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# A VANISHING CONJECTURE ON DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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Abstract. In the recent progress [4], [17] and [25], the well-known JC (Jacobian conjecture) ([2], [10]) has been reduced to a VC (vanishing conjecture) on the Laplace operators and HN (Hessian nilpotent) polynomials (the polynomials whose Hessian matrix are nilpotent). In this paper, we first show that the vanishing conjecture above, hence also the JC, is equivalent to a vanishing conjecture for all 2nd order homogeneous differential operators $\Lambda$ and $\Lambda$-nilpotent polynomials $P$ (the polynomials $P(z)$ satisfying $\Lambda^m P^m = 0$ for all $m \geqslant 1$). We then transform some results in the literature on the JC, HN polynomials and the VC of the Laplace operators to certain results on $\Lambda$-nilpotent polynomials and the associated VC for 2nd order homogeneous differential operators $\Lambda$. This part of the paper can also be read as a short survey on HN polynomials and the associated VC in the more general setting. Finally, we discuss a still-to-be-understood connection of $\Lambda$-nilpotent polynomials in general with the classical orthogonal polynomials in one or more variables. This connection provides a conceptual understanding for the isotropic properties of homogeneous $\Lambda$-nilpotent polynomials for 2nd order homogeneous full rank differential operators $\Lambda$ with constant coefficients.

## 1. Introduction

Let $z = (z_1, z_2, \ldots, z_n, \ldots)$ be a sequence of free commutative variables and $D = (D_1, D_2, \ldots, D_n, \ldots)$ with $D_i := \frac{\partial}{\partial z_i}$ $(i \geqslant 1)$. For any $n \geqslant 1$, denote by $\mathcal{A}_n$ (resp. $\bar{\mathcal{A}}_n$) the algebra of polynomials (resp. formal power series) in $z_i$ $(1 \leqslant i \leqslant n)$. Furthermore, we denote by $\mathcal{D}[\mathcal{A}_n]$ or $\mathcal{D}[n]$ (resp. $\mathbb{D}[\mathcal{A}_n]$ or $\mathbb{D}[n]$) the algebra of differential operators of the polynomial algebra $\mathcal{A}_n$ (resp. with constant coefficients). Note that, for any $k \geqslant n$, elements of $\mathcal{D}[n]$ are also differential operators of $\mathcal{A}_k$ and $\bar{\mathcal{A}}_k$. For any $d \geqslant 0$, denote by $\mathbb{D}_d[n]$ the set of *homogeneous differential operators of order $d$ with constants coefficients*. We let $\mathcal{A}$ (resp. $\bar{\mathcal{A}}$) be the union of $\mathcal{A}_n$ (resp. $\bar{\mathcal{A}}_n$) $(n \geqslant 1)$, $\mathcal{D}$ (resp. $\mathbb{D}$) the union of $\mathcal{D}[n]$ (resp. $\mathbb{D}[n]$) $(n \geqslant 1)$, and, for any $d \geqslant 1$, $\mathcal{D}_d$ the union of $\mathcal{D}_d[n]$ $(n \geqslant 1)$.

Recall that **JC** (the Jacobian conjecture) which was first proposed by Keller [14] in 1939, claims that, *for any polynomial map $F$ of $\mathbb{C}^n$ with Jacobian $j(F) = 1$,*

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*its formal inverse map $G$ must also be a polynomial map.* Despite intense study from mathematicians in more than sixty years, the conjecture is still open even for the case $n = 2$. For more history and known results before 2000 on **JC**, see [2], [10] and references there.

Based on the remarkable symmetric reduction achieved in [4], [17] and the classical celebrated homogeneous reduction [2] and [23] on **JC**, the author in [25] reduced **JC** further to the following *vanishing conjecture* on the Laplace operators $\Delta_n := \sum_{i=1}^{n} D_i^2$ of the polynomial algebra $\mathcal{A}_n$ and *HN* (*Hessian nilpotent*) polynomials $P(z) \in \mathcal{A}_n$, where we say a polynomial or formal power series $P(z) \in \bar{\mathcal{A}}_n$ is HN if its Hessian matrix $\operatorname{Hes}(P) := (\frac{\partial^2 P}{\partial z_i \partial z_j})_{n\times n}$ is nilpotent.

