ACTA MATHEMATICA VIETNAMICA
Volume 33, Number 1, 2008, pp. 39-44

ON A PROBLEM BY N. KALTON AND THE SPACE $l^p(I)$

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Abstract. We prove that each infinite dimensional quasi-Banach space has a closed, proper infinite dimensional subspace, so answering the Atomic Problem by N. Kalton in the affirmative. We obtain applications concerning the structure of the separable quasi-Banach spaces and of $l^p(I), 0 < p < 1$.

1. Introduction

The Atomic Problem of N. Kalton in [4] is the question whether each infinite dimensional quasi-Banach space X always has an infinite dimensional, proper closed subspace or not. In this paper we obtain the answer in the affirmative. In the Sec. 2, we give definitions and properties that we use to prove Theorem 3.1 and Theorem 3.2 in Sec. 3. Corollary 3.1 contains the result mentioned in the abstract and we obtain corollaries on the structure of the quasi-Banach space $l^p(I)$, I a set of indices whose cardinality is that of the continuum. All vector spaces are assumed to be either real or complex.

2. Preliminaries

Definition 2.1 ([5])**.** For a vector space X over $\mathbf{K}$ a real functional q on X having the properties

$$(qn1)\ q(x) \geqslant 0,\ q(x) = 0 \Leftrightarrow x = 0,$$

$$(qn2)\ q(\lambda x) = \mid \lambda \mid q(x),$$

$$(qn3)\ q(x+y) \leqslant c(q(x) + q(y)),$$

where $c \geqslant 1$ is a certain constant, for $x, y \in X, \lambda \in \mathbf{K}$ is said to be a quasi-norm on X. The pair (X, q) is said to be a quasi-normed space and the smallest suitable constant c in $(qn3)$ is the quasi-norm constant.

Remark 1. A quasi-normed space (X, q) is a metrizable topological vector space.

Proof. It follows easily from Theorem 9.2 and Theorem 13.1 in [11] (pp. 96, 115) that the class $\{\varepsilon U_q : \varepsilon > 0\}$, where $U_q = \{x \in X : q(x) \leqslant 1\}$ is the base of neighborhoods of 0 for a metrizable linear topology on X. □

Received April 5, 2007
2001 *Mathematics Subject Classification.* Primary 46A06, 46A10.
Key words and phrases. atom, quasi-norm, p-norm.

Definition 2.2 ([5])**.** For a vector space X the functional $\|.\| : X \to [0, +\infty)$ is said to be a p-norm on X where, $0 < p \leqslant 1$ if the following conditions are satified for $x, y \in X, \lambda \in \mathbf{K}$:

$$(1)\ \|x\| = 0 \Leftrightarrow x = 0,$$

$$(2)\ \|\lambda x\| = \mid \lambda \mid \|x\|,$$

$$(3)\ \|x + y\|^p \leqslant \|x\|^p + \|y\|^p.$$

Lemma 2.1. *Any p-norm on X is a quasi-norm, where we may take $c = 2^{1/p}$ for the constant in Definition 1. Conversely, if (X, q) is a quasi-normed space and c is the quasi-norm constant, there is a p-norm $\|.\|$ on X such that $\|.\| \leqslant q \leqslant 2^{1/p}\|.\|$, where $c = 2^{1/p-1}$.*

Proof. The first assertion follows easily. See [5], p. 47 to obtain the p-norm $\|.\|$ on X. □

Remark 2. The p-norm $\|.\|$ as in Propositon 1 defining a linear topology on X through the base of neighborhoods of zero $\{\varepsilon U : \varepsilon > 0\}$, where $U = \{x \in X : \|x\| \leqslant 1\}$, this topology coincides with the topology of (X, q) as in Remark 1.

