THE JACOBIAN CONJECTURE FOR RATIONAL POLYNOMIALS

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Abstract. In this paper we consider the Jacobian conjecture in dimension 2 over the field \( \mathbb{C} \) of complex numbers, when one of the two polynomial functions has generic rational fibers. We call rational polynomials polynomials whose general fiber is a non-singular rational curve.

1. Rational Polynomials

1.1. Smooth compactification of polynomials. We shall denote in the same way a complex polynomial function \( f : \mathbb{C}^2 \to \mathbb{C} \) and the corresponding polynomial \( f \in \mathbb{C}[X,Y] \). Let \( f \) be a complex polynomial of degree \( d \):

\[
f(X,Y) = \sum_{\alpha+\beta \leq d} a_{\alpha,\beta}X^\alpha Y^\beta
\]

with \( a_{\alpha,\beta} \in \mathbb{C} \). The corresponding homogeneized polynomial is:

\[
F(X,Y,T) = \sum_{\alpha+\beta \leq d} a_{\alpha,\beta}X^\alpha Y^\beta T^{d-\alpha-\beta}.
\]

This defines a rational function \( F/T^d : \mathbb{P}^2 \to \mathbb{P}^1 \) given by

\[
F/T^d(X : Y : T) := (F(X,Y,T) : T^d).
\]

The set of indeterminacy points of \( F/T^d \) is contained in the line at infinity \( T = 0 \) of \( \mathbb{C}^2 \). It is the set of asymptotic directions \( (x_i : y_i : 0) \) of \( f \) such that

\[
f_d(x_i, y_i) = \sum_{\alpha+\beta = d} a_{\alpha,\beta}x_i^\alpha y_i^\beta = 0.
\]

In e.g. \([9]\) we showed that a modification of \( \mathbb{P}^2 \), i.e. a composition of point blowing-ups of \( \mathbb{P}^2 \)

\[
\pi : Z \to \mathbb{P}^2,
\]

above the indeterminacy points of \( F/T^d \), we obtain an algebraic map \( \varphi : Z \to \mathbb{P}^1 \) which extends \( F/T^d \), i.e. for any point \( m \) of \( \mathbb{P}^2 \) different from the indeterminacy points of \( F/T^d \), we have \( F/T^d(m) = \varphi(\pi^{-1}(m)) \).

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**Definition 1.1.** A modification $\pi : Z \to \mathbb{P}^2$, such that $F/T^d$ extends into an algebraic map $\varphi : Z \to \mathbb{P}^1$, is said to lift the indeterminacy of the rational function $F/T^d$.

Let $\pi : Z \to \mathbb{P}^2$ be a modification which lifts the indeterminacy of the rational function $F/T^d$. We call $\mathcal{D} := \pi^{-1}(T = 0)$ the divisor at infinity of $Z$. The morphism $\pi$ induces an isomorphism of $Z \setminus \mathcal{D}$ over $\mathbb{P}^2 \setminus \{T = 0\} = \mathbb{C}^2$.

We denote by $D_\infty$ the strict transform of $T = 0$ by $\pi$. It is a component of $\mathcal{D}$.

A component of $D$ of $\mathcal{D}$ is one of the three following types:
1. The restriction of $\varphi$ to $D$ is a non-constant map over $\mathbb{P}^1$;
2. The restriction of $\varphi$ to $D$ is a constant map with value $\infty := (1 : 0) \in \mathbb{P}^1$;
3. The restriction of $\varphi$ to $D$ is a constant map with value $\neq \infty$.

The components of the type 1 are called **dicritical components** of $f$ (or $F/T^d$ or $\varphi$) in $Z$. We shall denote by $D_\infty$ the divisor $\varphi^{-1}(\infty)$. We have $D_\infty \subset \mathcal{D}_\infty$.

Zariski’s main theorem (see [6]) implies:

**Theorem 1.2.** The divisor $\mathcal{D}_\infty$ is connected.

Let us define:

**Definition 1.3.** A modification $\pi : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$ which lifts the indeterminacy of a rational function $F/T^d$ is called minimal if the only components of the divisor $\mathcal{D} := \pi^{-1}(T = 0)$ which have self-intersection $-1$ are among the dicritical components of $F/T^d$ or $D_\infty$, the strict transform by $\pi$ of $T = 0$.

One can prove that any modification $\pi : Z \to \mathbb{P}^2$ which lifts the indeterminacy of a rational function $F/T^d$ factorizes uniquely through a given minimal one. This shows that a minimal sequence of point blowing-ups which lifts the indeterminacy of the rational function $F/T^d$ is unique up to algebraic isomorphism.

