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# THE JACOBIAN CONJECTURE FOR RATIONAL POLYNOMIALS

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Abstract. In this paper we consider the Jacobian conjecture in dimension 2 over the field $\mathbb{C}$ of complex numbers, when one of the two polynomial functions has generic rational fibers. We call *rational polynomials* polynomials whose general fiber is a non-singular rational curve.

## 1. Rational Polynomials

1.1. **Smooth compactification of polynomials.** We shall denote in the same way a complex polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ and the corresponding polynomial $f \in \mathbb{C}[X, Y]$. Let $f$ be a complex polynomial of degree $d$:

$$f(X, Y) = \sum_{\alpha+\beta\leq d} a_{\alpha,\beta} X^\alpha Y^\beta$$

with $a_{\alpha,\beta} \in \mathbb{C}$. The corresponding homogeneized polynomial is:

$$F(X, Y, T) = \sum_{\alpha+\beta\leq d} a_{\alpha,\beta} X^\alpha Y^\beta T^{d-\alpha-\beta}.$$

This defines a rational function $F/T^d : \mathbb{P}^2 - - \to \mathbb{P}^1$ given by

$$F/T^d(X : Y : T) := (F(X, Y, T) : T^d).$$

The set of *indeterminacy points* of $F/T^d$ is contained in the line at infinity $T = 0$ of $\mathbb{C}^2$. It is the set of asymptotic directions $(x_i : y_i : 0)$ of $f$ such that

$$f_d(x_i, y_i) = \sum_{\alpha+\beta=d} a_{\alpha,\beta} x_i^\alpha y_i^\beta = 0.$$

In e.g. [9] we showed that a modification of $\mathbb{P}^2$, i.e. a composition of point blowing-ups of $\mathbb{P}^2$

$$\pi : Z \to \mathbb{P}^2,$$

above the indeterminacy points of $F/T^d$, we obtain an algebraic map $\varphi : Z \to \mathbb{P}^1$ *which extends* $F/T^d$, i.e. for any point $m$ of $\mathbb{P}^2$ different from the indeterminacy points of $F/T^d$, we have $F/T^d(m) = \varphi(\pi^{-1}(m))$.

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**Definition 1.1.** A modification $\pi : Z \to \mathbb{P}^2$, such that $F/T^d$ extends into an algebraic map $\varphi : Z \to \mathbb{P}^1$, is said to lift the indeterminacy of the rational function $F/T^d$.

Let $\pi : Z \to \mathbb{P}^2$ be a modification which lifts the indeterminacy of the rational function $F/T^d$. We call $\mathcal{D} := \pi^{-1}(T = 0)$ the *divisor at infinity* of $Z$. The morphism $\pi$ induces an isomorphism of $Z \setminus \mathcal{D}$ over $\mathbb{P}^2 \setminus \{T = 0\} = \mathbb{C}^2$.

We denote by $D_\infty$ the strict transform of $T = 0$ by $\pi$. It is a component of $\mathcal{D}$.

A component of $D$ of $\mathcal{D}$ is one of the three following types:

1. The restriction of $\varphi$ to $D$ is a non-constant map over $\mathbb{P}^1$;
2. The restriction of $\varphi$ to $D$ is a constant map with value $\infty := (1 : 0) \in \mathbb{P}^1$;
3. The restriction of $\varphi$ to $D$ is a constant map with value $\neq \infty$.

The components of the type 1 are called *dicritical components* of $f$ (or $F/T^d$ or $\varphi$) in $Z$. We shall denote by $\mathcal{D}_\infty$ the divisor $\varphi^{-1}(\infty)$. We have $D_\infty \subset \mathcal{D}_\infty$.

Zariski's main theorem (see [6]) implies:

**Theorem 1.2.** *The divisor $\mathcal{D}_\infty$ is connected.*

Let us define:

**Definition 1.3.** A modification $\pi : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$ which lifts the indeterminacy of a rational function $F/T^d$ is called minimal if the only components of the divisor $\mathcal{D} := \pi^{-1}(T = 0)$ which have self-intersection $-1$ are among the dicritical components of $F/T^d$ or $D_\infty$, the strict transform by $\pi$ of $T = 0$.

One can prove that any modification

$$\pi : Z \to \mathbb{P}^2$$

which lifts the indeterminacy of a rational function $F/T^d$ factorizes uniquely through a given minimal one. This shows that a minimal sequence of point blowing-ups which lifts the indeterminacy of the rational function $F/T^d$ is unique up to algebraic isomorphism.