**Conjecture 1.1.** *For any HN (homogeneous) polynomial $P(z) \in \mathcal{A}_n$ (of degree $d = 4$), we have $\Delta_n^m P^{m+1}(z) = 0$ when $m >> 0$.*

Note that, the following criteria of Hessian nilpotency were also proved in Theorem 4.3, [25].

**Theorem 1.1.** *For any $P(z) \in \bar{\mathcal{A}}_n$ with $o(P(z)) \geqslant 2$, the following statements are equivalent.*

1. *$P(z)$ is HN.*
2. *$\Delta^m P^m = 0$ for any $m \geqslant 1$.*
3. *$\Delta^m P^m = 0$ for any $1 \leqslant m \leqslant n$.*

Through the criteria in the proposition above, Conjecture 1.1 can be generalized to other differential operators as follows (see Conjecture 1.2 below).

First let us fix the following notion that will be used throughout the paper.

**Definition 1.1.** *Let $\Lambda \in \mathcal{D}[\mathcal{A}_n]$ and $P(z) \in \bar{\mathcal{A}}_n$. We say $P(z)$ is $\Lambda$-nilpotent if $\Lambda^m P^m = 0$ for any $m \geqslant 1$.*

Note that, when $\Lambda$ is the Laplace operator $\Delta_n$, by Theorem 1.1, a polynomial or formal power series $P(z) \in \mathcal{A}_n$ is $\Lambda$-nilpotent iff it is HN.

With the notion above, Conjecture 1.1 has the following natural generalization to differential operators with constant coefficients.

**Conjecture 1.2.** *For any $n \geqslant 1$ and $\Lambda \in \mathbb{D}[n]$, if $P(z) \in \mathcal{A}_n$ is $\Lambda$-nilpotent, then $\Lambda^m P^{m+1} = 0$ when $m >> 0$.*

We call the conjecture above the *vanishing conjecture* for differential operators with constant coefficients and denote it by **VC**. The special case of **VC** with $P(z)$ homogeneous is called the *homogeneous vanishing conjecture* and denoted by **HVC**. When the number $n$ of variables is fixed, **VC** (resp. **HVC**) is called (resp. *homogeneous*) *vanishing conjecture in $n$ variables* and denoted by **VC**$[n]$ (resp. **HVC**$[n]$).

Two remarks on **VC** are as follows. First, due to a counter-example given by M. de Bondt (see example 2.1), **VC** does not hold in general for differential operators with non-constant coefficients. Secondly, one may also allow $P(z)$ in

**VC** to be any $\Lambda$-nilpotent formal power series. No counter-example to this more general **VC** is known yet.

In this paper, we first apply certain linear automorphisms and Lefschetz's principle to show Conjecture 1.1, hence also **JC**, is equivalent to **VC** or **HVC** for all 2nd order homogeneous differential operators $\Lambda \in \mathbb{D}_2$ (see Theorem 2.1). We then in Section 3 transform some results on **JC**, HN polynomials and Conjecture 1.1 obtained in [22], [5], [6], [25], [26] and [11] to certain results on $\Lambda$-nilpotent ($\Lambda \in \mathbb{D}_2$) polynomials and **VC** for $\Lambda$. Another purpose of this section is to give a survey on recent study on Conjecture 1.1 and HN polynomials in the more general setting of $\Lambda \in \mathbb{D}_2$ and $\Lambda$-nilpotent polynomials. This is also why some results in the general setting, even though their proofs are straightforward, are also included here.

Even though, due to M. de Bondt's counter-example (see Example 2.1), **VC** does not hold for all differential operators with non-constant coefficients, it is still interesting to consider whether or not **VC** holds for higher order differential operators with constant coefficients; and if it also holds even for certain families of differential operators with non-constant coefficients. For example, when $\Lambda = D^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{N}^n$ and $|\mathbf{a}| \geqslant 2$, **VC**$[n]$ for $\Lambda$ is equivalent to a conjecture on Laurent polynomials (see Conjecture 3.1). This conjecture is very similar to a non-trivial theorem (see Theorem 3.11) on Laurent polynomials, which was first conjectured by O. Mathieu [16] and later proved by J. Duistermaat and W. van der Kallen [8].

In general, to consider the questions above, one certainly needs to get better understandings on the $\Lambda$-nilpotency condition, i.e. $\Lambda^m P^m = 0$ for any $m \geqslant 1$. One natural way to look at this condition is to consider the sequences of the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ for general differential operators $\Lambda$ and polynomials $P(z) \in \mathcal{A}$. What special properties do these sequences have so that **VC** wants them all vanish? Do they play any important roles in other areas of mathematics?