Proof. Analogously as in the proof of Remark 1 $\{\varepsilon U : \varepsilon > 0\}$ defines a linear metrizable topology on X. The remark follows from the fact that $\|x_n\| \to 0$ if and only if $q(x_n) \to 0$, where (x_n) is a sequence in X. □

In what follows, (X, q) being a quasi-normed space, we consider X equipped with the related p-norm $\|.\|$ as in Lemma 2.1. We also denote the space by $(X, \|.\|)$.

Definition 2.3 ([5])**.** A complete quasi-normed space (X, q) is called a quasi-Banach space. Letting $\|.\|$ stand for the p-norm on X as in Lemma 2.1, we also say that $(X, \|.\|)$ is a p-Banach space.

Example 1 ([8])**.** The space $L^p[0, 1[$, $0 < p < 1$ consisting of the scalar valued Lebesgue measurable functions f on $[0, 1[$, such that $\int_0^1 \mid f(t) \mid^p dt < \infty$, is a quasi-Banach space when equipped with the p-norm $\|f\| = (\int_0^1 \mid f(t) \mid^p)^{1/p}$.

Example 2 ([6])**.** Let $0 < p < 1$. The collection of all scalar-valued families $(\alpha_i)_{i\in[0,1]}$ such that $\sum_i \mid \alpha_i \mid^p < \infty$, where $\sum_i \mid \alpha_i \mid^p = s < \infty$, if for each $\varepsilon > 0$ we can find a finite subset F of $\mathbf{R}$ such that $\mid \sum_{i\in F} \mid \alpha_i \mid^p - s \mid \leqslant \varepsilon$, is a p-Banach space when equipped with the p-norm $\|(\alpha_i)\|_p = (\sum_i \mid \alpha_i \mid^p)^{1/p}$. We denote this p-Banach space by $l^p[0, 1]$.

Definition 2.4 ([8])**.** We say that a complete metrizable topological vector space is an F-space.

Remark 3. Each quasi-Banach space X is an F-space. If N is a closed subspace of X then the quotient X/N is an F-space, too.

Proof. The first assertion follows from above. The second one follows from Theorem 1.41 (d) in [8] (p.29). □

Lemma 2.2. *Each continuous linear bijection from an F-space onto another is a linear homeomorphism.*

Proof. This follows from Theorem 2.11 in [8] (p. 47). □

Recall [11] that, X being a vector space, $\mathbf{K}$ the scalar field, a Hamel basis of X is an infinite linearly independent set of vectors $\{u_\alpha : \alpha \in \mathcal{A}\} \subset X$ such that, each $x \in X$ has a unique representation as a finite sum $x = \sum_{n=1}^{n(x)} \lambda_n u_n$ ($n(x)$ depends on x). For s_α being nonzero scalars, $\{s_\alpha u_\alpha : \alpha \in \mathcal{A}\}$ is also a Hamel basis. A Hamel basis always exists; also all Hamel bases of a vector space have the same cardinality.

Remark 4. Denoting by $\dim X$ the cardinality of a Hamel basis of the vector space X and c the cardinality of the continuum, we have that the cardinality of X is c if $\dim X < c$ and it is $\dim X$ if $c \leqslant \dim X$.

Proof. This follows from 2. in [11] (p.43). □

Remark 5. For (X, q) a separable quasi-Banach space, each Hamel basis of X has the cardinality of the continuum.