In [9] (see also [8]) we stated the following result:

**Theorem 1.4.** a) *Let $\pi : Z \to \mathbb{P}^2$ be a modification which lifts the indeterminacy of a rational function $F/T^d$, the intersection graph of the divisor $\mathcal{D} := \pi^{-1}(T = 0)$ is a tree.*

b) *Let a minimal sequence of point blowing-ups $\pi : Z \to \mathbb{P}^2$ which lifts the indeterminacy of a rational function $F/T^d$. Consider the finite set $(\mathcal{A}_k)_{k\in K}$ of closures of connected components of $\mathcal{D} \setminus \mathcal{D}_\infty$, then*

1. *The sets $\mathcal{A}_k$ $(k \in K)$ are pairwise distinct.*
2. *Each $\mathcal{A}_k$ contains exactly one dicritical component and this dicritical component contains the intersection point of $\mathcal{A}_k$ and $\mathcal{D}_\infty$.*
3. *The intersection graph $\mathcal{B}_k$ of $\mathcal{A}_k$ is a bamboo and one of its extremity is the dicritical component $D_k$. In particular the intersection graph of the closure of $\mathcal{A}_k \setminus D_k$ is a sub-bamboo $\mathcal{B}'_k$ of $\mathcal{B}_k$ which might be empty in some cases.*

1.2. **Rational polynomials.**

**Definition 1.5.** A complex polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ is said to be *rational* if its general fiber is a rational curve, i.e. diffeomorphic to a punctured 2-sphere.

As above, one can consider the homogeneisation $F$ of the rational polynomial $f$ and the rational function $F/T^d$ that it defines on $\mathbb{P}^2$. Let $\pi : Z \to \mathbb{P}^2$ be a sequence of point blowing-ups over the asymptotic directions of $f$ which lifts the indeterminacy of $F/T^d$ and defines the map $\varphi : Z \to \mathbb{P}^1$ which extends the polynomial function $f$. A classical result (see e.g. [4] p. 521) shows that:

**Proposition 1.6.** *There exists a composition of a finite sequence of contractions on points*

$$q : Z \to Z_1$$

*and a locally trivial fibration* $\varphi_1 : Z_1 \to \mathbb{P}_1$ *such that* $\varphi_1 \circ q = \varphi$.

We have the following corollaries:

**Corollary 1.7.** *The fibers of a complex rational polynomial are normal crossing divisors of non-singular rational curves.*

*Proof.* By assumption, this is already true for general fibers. Let $f = \lambda$ be a fiber which is not general. The fiber $\varphi^{-1}(\lambda)$ retracts on the rational curve $\varphi_1^{-1}(\lambda)$, so it is a normal crossing divisor whose components are all non-singular rational curves. This easily implies our result. □

**Corollary 1.8.** *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a rational polynomial function and let* $\pi : Z \to \mathbb{P}^2$ *be the minimal modification which lifts the indeterminacy of the corresponding rational function* $F/T^d$. *If the degree of the polynomial* $f$ *is* $\geq 2$, *the component* $D_\infty$, *strict transform of the line at infinity* $T = 0$, *has self-intersection* $-1$.

*Proof.* According to the preceding Proposition 1.6 there is a finite sequence of contractions on points $q : Z \to Z_1$ and a locally trivial fibration $\varphi_1 : Z_1 \to \mathbb{P}_1$, such that $\varphi_1 \circ q = \varphi$, so the divisor $\mathcal{D}_\infty := \varphi^{-1}(\infty)$ contracts onto $\varphi_1^{-1}(\infty)$. Since the morphism $\pi$ is a minimal sequence of point blowing-ups which lifts the indeterminacy of $F/T^d$, the only possible component of $\mathcal{D}_\infty$ which might have self-intersection $-1$ and therefore might contract is $D_\infty$. If the degree of $f$ is $\geq 2$, the morphism $q$ is not an isomorphism, so necessarily $D_\infty$ has self-intersection $-1$. □

1.3. **Examples.** Let $(f, g)$ be an automorphism of $\mathbb{C}$. Obviously $f$ and $g$ are rational polynomials. In the class of rational polynomials they are particular rational polynomials. For instance, they are locally trivial fibrations over $\mathbb{C}$. Their compactifications have been studied in [3].

Consider $f(X, Y) = X(XY - 1)$. It is a polynomial of degree 3. For $\lambda \neq 0$ the fiber $f^{-1}(\lambda)$ is general and diffeomorphic to a cylinder, i.e. to a 2-sphere minus two distinct points, the asymptotic directions of $f$. So, $f$ is a rational polynomial. Also $f^{-1}(0)$ is the union of a line and a cylinder given by $XY = 1$. We can observe that it is not a locally trivial fibration over $\mathbb{C}$.

## 2. Rational Polynomials in Jacobian pairs

Let us call Jacobian pair a pair $(f, g)$ of polynomial functions on $\mathbb{C}^2$ such that the determinant of the Jacobian matrix $J(f, g)$ is a non-zero constant.

In this section we shall consider rational polynomials $f$ which belong to a Jacobian pair $(f, g)$.