The answer to the first question above is still not clear. The answer to the later seems to be "No". It seems that the sequences of the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ do not appear very often in mathematics. But the answer turns out to be "Yes" if one considers the question in the setting of some localizations $\mathcal{B}$ of $\mathcal{A}_n$. Actually, as we will discuss in some detail in subsection 4.1, all classical orthogonal polynomials in one variable have the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ except there one often chooses $P(z)$ from some localizations $\mathcal{B}$ of $\mathcal{A}_n$ and $\Lambda$ a differential operators of $\mathcal{B}$. Some classical polynomials in several variables can also be obtained from sequences of the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ by a slightly modified procedure.

Note that, due to their applications in many different areas of mathematics, especially in ODE, PDE, the eigenfunction problems and representation theory, orthogonal polynomials have been under intense study by mathematicians in the last two centuries. For example, in [18] published in 1940, about 2000 published articles mostly on one-variable orthogonal polynomials have been included. The classical reference for one-variable orthogonal polynomials is [20] (see also [1],

[7], [19]). For multi-variable orthogonal polynomials, see [9], [15] and references there.

It is hard to believe that the connection discussed above between $\Lambda$-nilpotent polynomials or formal power series and classical orthogonal polynomials is just a coincidence. But a precise understanding of this connection still remains mysterious. What is clear is that, $\Lambda$-nilpotent polynomials or formal power series and the polynomials or formal power series $P(z) \in \bar{\mathcal{A}}_n$ such that the sequence $\{\Lambda^m P^m \mid m \geqslant 1\}$ for some differential operator $\Lambda$ provides a sequence of orthogonal polynomials lie in two opposite extreme sides, since, from the same sequence $\{\Lambda^m P^m \mid m \geqslant 1\}$, the former provides nothing but zero; while the later provides an orthogonal basis for $\mathcal{A}_n$.

Therefore, one naturally expects that $\Lambda$-nilpotent polynomials $P(z) \in \mathcal{A}_n$ should be isotropic with respect to a certain $\mathbb{C}$-bilinear form of $\mathcal{A}_n$. It turns out that, as we will show in Theorem 4.1 and Corollary 4.1, it is indeed the case when $P(z)$ is homogeneous and $\Lambda \in \mathbb{D}_2[n]$ is of full rank. Actually, in this case $\Lambda^m P^{m+1}$ ($m \geqslant 0$) are all isotropic with respect to same properly defined $\mathbb{C}$-bilinear form. Note that, Theorem 4.1 and Corollary 4.1 are just transformations of the isotropic properties of HN nilpotent polynomials, which were first proved in [25]. But the proof in [25] is very technical and lacks any convincing interpretations. From the "formal" connection of $\Lambda$-nilpotent polynomials and orthogonal polynomials discussed above, the isotropic properties of homogeneous $\Lambda$-nilpotent polynomials with $\Lambda \in \mathbb{D}_2[n]$ of full rank become much more natural.

The arrangement of the paper is as follows. In Section 2, we mainly show that Conjecture 1.1, hence also **JC**, is equivalent to **VC** or **HVC** for all $\Lambda \in \mathbb{D}_2$ (see Theorem 2.1). One consequence of this equivalence is that, to prove or disprove **VC** or **JC**, the Laplace operators are not the only choices, even though they are the best in many situations. Instead, one can choose any sequence $\Lambda_{n_k} \in \mathbb{D}_2$ with strictly increasing ranks (see Proposition 2.3). For example, one can choose the "Laplace operators" with respect to the Minkowski metric or symplectic metric, or simply choose $\Lambda$ to be the complex $\bar{\partial}$-Laplace operator $\Delta_{\bar{\partial},k}$ ($k \geqslant 1$) in Eq. (2.11).

In Section 3, we transform some results on **JC**, HN polynomials and Conjecture 1.1 in the literature to certain results on $\Lambda$-nilpotent ($\Lambda \in \mathbb{D}_2$) polynomials $P(z)$ and **VC** for $\Lambda$. In subsection 3.1, we discuss some results on the polynomial maps and PDEs associated with $\Lambda$-nilpotent polynomials for $\Lambda \in \mathbb{D}_2[n]$ of full rank (see Theorems 3.1–3.3). The results in this subsection are transformations of those in [24] and [25] on HN polynomials and their associated symmetric polynomial maps.