Proof. We first prove that X has no countable Hamel basis. In fact, if $\{u_n : n \in \mathbf{N}\}$ is a countably infinite Hamel basis of X, then $e_n = u_n/\|u_n\|$ where, $\|.\|$ is the p-norm on X as in Lemma 2.1, also form a Hamel basis. The series $\sum_{n=1}^{\infty} e_n/2^n$ is convergent in the subspace $E = \{\sum_{n=1}^{\infty} \lambda_n e_n : \sum_{n=1}^{\infty} \mid \lambda_n \mid^p < \infty\}$. We may consider the p-norm $\|.\|_p$ on E defined by $\| \sum_{n=1}^{\infty} \lambda_n e_n \|_p = (\sum_{n=1}^{\infty} \mid \lambda_n \mid^p)^{1/p}$. Clearly that each linear functional $f_k(\sum_{n=1}^{\infty} \lambda_n e_n) = \lambda_k$ $(k = 1, 2, ...)$ is continuous on $(E, \|.\|_p)$ and, the identity injection of $(E, \|.\|_p)$ to $(E, \|.\|)$ is continuous. Hence we have that $f_k(x) = 1/2^k \neq 0$ for infinitely many k, where $x = \sum_{n=1}^{\infty} e_n/2^n$ and it follows that x is not a finite linear combination of the vectors e_n, contradiction. Since (X, q) is separable, there exists a countably infinite dense subset $S = \{x_n : n \in \mathbf{N}\}$. It follows that the cardinality of X is not greater than the cardinality of the set of all sequences in S, that is, of the continuum. Applying Remark 4, the case where $c \leqslant \dim X$ we conclude that the cardinality of X is c, hence $\dim X = c$. □

Remark 6. The linear span of the set of families $\{(\delta_{k,i})_{i \in [0,1]} : k \in [0, 1]\}, \delta_{k,i}$ being the Kronecker symbol, is dense in the p-Banach space $l^p[0, 1]$.

Proof. It is clear that $\sum_i \mid \alpha_i \mid = \sup\{\sum_{i \in F} \mid \alpha_i \mid : F \subset [0, 1], F \text{ finite}\}$ for $(\alpha_i)_{i \in [0,1]} \in l^p[0, 1]$, hence it follows by [7] (20., p. 38) that $(\alpha_i) \in l^p[0, 1]$ only if the set of indices i such that $\alpha_i \neq 0$ is countable. Now the limit in $l^p[0, 1]$ of the family $(\alpha_i \chi_F(i))$, where $\chi_F(i) = 1$ $(i \in F), \chi_F(i) = 0$ $(i \notin F)$ and F ranges over the set of finite subsets of $[0, 1]$, is precisely $(\alpha_i)_{i \in [0,1]}$ and the remark follows. □

Remark 7. The space $l^p[0,1]$ is the closure of a subspace having a Hamel basis whose cardinality is that of the continuum.

Proof. This follows from Remark 6, since the set $\{(\delta_{k,i})_{i\in[0,1]} : k \in [0,1]\}$ is a linearly independent subset of $l^p[0,1]$ which has cardinality c. $\square$

3. The Results

Theorem 3.1. *If (X,q) is a non separable quasi-Banach space, then X has a proper closed infinite dimensional subspace.*

Proof. If X is not separable it follows that X is not the closure of any subspace Y that has a countable Hamel basis (otherwise the set of all linear combinations of vectors in the basis with rational coefficients would be a countable dense subset of X). Hence we may consider an uncountable Hamel basis $\mathcal{B} = \{e_\alpha : \alpha \in \mathcal{A}\}$ of X. Some countably infinite subset B of $\mathcal{B}$ is a Hamel basis of $span(B)$ and $\overline{span(B)}$ is such a closed, proper subspace of X. $\square$

Theorem 3.2. *Let (X,q) be an infinite dimensional separable quasi-Banach space. Then there exists a proper closed infinite dimensional subspace of X.*