2.1. **Finite fibers.** The first obvious observation is that a rational polynomial function which belongs to a Jacobian pair has no critical points. This implies that all the fibers of such rational polynomials are non-singular (in particular reduced). We have the following theorem (see [7] §2. 1):

**Theorem 2.1.** *Let $f$ be a rational polynomial function of $\mathbb{C}^2$ with non-singular fibers. Let $\pi : Z \to \mathbb{P}^2$ be a minimal sequence of point blowing-ups which lifts the indeterminacy of the corresponding rational function $F/T^d$. Let $\varphi$ be the extension of $f$ to $Z$. The fibers $\varphi^{-1}(\lambda)$ for $\lambda \neq \infty$ are reduced normal crossing divisors.*

2.2. **Fiber over $\infty$.** The fiber of a compactification of a polynomial over $\infty$ is in general non-reduced:

**Proposition 2.2.** *Let $\pi : Z \to \mathbb{P}^2$ be a modification of $\mathbb{P}^2$ which lifts the indeterminacy of the rational function $F/T^d$ associated to a polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$. The multiplicity of the strict transform $D_\infty$ of the line at infinity $T = 0$ of $\mathbb{C}^2$ by $\pi$ equals the degree $d$ of $f$.*

When the degree is $\geq 2$, the fiber over $\infty$ of a minimal compactification being obtained from a non-singular curve of self-intersection 0 by a sequence of point blowing-ups, one starts by blowing-up a point of this non-singular curve, say $C$, of self-intersection 0 which becomes a non-singular curve of self-intersection $-1$ intersected transversally by the new exceptional divisor. Then, we have to blow-up the intersection of these two non-singular curves of self-intersection $-1$, because the fiber over $\infty$ of the minimal compactification has only one component of self-intersection $-1$. Now to continue we have to blow-up points over this last curve which has multiplicity 2 in the divisor inverse image of the original curve $C$, so that all the following curves that appear in the successive blowing-ups must have multiplicity $\geq 2$. As a consequence, we have:

**Proposition 2.3.** *Let $f$ be a rational polynomial. Let $\pi : Z \to \mathbb{P}^2$ be a modification of $\mathbb{P}^2$ on which $f$ extends into a map $\varphi : Z \to \mathbb{P}^1$. Assume it is a minimal compactification of $f$. Then in the fiber $\varphi^{-1}(\infty)$ of $\varphi$ over $\infty$, there are only two components with multiplicity* 1*.*

## 3. The Jacobian Conjecture

In this section we shall give indications on why a Jacobian pair, in which a polynomial is rational, might be an automorphism of $\mathbb{C}^2$.

3.1. **A geometric approach.** Let us suppose that after composition with any automorphism of $\mathbb{C}^2$ the polynomial $f$ is not a coordinate function. In [5] we have observed that it is equivalent to say that $f$ is not a locally trivial fibration over $\mathbb{C}$.

We have seen above that there is a modification $\pi : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$, a sequence of point blowing-ups over the asymptotic directions of $f$, on which $f$ extends into a map $\varphi : Z \to \mathbb{P}^1$, called a compactification of $f$. We have seen that

$$\pi^{-1}(T = 0) = \mathcal{D} = Z \setminus \pi^{-1}(\mathbb{C}^2).$$

In the divisor $\mathcal{D}$ we have distinguished three types of components, the ones over which the restriction of $\varphi$ is constant with value $\infty$, the ones over which $\varphi$ has a constant value $\lambda \neq \infty$, the ones over which $\varphi$ is not constant and induces a map over $\mathbb{P}^1$ with degree $\geq 1$ that we have called dicritical components of $f$. The degree of the restriction of $\varphi$ to a dicritical component is called the *degree* of the dicritical component.

In [7] we have proved that if $f$ is *simple* (see [11]), i.e. for all the dicritical components of $f$ the restrictions of the compactification $\varphi$ of $f$ have degree 1, a Jacobian pair in which a polynomial $f$ is rational, is an automorphism of $\mathbb{C}^2$.

Let us assume that the polynomial $f$ is not a locally trivial fibration over $\mathbb{C}$. It means that at least one of the dicritical components of $f$ is non-equisingular. Recall that a dicritical component of $f$ is *non-equisingular* in the minimal compactification $Z_0$ of $f$ (see [8]) if, either the degree of this dicritical component of $f$ is $> 1$, or it has degree one and it is intersected by the components at infinity contained in the inverse image by the compactification $\varphi$ of an atypical value $\lambda \in \mathbb{C}$ of $f$. A dicritical component of $f$ in a compactification $Z$ is non-equisingular if its image in a minimal compactification is non-equisingular in the preceding sense.

Above we have studied non-general fibers of a minimal compactification of a rational polynomial which belongs to a Jacobian pair. We found that such fibers are reduced normal crossing divisors (see Theorem 2.1).