In subsection 3.2, we give four criteria of $\Lambda$-nilpotency ($\Lambda \in \mathbb{D}_2$) (see Propositions 3.1, 3.2, 3.3 and 3.4). The criteria in this subsection are transformations of the criteria of Hessian nilpotency derived in [25] and [26]. In subsection 3.3, we transform some results in [2], [22] and [23] on **JC**; [5] and [6] on symmetric polynomial maps; [25], [26] and [11] on HN polynomials to certain results on **VC** for $\Lambda \in \mathbb{D}_2$. Finally, we recall a result in [26] which says, **VC** over fields $k$ of characteristic $p > 0$, even under some conditions weaker than $\Lambda$-nilpotency, actually holds for any differential operators $\Lambda$ of $k[z]$ (see Proposition 3.6 and Corollary 3.5).

In subsection 3.4, we consider **VC** for high order differential operators with constant coefficients. In particular, we show in Proposition 3.5 **VC** holds for $\Lambda = \delta^k$ $(k \geqslant 1)$, where $\delta$ is a derivation of $\mathcal{A}$. In particular, **VC** holds for any $\Lambda \in \mathbb{D}_1$ (see Corollary 3.4). We also show that, when $\Lambda = D^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{N}^n$ and $|\mathbf{a}| \geqslant 2$, **VC** is equivalent to a conjecture, Conjecture 3.1, on Laurent polynomials. This conjecture is very similar to a non-trivial theorem (see Theorem 3.11) first conjectured by O. Mathieu [16] and later proved by J. Duistermaat and W. van der Kallen [8].

In subsection 4.1, by using Rodrigues' formulas Eq. (4.1), we show that all classical orthogonal polynomials in one variable have the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ for some $P(z)$ in certain localizations $\mathcal{B}$ of $\mathcal{A}_n$ and $\Lambda$ a differential operators of $\mathcal{B}$. We also show that some classical polynomials in several variables can also be obtained from sequences of the form $\{\Lambda^m P^m \mid m \geqslant 1\}$ with a slight modification. Some of the most classical orthogonal polynomials in one or more variables are briefly discussed in Examples 4.1–4.4, 4.5 and 4.6. In subsection 4.2, we transform the isotropic properties of homogeneous HN homogeneous polynomials derived in [25] to homogeneous $\Lambda$-nilpotent polynomials for $\Lambda \in \mathbb{D}_2[n]$ of full rank (see Theorem 4.1 and Corollary 4.1).

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## 2. The vanishing conjecture for the 2nd order homogeneous differential operators with constant coefficients

In this section, we apply certain linear automorphisms and Lefschetz's principle to show Conjecture 1.1, hence also **JC**, is equivalent to **VC** or **HVC** for all $\Lambda \in \mathbb{D}_2$ (see Theorem 2.1). In subsection 2.1, we fix some notation and recall some lemmas that will be needed throughout this paper. In subsection 2.2, we prove the main results of this section, Theorem 2.1 and Proposition 2.3.

2.1. **Notation and Preliminaries.** Throughout this paper, unless stated otherwise, we will keep using the notations and terminology introduced in the previous section and also the ones fixed as below.

1. For any $P(z) \in \mathcal{A}_n$, we denote by $\nabla P$ the *gradient* of $P(z)$, i.e. we set

$$\nabla P(z) := (D_1 P,\ D_2 P, \ldots, D_n P). \tag{2.1}$$

2. For any $n \geqslant 1$, we let $SM(n, \mathbb{C})$ (resp. $SGL(n, \mathbb{C})$) denote the symmetric complex $n \times n$ (resp. invertible) matrices.

3. For any $A = (a_{ij}) \in SM(n, \mathbb{C})$, we set

$$\Delta_A := \sum_{i,j=1}^{n} a_{ij} D_i D_j \in \mathbb{D}_2[n]. \tag{2.2}$$

   Note that, for any $\Lambda \in \mathbb{D}_2[n]$, there exists a unique $A \in SM(n, \mathbb{C})$ such that $\Lambda = \Delta_A$. We define the *rank* of $\Lambda = \Delta_A$ simply to be the rank of the matrix $A$.
4. For any $n \geqslant 1$, $\Lambda \in \mathbb{D}_2[n]$ is said to be *full rank* if $\Lambda$ has rank $n$. The set of full rank elements of $\mathbb{D}_2[n]$ will be denoted by $\mathbb{D}_2^\circ[n]$.
5. For any $r \geqslant 1$, we set

$$\Delta_r := \sum_{i=1}^{r} D_i^2. \tag{2.3}$$

   Note that $\Delta_r$ is a full rank element in $\mathbb{D}_2[r]$ but not in $\mathbb{D}_2[n]$ for any $n > r$.