In [9] (see also [8]) we stated the following result:

**Theorem 1.4.** a) Let $\pi : Z \to \mathbb{P}^2$ be a modification which lifts the indeterminacy of a rational function $F/T^d$, the intersection graph of the divisor $\mathcal{D} := \pi^{-1}(T = 0)$ is a tree.

b) Let a minimal sequence of point blowing-ups $\pi : Z \to \mathbb{P}^2$ which lifts the indeterminacy of a rational function $F/T^d$. Consider the finite set $(A_k)_{k \in K}$ of closures of connected components of $\mathcal{D} \setminus D_\infty$, then

1. The sets $A_k$ ($k \in K$) are pairwise distinct.
2. Each $A_k$ contains exactly one dicritical component and this dicritical component contains the intersection point of $A_k$ and $D_\infty$.
3. The intersection graph $B_k$ of $A_k$ is a bamboo and one of its extremity is the dicritical component $D_k$. In particular the intersection graph of the closure of $A_k \setminus D_k$ is a sub-bamboo $B'_k$ of $B_k$ which might be empty in some cases.
1.2. Rational polynomials.

**Definition 1.5.** A complex polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ is said to be **rational** if its general fiber is a rational curve, i.e. diffeomorphic to a punctured 2-sphere.

As above, one can consider the homogeneisation $F$ of the rational polynomial $f$ and the rational function $F/T^d$ that it defines on $\mathbb{P}^2$. Let $\pi : Z \to \mathbb{P}^2$ be a sequence of point blowing-ups over the asymptotic directions of $f$ which lifts the indeterminacy of $F/T^d$ and defines the map $\varphi : Z \to \mathbb{P}^1$ which extends the polynomial function $f$. A classical result (see e.g. [4] p. 521) shows that:

**Proposition 1.6.** There exists a composition of a finite sequence of contractions on points

$q : Z \to Z_1$

and a locally trivial fibration $\varphi_1 : Z_1 \to \mathbb{P}^1$ such that $\varphi_1 \circ q = \varphi$.

We have the following corollaries:

**Corollary 1.7.** The fibers of a complex rational polynomial are normal crossing divisors of non-singular rational curves.

*Proof.* By assumption, this is already true for general fibers. Let $f = \lambda$ be a fiber which is not general. The fiber $\varphi^{-1}(\lambda)$ retracts on the rational curve $\varphi_1^{-1}(\lambda)$, so it is a normal crossing divisor whose components are all non-singular rational curves. This easily implies our result. \(\square\)

**Corollary 1.8.** Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a rational polynomial function and let $\pi : Z \to \mathbb{P}^2$ be the minimal modification which lifts the indeterminacy of the corresponding rational function $F/T^d$. If the degree of the polynomial $f$ is $\geq 2$, the component $D_\infty$, strict transform of the line at infinity $T = 0$, has self-intersection $-1$.

*Proof.* According to the preceding Proposition 1.6 there is a finite sequence of contractions on points $q : Z \to Z_1$ and a locally trivial fibration $\varphi_1 : Z_1 \to \mathbb{P}^1$, such that $\varphi_1 \circ q = \varphi$, so the divisor $D_\infty := \varphi^{-1}(\infty)$ contracts onto $\varphi_1^{-1}(\infty)$. Since the morphism $\pi$ is a minimal sequence of point blowing-ups which lifts the indeterminacy of $F/T^d$, the only possible component of $D_\infty$ which might have self-intersection $-1$ and therefore might contract is $D_\infty$. If the degree of $f$ is $\geq 2$, the morphism $q$ is not an isomorphism, so necessarily $D_\infty$ has self-intersection $-1$. \(\square\)

1.3. Examples. Let $(f,g)$ be an automorphism of $\mathbb{C}$. Obviously $f$ and $g$ are rational polynomials. In the class of rational polynomials they are particular rational polynomials. For instance, they are locally trivial fibrations over $\mathbb{C}$. Their compactifications have been studied in [3].

Consider $f(X,Y) = X(XY - 1)$. It is a polynomial of degree 3. For $\lambda \neq 0$ the fiber $f^{-1}(\lambda)$ is general and diffeomorphic to a cylinder, i.e. to a 2-sphere minus two distinct points, the asymptotic directions of $f$. So, $f$ is a rational polynomial. Also $f^{-1}(0)$ is the union of a line and a cylinder given by $XY = 1$. We can observe that it is not a locally trivial fibration over $\mathbb{C}$. 
2. Rational Polynomials in Jacobian pairs

Let us call Jacobian pair a pair \((f, g)\) of polynomial functions on \(\mathbb{C}^2\) such that the determinant of the Jacobian matrix \(J(f, g)\) is a non-zero constant.