Proof. Consider a separable infinite dimensional quasi-Banach space (X,q). Let $\|.\|$ be the p-norm on X defining the topology as in Lemma 2.1, $0 < p < 1$. Following Remark 5, X has a Hamel basis $\{e_k : k \in [0,1]\}$ such that $\|e_k\| = 1$. Denoting $u_k = (\delta_{k,i})_{i\in[0,1]}$ in the proof of Remark 7, the linear operator $T\sum_{n=1}^m \lambda_{k(n)}u_{k(n)} = \sum_{n=1}^m \lambda_{k(n)}e_{k(n)}$ $(k(1),...,k(m) \in [0,1])$ is a continuous surjection on $span\{u_k : k \in [0,1]\}$ equipped with the induced topology onto $(X,\|.\|)$. Therefore, following [10] (pp.18, 78), it extends to the continuous linear surjection $\widetilde{T} : (l^p[0,1],\|.\|_p) \to (X,\|.\|)$. Hence the quotient map $\widetilde{T}/N(\widetilde{T}) : l^p[0,1]/N(\widetilde{T}) \to (X,\|.\|)$ is a linear homeomorphism. Let $f : l^p[0,1] \to \mathbf{K}$, $f(\alpha_i)_{i\in[0,1]} = \alpha_0$. Then f is a continuous linear functional and denoting $N(f) = \{(\alpha_i)_{i\in[0,1]} : f(\alpha_i)_{i\in[0,1]} = 0\}$ we have that $N(f)$ is closed. Hence $N(f)\cap N(\widetilde{T})$ is closed. We find that $u_{1/n} \in N(f)$, $n = 1,2,...$ in the notation as above and, $T\sum_{n=1}^m \lambda_n u_{1/n} = \sum_{n=1}^m \lambda_n e_{1/n} = 0$ implies $\lambda_1 = ... = \lambda_m = 0$ for each m. Hence the dimension of the subspace $N(f)/N(f)\cap N(\widetilde{T})$ of $l^p[0,1]/N(f)\cap N(\widetilde{T})$ is infinite. Also $N(f)/N(f)\cap N(\widetilde{T})$ is complete (Remark 3), hence it is a closed subspace of $l^p[0,1]/N(f)\cap N(\widetilde{T})$. It follows that $\widetilde{T}/N(\widetilde{T})(N(f)/N(f)\cap N(\widetilde{T}))$ is a closed, proper infinite dimensional subspace of (X,q) and the proof is complete. $\square$

Corollary 3.1. *For (X,q), an infinite dimensional quasi-Banach space, there exists a proper infinite dimensional closed subspace of X*

Proof. This follows from Theorem 3.1, Theorem 3.2. $\square$

Corollary 3.2. *Each separable quasi-Banach space (X,q) with quasi-norm constant C is linearly homeomorphic to a quotient of $l^p[0,1]$, $C = 2^{1/p-1}$. If (X,q) is a separable Banach space, it is linearly homeomorphic to a quotient of $l^1[0,1]$.*

Proof. This follows from the proof of Theorem 3.2. □

Corollary 3.3. *If $0 < p \leqslant 1$, the space $L^p[0,1]$ is linearly homeomorphic to a quotient of $l^p(0,1)$.*

Proof. In fact, the proof that each continuous real-valued function on $[0,1]$ is the uniform limit of a sequence of step functions in [3] and the Lebesgue's Dominated Convergence Theorem in [9] applies to conclude that the set of all step functions $s = \sum_{i=1}^{m} s_i \chi_{[a(i),b(i)]}$ where, $s_i \in \mathbf{Q}$, $0 \leqslant a(i) < b(i) \leqslant 1, a(i), b(i)$ are rational numbers, is a countable dense subset of $L^p[0,1]$. The corollary follows from Corollary 3.2. □

Recall [2] the Rademacher functions $r_0(t) = 1$, $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$ $(t \in [0,1])$, $r_n \in L^p[0,1]$ $(p > 0)$.

Remark 8. If $0 < p < 1$, the inequality

$$A_p(\sum_{n=1}^{\infty} \mid a_n \mid^2)^{1/2} \leqslant (\int_0^1 \mid \sum_{n=1}^{\infty} a_n r_n(t) \mid^p dt)^{1/p} \leqslant (\sum_{n=1}^{m} \mid a_n \mid^2)^{1/2}$$

holds. $L^p[0,1]$ contains a basic sequence.