For $U \in GL(n, \mathbb{C})$, we define

$$\begin{aligned} \Phi_U : \bar{\mathcal{A}}_n &\to \quad \bar{\mathcal{A}}_n \\ P(z) &\to P(U^{-1}z) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \Psi_U : \mathcal{D}[n] \quad &\to \qquad \mathcal{D}[n] \\ \Lambda \quad &\to \Phi_U \circ \Lambda \circ \Phi_U^{-1}. \end{aligned} \tag{2.5}$$

It is easy to see that, $\Phi_U$ (resp. $\Psi_U$) is an algebra automorphism of $\mathcal{A}_n$ (resp. $\mathcal{D}[n]$). Moreover, the following standard facts are also easy to check directly.

**Lemma 2.1.** (a) *For any $U = (u_{ij}) \in GL(n, \mathbb{C})$, $P(z) \in \bar{\mathcal{A}}_n$ and $\Lambda \in \mathbb{D}[n]$, we have*

$$\Phi_U(\Lambda P) = \Psi_U(\Lambda)\Phi_U(P). \tag{2.6}$$

(b) *For any $1 \leqslant i \leqslant n$ and $f(z) \in \mathcal{A}_n$ we have*

$$\begin{aligned} \Psi_U(D_i) &= \sum_{j=1}^{n} u_{ji} D_j, \\ \Psi_U(f(D)) &= f(U^\tau D). \end{aligned}$$

*In particular, for any $A \in SM(n, \mathbb{C})$, we have*

$$\Psi_U(\Delta_A) = \Delta_{UAU^\tau}. \tag{2.7}$$

The following lemma will play a crucial role in our later arguments. Actually the lemma can be stated in a stronger form (see [13], for example) which we do not need here.

**Lemma 2.2.** *For any $A \in SM(n, \mathbb{C})$ of rank $r > 0$, there exists $U \in GL(n, \mathbb{C})$ such that*

$$A = U \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} U^{\tau}. \tag{2.8}$$

Combining Lemmas 2.1 and 2.2, it is easy to see we have the following corollary.

**Corollary 2.1.** *For any $n \geqslant 1$ and $\Lambda, \Xi \in \mathbb{D}_2[n]$ of same rank, there exists $U \in GL(n, \mathbb{C})$ such that $\Psi_U(\Lambda) = \Xi$.*

2.2. **The vanishing conjecture for the 2nd order homogeneous differential operators with constant coefficients.** In this subsection, we show that Conjecture 1.1, hence also **JC**, is actually equivalent to **VC** or **HVC** for all 2nd order homogeneous differential operators $\Lambda \in \mathbb{D}_2$ (see Theorem 2.1). We also show that the Laplace operators are not the only choices in the study of **VC** or **JC** (see Proposition 2.3 and Example 2.2).

First, let us point out that **VC** fails badly for differential operators with non-constant coefficients. The following counter-example was given by M. de Bondt [3].

**Example 2.1.** Let $x$ be a free variable and $\Lambda = x\frac{d^2}{dx^2}$. Let $P(x) = x$. Then one can check inductively that $P(x)$ is $\Lambda$-nilpotent, but $\Lambda^m P^{m+1} \neq 0$ for any $m \geqslant 1$.

**Lemma 2.3.** *For any $\Lambda \in \mathcal{D}[n]$, $U \in GL(n, \mathbb{C})$, $A \in SM(n, \mathbb{C})$ and $P(z) \in \bar{\mathcal{A}}_n$, we have*

(a) *$P(z)$ is $\Lambda$-nilpotent iff $\Phi_U(P)$ is $\Psi_U(\Lambda)$-nilpotent. In particular, $P(z)$ is $\Delta_A$-nilpotent iff $\Phi_U(P) = P(U^{-1}z)$ is $\Delta_{UAU^{\tau}}$-nilpotent.*
(b) ***VC**$[n]$ (resp. **HVC**$[n]$) holds for $\Lambda$ iff it holds for $\Psi_U(\Lambda)$. In particular, **VC**$[n]$ (resp. **HVC**$[n]$) holds for $\Delta_A$ iff it holds for $\Delta_{UAU^{\tau}}$.*