In this section we shall consider rational polynomials \(f\) which belong to a Jacobian pair \((f, g)\).

2.1. Finite fibers. The first obvious observation is that a rational polynomial function which belongs to a Jacobian pair has no critical points. This implies that all the fibers of such rational polynomials are non-singular (in particular reduced). We have the following theorem (see [7] §2.1):

**Theorem 2.1.** Let \(f\) be a rational polynomial function of \(\mathbb{C}^2\) with non-singular fibers. Let \(\pi : Z \rightarrow \mathbb{P}^2\) be a minimal sequence of point blowing-ups which lifts the indeterminacy of the corresponding rational function \(F/T^{d}\). Let \(\varphi\) be the extension of \(f\) to \(Z\). The fibers \(\varphi^{-1}(\lambda)\) for \(\lambda \neq \infty\) are reduced normal crossing divisors.

2.2. Fiber over \(\infty\). The fiber of a compactification of a polynomial over \(\infty\) is in general non-reduced:

**Proposition 2.2.** Let \(\pi : Z \rightarrow \mathbb{P}^2\) be a modification of \(\mathbb{P}^2\) which lifts the indeterminacy of the rational function \(F/T^{d}\) associated to a polynomial function \(f : \mathbb{C}^2 \rightarrow \mathbb{C}\). The multiplicity of the strict transform \(D_{\infty}\) of the line at infinity \(T = 0\) of \(\mathbb{C}^2\) by \(\pi\) equals the degree \(d\) of \(f\).

When the degree is \(\geq 2\), the fiber over \(\infty\) of a minimal compactification being obtained from a non-singular curve of self-intersection 0 by a sequence of point blowing-ups, one starts by blowing-up a point of this non-singular curve, say \(C\), of self-intersection 0 which becomes a non-singular curve of self-intersection \(-1\) intersected transversally by the new exceptional divisor. Then, we have to blow-up the intersection of these two non-singular curves of self-intersection \(-1\), because the fiber over \(\infty\) of the minimal compactification has only one component of self-intersection \(-1\). Now to continue we have to blow-up points over this last curve which has multiplicity \(2\) in the divisor inverse image of the original curve \(C\), so that all the following curves that appear in the successive blowing-ups must have multiplicity \(\geq 2\). As a consequence, we have:

**Proposition 2.3.** Let \(f\) be a rational polynomial. Let \(\pi : Z \rightarrow \mathbb{P}^2\) be a modification of \(\mathbb{P}^2\) on which \(f\) extends into a map \(\varphi : Z \rightarrow \mathbb{P}^1\). Assume it is a minimal compactification of \(f\). Then in the fiber \(\varphi^{-1}(\infty)\) of \(\varphi\) over \(\infty\), there are only two components with multiplicity 1.

3. The Jacobian Conjecture

In this section we shall give indications on why a Jacobian pair, in which a polynomial is rational, might be an automorphism of \(\mathbb{C}^2\).
3.1. A geometric approach. Let us suppose that after composition with any automorphism of \( \mathbb{C}^2 \) the polynomial \( f \) is not a coordinate function. In [5] we have observed that it is equivalent to say that \( f \) is not a locally trivial fibration over \( \mathbb{C} \).

We have seen above that there is a modification \( \pi : Z \to \mathbb{P}^2 \) of \( \mathbb{P}^2 \), a sequence of point blowing-ups over the asymptotic directions of \( f \), on which \( f \) extends into a map \( \varphi : Z \to \mathbb{P}^1 \), called a compactification of \( f \). We have seen that 

\[
\pi^{-1}(T = 0) = D = Z \setminus \pi^{-1}(\mathbb{C}^2).
\]

In the divisor \( D \) we have distinguished three types of components, the ones over which the restriction of \( \varphi \) is constant with value \( \infty \), the ones over which \( \varphi \) has a constant value \( \lambda \neq \infty \), the ones over which \( \varphi \) is not constant and induces a map over \( \mathbb{P}^1 \) with degree \( \geq 1 \) that we have called dicritical components of \( f \). The degree of the restriction of \( \varphi \) to a dicritical component is called the degree of the dicritical component.

In [7] we have proved that if \( f \) is simple (see [11]), i.e. for all the dicritical components of \( f \) the restrictions of the compactification \( \varphi \) of \( f \) have degree 1, a Jacobian pair in which a polynomial \( f \) is rational, is an automorphism of \( \mathbb{C}^2 \).