Proof. We easily conclude the inequalities using the method for the proof of Proposition 1 in [2], pp. 141-3. We conclude that (r_n) is also a basic sequence in $L^p[0,1]$. Whence it follows that $(\int_0^1 \mid \sum_{n=1}^{m(1)} a_n r_n(t) \mid^p dt)^{1/p} \leqslant (\sum_{n=1}^{m} \mid a_n \mid^2)^{1/2} \leqslant (\sum_{n=1}^{m(2)} \mid \alpha_n \mid^2)^{1/2} \leqslant A_p^{-1}(\int_0^1 \mid a_n r_n(t) \mid^p dt)^{1/p}$ for all $m(1) < m(2)$ (see [2], Proposition 1, p. 81). □

Remark 9. If the cardinality of I is that of the continuum, there is a quotient of $l^p(I)$ having a basic sequence.

Proof. This follows, as in the proof of Remark 8, from Corollary 3.3. □

Following Theorem 3.6 in [4], the above remark applies to conclude that the space $L^p[0,1]$ has a descending sequence (L_n) of infinite dimensional closed subspaces, such that $\bigcap_{n=1}^{\infty}\{L_n\} = \{0\}$. Also, following [1], $L^p[0,1]$ is primary if $1 \leqslant p \leqslant \infty$, that is, whenever $L^p[0,1] = Y \oplus Z$ a topological direct sum, either Y is linearly homeomorphic to $L^p[0,1]$ or Z is linearly homeomorphic to $L^p[0,1]$. If $0 < p < 1$, then $L^p[0,1]$ is also primary. In fact, if $L^p[0,1] = Y \oplus Z$ we have that Y is not a closed hyperplane (otherwise $L^p[0,1] = Y \oplus \mathbf{K}e$ and the linear functional $f(y + \lambda e) = \lambda$, $y \in Y$, $\lambda \in \mathbf{K}$ is continuous by means of Theorem 3.1.4. p.124 in [11], which is impossible, since there is no nonzero continuous linear functional on $L^p[0,1]$, according to [8], p. 36). The closed graph theorem in [8] applies to conclude that, taking $z_0 \in Z$, we have $L^p[0,1] = (Y \oplus \mathbf{K}z_0) \oplus W$, where $W = span(Z \backslash \{z_0\})$ and the linear projection $P : L^p[0,1] \to Y \oplus \mathbf{K}z_0$ defined by $P(y + \lambda z_0 + w) = y + \lambda z_0$ is continuous. Since $f(y + \lambda z_0) = \lambda$ is a nonzero continuous linear functional on $Y \oplus \mathbf{K}z_0$, the composite $foP : L^p[0,1] \to \mathbf{K}$ would be continuous, which is a contradiction as we saw. Therefore $L^p[0,1] = Y \oplus Z$ implies $Y = L^p[0,1], Z = \{0\}$.

Recall [4] that a quasi-Banach space X has a proper closed weakly dense subspace (PCWD) if there is a proper closed subspace E such that the quotient X/E has a trivial dual. We conclude

Corollary 3.4. *If $0 < p < 1$ and I is a set of indices whose cardinality is* c, *then the space $l^p(I)$ has a primary quotient space. Also $l^p(I)$ has a PCWD subspace.*

Proof. Clearly $l^p(I)$ is $l^p[0,1]$. The corollary follows from Corollary 3.1, namely, the proof of Theorem 3.2. In fact, as for the first assertion, the dual of $l^p[0,1]/N(f)\cap N(\widetilde{T})$ reduces to $\{0\}$, the space being linearly homeomorphic to $L^p[0,1]$ with a trivial dual. $L^p[0,1]$ being primary as we saw, hence is $l^p[0,1]/N(f)\cap N(\widetilde{T})$. Analogously $N(f)\cap N(\widetilde{T})$ is a PCWD subspace of $l^p[0,1]$. The corollary is proved. $\square$

Acknowledgement

This work was developed in CIMA-UE with financial support from FCT (Programa TOCTI-FEDER).

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