*Proof.* Note first that, for any $m, k \geqslant 1$, we have

$$\begin{aligned} \Phi_U\left(\Lambda^m P^k\right) &= (\Phi_U \Lambda^m \Phi_U^{-1})\, \Phi_U P^k \\ &= (\Phi_U \Lambda \Phi_U^{-1})^m (\Phi_U P)^k \\ &= [\Psi_U(\Lambda)]^m (\Phi_U P)^k. \end{aligned}$$

When $\Lambda = \Delta_A$, by Eq. (2.7), we further have

$$\Phi_U\left(\Delta_A^m P^k\right) = \Lambda_{UAU^{\tau}}^m (\Phi_U P)^k.$$

Since $\Phi_U$ (resp. $\Psi_U$) is an automorphism of $\bar{\mathcal{A}}_n$ (resp. $\mathcal{D}[n]$), it is easy to check directly that both (a) and (b) follow from the equations above. ∎

Combining the lemma above with Corollary 2.1, we immediately have the following corollary.

**Corollary 2.2.** *Suppose* $\mathbf{HVC}[n]$ (*resp.* $\mathbf{VC}[n]$) *holds for a differential operator* $\Lambda \in \mathbb{D}_2[n]$ *of rank* $r \geqslant 1$*. Then* $\mathbf{HVC}[n]$ (*resp.* $\mathbf{VC}[n]$) *holds for all differential operators* $\Xi \in \mathbb{D}_2[n]$ *of rank* $r$*.*

Actually we can derive much more (as follows) from the conditions in the corollary above.

**Proposition 2.1.** (a) *Suppose* $\mathbf{HVC}[n]$ *holds for a full rank* $\Lambda \in \mathbb{D}_2^{\circ}[n]$*. Then, for any* $k \leqslant n$*,* $\mathbf{HVC}[k]$ *holds for all full rank* $\Xi \in \mathbb{D}_2^{\circ}[k]$*.*

(b) *Suppose* $\mathbf{VC}[n]$ *holds for a full rank* $\Lambda \in \mathbb{D}_2^{\circ}[n]$*. Then, for any* $m \geqslant n$*,* $\mathbf{VC}[m]$ *holds for all* $\Xi \in \mathbb{D}_2[m]$ *of rank* $n$*.*

*Proof.* Note first that, the cases $k = n$ in (a) and $m = n$ in (b) follow directly from Corollary 2.2. So we may assume $k < n$ in (a) and $m > n$ in (b). Secondly, by Corollary 2.2, it will be enough to show $\mathbf{HVC}[k]$ ($k < n$) holds for $\Delta_k$ for (a) and $\mathbf{VC}[m]$ ($m > n$) holds for $\Delta_n$ for (b).

(a) Let $P \in \mathcal{A}_k$ a homogeneous $\Delta_k$-nilpotent polynomial. We view $\Delta_k$ and $P$ as elements of $\mathbb{D}_2[n]$ and $\mathcal{A}_n$, respectively. Since $P$ does not depend on $z_i$ ($k+1 \leqslant i \leqslant n$), for any $m, \ell \geqslant 0$, we have

$$\Delta_k^m P^\ell = \Delta_n^m P^\ell.$$

Hence, $P$ is also $\Delta_n$-nilpotent. Since $\mathbf{HVC}[n]$ holds for $\Delta_n$ (as pointed out at the beginning of the proof), we have $\Delta_k^m P^{m+1} = \Delta_n^m P^{m+1} = 0$ when $m >> 0$. Therefore, $\mathbf{HVC}[k]$ holds for $\Delta_k$.

(b) Let $K$ be the rational function field $\mathbb{C}(z_{n+1}, \ldots, z_m)$. We view $\mathcal{A}_m$ as a subalgebra of the polynomial algebra $K[z_1, \ldots, z_n]$ in the standard way. Note that the differential operator $\Delta_n = \sum_{i=1}^n D_i^2$ of $\mathcal{A}_m$ extends canonically to a differential operator of $K[z_1, \ldots, z_n]$ with constant coefficients.

