x-x_0|^{n-\delta}}]|a(y)|dy \\ &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta-\beta}}\right]. \end{aligned}$$

For S_ψ^A, by the vanishing moment of a, we write

$$\begin{aligned} F_t^A(a)(x,y) &= \int_{R^n}\left[\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{\psi_t(y-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]dz \\ &\quad -\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{\psi_t(y-z)(x-z)^\alpha D^\alpha\tilde{A}(z)}{|x-z|^m}a(z)dz. \end{aligned}$$

Similarly as in the proof of Lemma 2.2, we have

$$\begin{aligned} S_\psi^A(a)(x) &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\int_{R^n}[\frac{|y-x_0|}{|x-x_0|^{n+1-\delta-\beta}}+\frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\delta-\beta}} \\ &\quad+\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|-\delta}}+\frac{|y-x_0|^\beta}{|x-x_0|^{n-\delta}}]|a(y)|dy \\ &\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta-\beta}}\right].\end{aligned}$$

For g_μ^A, it holds

$$g_\mu^A(a)(x)\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta-\beta}}\right].$$

Thus,

$$\begin{aligned} II &\leqslant \sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}[T^A(a)(x)]^q dx \\ &\leqslant C\left(\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\right)^q\sum_{k=1}^\infty[2^{kqn(1/p-(n+\beta)/n)}+2^{kqn(1/p-(n+\varepsilon)/n)}] \\ &\leqslant C\left(\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\right)^q,\end{aligned}$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We only give the proof of the boundedness of $\tilde g_\psi^A$. The proofs of the boundedness of $\tilde S_\psi^A$ and $\tilde g_\mu^A$ are omitted for their similarity to the first one. It suffices to show that there exists a constant $C>0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q=Q(x_0,r)$, we have

$$||\tilde g_\psi^A(a)||_{L^{n/(n-\delta)}}\leqslant C.$$

Write

$$\int_{R^n}[\tilde g_\psi^A(a)(x)]^{n/(n-\delta)}dx=\left[\int_{|x-x_0|\leqslant 2r}+\int_{|x-x_0|>2r}\right][\tilde g_\psi^A(a)(x)]^{n/(n-\delta)}dx:=J+JJ.$$

For J, by the equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y)),$$

we have, similarly as in the proof of Lemma 2.2,

$$\tilde{g}_\psi^A(a)(x) \leqslant g_\psi^A(a)(x) + C\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|a(y)|dy,$$

thus $\tilde{g}_\psi^A$ is (L^p, L^q)-bounded by Lemma 2.1 and [8], where $1 < p < n/(\delta+\beta)$ and $1/q = 1/p - (\delta+\beta)/n$. We see that

$$J \leqslant C||\tilde{g}_\psi^A(a)||_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)} \leqslant C||a||_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)} \leqslant C.$$

To obtain the estimate of JJ, we denote

$$\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{2Q}x^\alpha.$$

Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. For $x \in (2Q)^c$, by the vanishing moment of a and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x),$$

we can write

$$\begin{aligned}
&\tilde{F}_t^A(a)(x)\\
= &\int_{R^n}\frac{\psi_t(x-y)R_m(\tilde{A};x,y)}{|x-y|^m}a(y)dy\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{\psi_t(x-y)D^\alpha\tilde{A}(x)(x-y)^\alpha}{|x-y|^m}a(y)dy\\
= &\int_{R^n}\left[\frac{\psi_t(x-y)R_m(\tilde{A};x,y)}{|x-y|^m}-\frac{\psi_t(x-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]a(y)dy\\
&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{\psi_t(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right]\\
&\times D^\alpha\tilde{A}(x)a(y)dy.
\end{aligned}$$

Thus, similarly to the proof of Theorem 1.1, for each $x \in (2Q)^c$ we obtain

$$\begin{aligned}
|\tilde{g}_\psi^A(a)(x)| \leqslant\ & C|Q|^{-\beta/n}\sum_{|\alpha|=m}\Bigg(||D^\alpha A||_{Lip_\beta}\left[\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta-\beta}}+\frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon-\delta-\beta}}\right]\\
&+|D^\alpha\tilde{A}(x)|\left[\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon-\delta}}\right]\Bigg),
\end{aligned}$$

so that,

$$JJ \leqslant C\left(\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\right)^{n/(n-\delta)}\sum_{k=1}^\infty[2^{kn(\beta-1)/(n-\delta)}+2^{kn(\beta-\varepsilon)/(n-\delta)}] \leqslant C,$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. We only give the proof for g_ψ^A. By the equality

$$R_{m+1}(A;x,y)=Q_{m+1}(A;x,y)+\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y)),$$

as in the proof of Lemma 2.2 we get

$$g_\psi^A(f)(x)\leqslant \tilde{g}_\psi^A(f)(x)+C\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|f(y)|dy.$$