Let us assume that the polynomial \( f \) is not a locally trivial fibration over \( \mathbb{C} \).

Let us summarize our observations in the following

**Proposition 3.1.** Non-general fibers over atypical values of the minimal compactification of a rational polynomial are reduced divisors with normal crossings whose components at \( \infty \) have self-intersection \( \leq -2 \).

3.2. Conjecture. We conjecture:

A rational polynomial which is not a locally trivial fibration over \( \mathbb{C} \) cannot belong to a Jacobian pair.

In [7] we proved that this conjecture is true if the rational polynomial is simple.

Let \( \pi : Z \to \mathbb{P}^2 \) be a modification of \( \mathbb{P}^2 \) on which \( f \) has a compactification \( \varphi : Z \to \mathbb{P}^1 \).

Consider the canonical divisor \( K_Z \) of \( Z \). The components of \( \pi^{-1}(T = 0) = D \) give a basis of the free abelian group \( H_2(Z) \), so the multiplicities of \( K_Z \) along these components are well defined. A dicritical component of \( f \) is said negative.
Remark 3.2. Observe that if $D$ is a dicritical component of $f$ in a modification $\pi: Z \to \mathbb{P}^2$ of $\mathbb{P}^2$, it is the strict transform in $Z$ of a dicritical component of $f$ in the minimal modification on which $f$ has a compactification. It is easy to see that the notion of positivity or negativity does not depend on the compactification of $f$ chosen (see below).

We may assume that $Z$ is a modification of $\mathbb{P}^2$ on which $g$ has also a compactification $\psi: Z \to \mathbb{P}^1$. The Jacobian hypothesis implies that the support of the differential $\omega := d\varphi \wedge d\psi$ is on $D$. Therefore, along a negative dicritical component $D$ of $f$ in $Z$, the differential $\omega$ has a pole. Since $\varphi$ along $D$ defines a non-constant function, $\psi$ has a pole along $D$. In particular this implies (see [7] Lemma 3.4):

**Lemma 3.3.** Let $D$ be a negative dicritical component of $f$ along which $\psi$ is constant. If $D$ is non-equisingular, the pair $(f,g)$ is not a Jacobian pair.

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This lemma is based on the following lemma (see [7] Lemma 3.7)

**Lemma 3.5.** Let $X$ be a modification of $\mathbb{P}^2$. Let $\mathcal{D}$ be the divisor at $\infty$ of $X$. Let $\mathcal{B} = \{D_1, \ldots, D_\ell\}$ be a bamboo of components contained in $\mathcal{D}$. Assume that

1. the extremity $D_1$ of $\mathcal{B}$ does not intersect any other component of $\mathcal{D}$ but $D_2$,
2. no other component but $D_\ell$ in $\mathcal{D}$ intersect the sub-bamboo $\mathcal{B}' = \{D_1, \ldots, D_{\ell-1}\}$
3. the self-intersections of $D_1, \ldots, D_{\ell-1}$ are $\leq -2$.

Then $D_\ell$ is negative (resp. positive) in the canonical divisor of $X$ if and only if $D_1$ is negative (resp. positive) in the canonical divisor of $X$.

In summary, if $(f,g)$ is a Jacobian pair, the negative dicritical $D$ of $f$ has to be equisingular. In particular if $D$ is a negative dicritical component of $f$, the restriction of the extension $\varphi$ of $f$ to $D$ has degree 1. In particular the component of $\varphi^{-1}(\infty)$ which intersects $D$ has multiplicity 1 in the divisor defined by $\varphi^{-1}(\infty)$.

Therefore $D$ intersects only one of the two components of $\varphi^{-1}(\infty)$ which have multiplicity 1 (cf. Lemma 2.3). Using arguments as in the proof of Lemma 3.5, since $D$ is a negative equisingular component of $f$, all the tree $\mathcal{A}$ of components of $\varphi^{-1}(\infty)$ which intersects $D$ and the strict transform of $\{T = 0\}$ only contains negative components.

Then, we conjecture that the tree $\mathcal{A}$ is only intersected by one negative dicritical component $f$ which intersects one of the two components of $\varphi^{-1}(\infty)$ of multiplicity 1.
The main conjecture above would follow because a fiber of an atypical fiber of $\varphi$ intersecting a non-equisingular positive component would have at least two components with a non-empty affine Zariski open subset, one of which intersecting the negative dicritical component, the other one not intersecting the negative dicritical component. Since these two components are non-singular rational curves, this would contradict Riemann-Roch theorem.

**References**