Since $\mathbf{VC}[n]$ holds for $\Delta_n$ over the complex field (as pointed out at the beginning of the proof), by Lefschetz's principle, we know that $\mathbf{VC}[n]$ also holds for $\Delta_n$ over the field $K$. Therefore, for any $\Delta_n$-nilpotent $P(z) \in \mathcal{A}_m$, by viewing $\Delta_n$ as an element of $\mathbb{D}_2(K[z_1, \ldots, z_n])$ and $P(z)$ an element of $K[z_1, \ldots, z_n]$ (which is still $\Delta_n$-nilpotent in the new setting), we have $\Delta_n^k P^{k+1} = 0$ when $k >> 0$. Hence $\mathbf{VC}[m]$ holds for $P(z) \in \mathcal{A}_m$ and $\Delta_n \in \mathbb{D}_2[m]$. □

**Proposition 2.2.** *Suppose* $\mathbf{HVC}[n]$ *holds for a differential operator* $\Lambda \in \mathbb{D}_2[n]$ *with rank* $r < n$*. Then, for any* $k \geqslant r$*,* $\mathbf{VC}[k]$ *holds for all* $\Xi \in \mathbb{D}_2[k]$ *of rank* $r$*.*

*Proof.* First, by Corollary 2.2, we know $\mathbf{HVC}[n]$ holds for $\Delta_r$. To show Proposition 2.2, by Proposition 2.1, (b), it will be enough to show that $\mathbf{VC}[r]$ holds for $\Delta_r$.

Let $P \in \mathcal{A}_r \subset \mathcal{A}_n$ be a $\Delta_r$-nilpotent polynomial. If $P$ is homogeneous, there is nothing to prove since, as pointed out above, $\mathbf{HVC}[n]$ holds for $\Delta_r$. Otherwise, we homogenize $P(z)$ to $\widetilde{P} \in \mathcal{A}_{r+1} \subseteq \mathcal{A}_n$. Since $\Delta_r$ is a homogeneous differential operator, it is easy to see that, for any $m, k \geqslant 1$, $\Delta_r^m P^k = 0$ iff $\Delta_r^m \widetilde{P}^k = 0$. Therefore, $\widetilde{P} \in \mathcal{A}_n$ is also $\Delta_r$-nilpotent when we view $\Delta_r$ as a differential operator of $\mathcal{A}_n$. Since **HVC**$[n]$ holds for $\Delta_r$, we have that $\Delta_r^m \widetilde{P}^{m+1} = 0$ when $m >> 0$. Then, by the observation above again, we also have $\Delta_r^m P^{m+1} = 0$ when $m >> 0$. Therefore, **VC**$[r]$ holds for $\Delta_r$. □

Now we are ready to prove our main result of this section.

**Theorem 2.1.** *The following statements are equivalent to each other.*

1. **JC** *holds.*
2. **HVC**$[n]$ $(n \geqslant 1)$ *hold for the Laplace operator* $\Delta_n$.
3. **VC**$[n]$ $(n \geqslant 1)$ *hold for the Laplace operator* $\Delta_n$.
4. **HVC**$[n]$ $(n \geqslant 1)$ *hold for all* $\Lambda \in \mathbb{D}_2[n]$.
5. **VC**$[n]$ $(n \geqslant 1)$ *hold for all* $\Lambda \in \mathbb{D}_2[n]$.

*Proof.* First, the equivalences of (1), (2) and (3) have been established in Theorem 7.2 in [25]. While (4) $\Rightarrow$ (2), (5) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) are trivial. Therefore, it will be enough to show (3) $\Rightarrow$ (5).

To show (3) $\Rightarrow$ (5), we fix any $n \geqslant 1$. By Corollary 2.2, it will be enough to show **VC**$[n]$ holds for $\Delta_r$ $(1 \leqslant r \leqslant n)$. But under the assumption of (3) (with $n = r$), we know that **VC**$[r]$ holds for $\Delta_r$. Then, by Proposition 2.1, (b), we know **VC**$[n]$ also holds for $\Delta_r$. □

Next, we show that, to study **HVC**, equivalently **VC** or **JC**, the Laplace operators are not the only choices, even though they are the best in many situations.