Let us summarize our observations in the following

**Proposition 3.1.** *Non-general fibers over atypical values of the minimal compactification of a rational polynomial are reduced divisors with normal crossings whose components at $\infty$ have self-intersection $\leq -2$.*

3.2. **Conjecture.** We conjecture:

*A rational polynomial which is not a locally trivial fibration over $\mathbb{C}$ cannot belong to a Jacobian pair.*

In [7] we proved that this conjecture is true if the rational polynomial is simple.

Let $\pi : Z \to \mathbb{P}^2$ be a modification of $\mathbb{P}^2$ on which $f$ has a compactification $\varphi : Z \to \mathbb{P}^1$.

Consider the canonical divisor $K_Z$ of $Z$. The components of $\pi^{-1}(T = 0) = \mathcal{D}$ give a basis of the free abelian group $H_2(Z)$, so the multiplicities of $K_Z$ along these components are well defined. A dicritical component of $f$ is said *negative* in $Z$ if its multiplicity in $K_Z$ is $< 0$, it is said *positive* if its multiplicity in $K_Z$ is $\geq 0$.

**Remark 3.2.** Observe that if $D$ is a dicritical component of $f$ in a modification $\pi : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$, it is the strict transform in $Z$ of a dicritical component of $f$ in the minimal modification on which $f$ has a compactification. It is easy to see that the notion of positivity or negativity does not depend on the compactification of $f$ chosen (see below).

We may assume that $Z$ is a modification of $\mathbb{P}^2$ on which $g$ has also a compactification $\psi : Z \to \mathbb{P}^1$. The Jacobian hypothesis implies that the support of the differential $\omega := d\varphi \wedge d\psi$ is on $\mathcal{D}$. Therefore, along a negative dicritical component $D$ of $f$ in $Z$, the differential $\omega$ has a pole. Since $\varphi$ along $D$ defines a non-constant function, $\psi$ has a pole along $D$. In particular this implies (see [7] Lemma 3.4):

**Lemma 3.3.** *If a pair $(f, g)$ is a Jacobian pair, a negative dicritical for $f$ has multiplicity $\leq -2$ in the canonical divisor on a modification of $\mathbb{P}^2$ on which $f$ extends.*

Then, we have the following result (see [7] Lemma 3.6):

**Lemma 3.4.** *Let $D$ be a negative dicritical component of $f$ along which $\psi$ is constant. If $D$ is non-equisingular, the pair $(f, g)$ is not a Jacobian pair.*

This lemma is based on the following lemma (see [7] Lemma 3.7)

**Lemma 3.5.** *Let $X$ be a modification of $\mathbb{P}^2$. Let $\mathcal{D}$ be the divisor at $\infty$ of $X$. Let $\mathcal{B} = \{D_1, \ldots, D_\ell\}$ be a bamboo of components contained in $\mathcal{D}$. Assume that*

1. *the extremity $D_1$ of $\mathcal{B}$ does not intersect any other component of $\mathcal{D}$ but $D_2$,*
2. *no other component but $D_\ell$ in $\mathcal{D}$ intersect the sub-bamboo $\mathcal{B}' = \{D_1, \ldots, D_{\ell-1}\}$*
3. *the self-intersections of $D_1, \ldots, D_{\ell-1}$ are $\leq -2$.*

*Then $D_\ell$ is negative (resp. positive) in the canonical divisor of $X$ if and only if $D_1$ is negative (resp. positive) in the canonical divisor of $X$.*

In summary, if $(f, g)$ is a Jacobian pair, the negative dicritical $D$ of $f$ has to be equisingular. In particular if $D$ is a negative dicritical component of $f$, the restriction of the extension $\varphi$ of $f$ to $D$ has degree 1. In particular the component of $\varphi^{-1}(\infty)$ which intersects $D$ has multiplicity 1 in the divisor defined by $\varphi^{-1}(\infty)$.

Therefore $D$ intersects only one of the two components of $\varphi^{-1}(\infty)$ which have multiplicity 1 (cf. Lemma 2.3). Using arguments as in the proof of Lemma 3.5, since $D$ is a negative equisingular component of $f$, all the tree $\mathcal{A}$ of components of $\varphi^{-1}(\infty)$ which intersects $D$ and the strict transform of $\{T = 0\}$ only contains negative components.

Then, we conjecture that the tree $\mathcal{A}$ is only intersected by one negative dicritical component $f$ which intersects one of the two components of $\varphi^{-1}(\infty)$ of multiplicity 1.

The main conjecture above would follow because a fiber of an atypical fiber of $\varphi$ intersecting a non-equisingular positive component would have at least two components with a non-empty affine Zariski open subset, one of which intersecting the negative dicritical component, the other one not intersecting the negative dicritical component. Since these two components are non-singular rational curves, this would contradict Riemann-Roch theorem.

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