From Theorems 1.1, 1.2 and from [8] we obtain

$$\begin{aligned}
&|\{x\in R^n: g_\psi^A(f)(x)>\lambda\}|\\
\leqslant\ &|\{x\in R^n:\tilde{g}_\psi^A(f)(x)>\lambda/2\}|\\
&+|\{x\in R^n:\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^{n-\delta}}|f(y)|dy>C\lambda\}|\\
\leqslant\ &C(||f||_{H^{n/(n+\beta)}}/\lambda)^{n/(n-\delta)}.
\end{aligned}$$

This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4. We only give the proof for g_ψ^A. Let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x)=\sum_{j=-\infty}^\infty\lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1.1. We write

$$\begin{aligned}
||g_\psi^A(f)||_{\dot{K}_{q_2}^{\alpha,p}}^p &\leqslant \sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||g_\psi^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\
&\quad+\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||g_\psi^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\
&= L_1+L_2.
\end{aligned}$$

For L_2, by the (L^{q_1},L^{q_2}) boundedness of g_ψ^A (see Lemma 2.2), we have

$$\begin{aligned}
L_2 &\leqslant C\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=k-2}^\infty|\lambda_j|||a_j||_{L^{q_1}}\right)^p\\
&\leqslant C\sum_{j=-\infty}^\infty|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right)
\end{aligned}$$

whenever $0 < p \leqslant 1$. If $p > 1$, then

$$\begin{aligned} L_2 &\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2} \right)^{p/p'} \\ &\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leqslant C||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}. \end{aligned}$$

For L_1, similarly as in the proof of Theorem 1.1, we have

$$\begin{aligned} g^A_\psi(a_j)(x) &\leqslant C \left(\frac{|B_j|^{\beta/n}}{|x|^{n-\delta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-\beta}} \right) \int |a_j(y)| dy \\ &\leqslant C \left(2^{j(\beta+n(1-1/q_1)-\alpha)} |x|^{\delta-n} + 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} |x|^{\delta+\beta-n-\varepsilon} \right) \end{aligned}$$

whenever $x \in C_k$, $j \leqslant k-3$. Thus

$$||g^A_\psi(a_j)\chi_k||_{L^{q_2}} \leqslant C 2^{-k\alpha} (2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)})$$

and

$$\begin{aligned} L_1 &\leqslant C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| (2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)} \right)^p \\ &\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} \left(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)} \right)^p \end{aligned}$$

whenever $0 < p \leqslant 1$. If $p > 1$, then

$$\begin{aligned} L_1 &\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left[\sum_{k=j+3}^{\infty} \left(2^{(j-k)p(\beta+n(1-1/q_1)-\alpha)/2} + 2^{(j-k)p(\varepsilon+n(1-1/q_1)-\alpha)/2} \right) \right] \\ &\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \\ &\leqslant C||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}. \end{aligned}$$

This finishes the proof of Theorem 1.4. $\square$

References

[1] J. Alvarez, R. J. Babgy, D. S. Kurtz and C. Perez, *Weighted estimates for commutators of linear operators*, Studia Math. **104** (1993), 195-209.

[2] W. G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica **16** (2000), 613-626.

[3] J. Cohen, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J. **30** (1981), 693-702.

[4] J. Cohen and J. Gosselin, *On multilinear singular integral operators on R^n*, Studia Math. **72** (1982), 199-223.

[5] J. Cohen and J. Gosselin, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math. **30** (1986), 445-465.

[6] Y. Ding and S. Lu, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica **17** (2001), 517-526.
[7] J. Garcia-Cuerva and M. J. L. Herrero, *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc. **69** (1994), 605-628.
[8] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math. **16** (1978), 263-270.
[9] L. Z. Liu, *Continuity for commutators of Littlewood-Paley operator on certain Hardy spaces*, J. Korean Math. Soc. **40** (2003), 41-60.
[10] S. Z. Lu, Q. Wu and D. C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. in China(ser. A) **45** (2002), 984-997.
[11] S. Z. Lu and D. C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (ser. A) **38** (1995), 147-158.
[12] S. Z. Lu and D. C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. in China (ser. A) **38** (1995), 662-673.
[13] M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. **44** (1995), 1-17.
[14] A. Torchinsky, *The Real Variable Methods in Harmonic Analysis*, Pure and Applied Mathematics **123**, Academic Press, 1986.
[15] Q. Wu, *Lipschitz estimates for multilinear singular integrals*, preprint.

College of Mathematics
Changsha University of Science and Technology
Changsha 410077, P.R. of China

E-mail address: lanzheliu@163.com