**Proposition 2.3.** *Let* $\{n_k \,|\, k \geqslant 1\}$ *be a strictly increasing sequence of positive integers and* $\{\Lambda_{n_k} \,|\, k \geqslant 1\}$ *a sequence of differential operators in* $\mathbb{D}_2$ *with* rank$(\Lambda_{n_k}) = n_k$ $(k \geqslant 1)$. *Suppose that, for any* $k \geqslant 1$, **HVC**$[N_k]$ *holds for* $\Lambda_{n_k}$ *for some* $N_k \geqslant n_k$. *Then, the equivalent statements in Theorem 2.1 hold.*

*Proof.* We show, under the assumption in the proposition, the statement (2) in Theorem 2.1 holds, i.e. for any $n \geqslant 1$, **HVC**$[n]$ $(n \geqslant 1)$ holds for the Laplace operator $\Delta_n \in \mathbb{D}_2[n]$.

For any fixed $n \geqslant 1$, let $k \geqslant 1$ such that $n_k \geqslant n$. If $N_k = n_k$, then, by Proposition 2.1, (a), we have **HVC**$[n]$ $(n \geqslant 1)$ holds for the Laplace operator $\Delta_n \in \mathbb{D}_2[n]$. If $N_k > n_k$, then, by Proposition 2.2, we know **VC**$[n_k]$ (hence also **HVC**$[n_k]$) holds for $\Delta_{n_k}$. Since $n_k \geqslant n$, by Proposition 2.1, (a) again, we know **HVC**$[n]$ does hold for the Laplace operator $\Delta_n$. □

**Example 2.2.** Besides the Laplace operators, by Proposition 2.3, the following sequences of differential operators are also among the most natural choices.

1. Let $n_k = k$ $(k \geqslant 2)$ (or any other strictly increasing sequence of positive integers). Let $\Lambda_{n_k}$ be the "Laplace operator" with respect to the standard Minkowski metric of $\mathbb{R}^{n_k}$. Namely, choose

$$\Lambda_k = D_1^2 - \sum_{i=2}^{k} D_i^2. \tag{2.9}$$

2. Choose $n_k = 2k$ ($k \geqslant 1$) (or any other strictly increasing sequence of positive even numbers). Let $\Lambda_{2k}$ be the "Laplace operator" with respect to the standard symplectic metric on $\mathbb{R}^{2k}$, i.e. choose

$$\Lambda_k = \sum_{i=1}^{k} D_i D_{i+k}. \tag{2.10}$$

3. We may also choose the complex Laplace operators $\Delta_{\bar{\partial}}$ instead of the real Laplace operator $\Delta$. More precisely, we choose $n_k = 2k$ for any $k \geqslant 1$ and view the polynomial algebra of $w_i$ ($1 \leqslant i \leqslant 2k$) over $\mathbb{C}$ as the polynomial algebra $\mathbb{C}[z_i, \bar{z}_i \,|\, 1 \leqslant i \leqslant k]$ by setting $z_i = w_i + \sqrt{-1}\, w_{i+k}$ for any $1 \leqslant i \leqslant k$. Then, for any $k \geqslant 1$, we set

$$\Lambda_k = \Delta_{\bar{\partial},k} := \sum_{i=1}^{k} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}. \tag{2.11}$$

4. More generally, we may also choose $\Lambda_k = \Delta_{A_{n_k}}$, where $n_k \in \mathbb{N}$ and $A_{n_k} \in SM(n_k, \mathbb{C})$ (not necessarily invertible) ($k \geqslant 1$) with strictly increasing ranks.

## 3. Some properties of $\Delta_A$-nilpotent polynomials

As pointed earlier in Section 1 (see page 260), for the Laplace operators $\Delta_n$ ($n \geqslant 1$), the notion $\Delta_n$-nilpotency coincides with the notion of Hessian nilpotency. HN (Hessian nilpotent) polynomials or formal power series, their associated symmetric polynomial maps and Conjecture 1.1 have been studied in [5], [6], [24]–[26] and [11]. In this section, we apply Corollary 2.1, Lemma 2.3 and also Lefschetz's principle to transform some results obtained in the references above to certain results on $\Lambda$-nilpotent ($\Lambda \in \mathbb{D}_2$) polynomials or formal power series, **VC** for $\Lambda$ and also associated polynomial maps. Another purpose of this section is to give a short survey on some results on HN polynomials and Conjecture 1.1 in the more general setting of $\Lambda$-nilpotent polynomials and **VC** for differential operators $\Lambda \in \mathbb{D}_2$.

In subsection 3.1, we transform some results in [24] and [25] to the setting of $\Lambda$-nilpotent polynomials for $\Lambda \in \mathbb{D}_2[n]$ of full rank (see Theorems 3.1–3.3). In subsection 3.2, we derive four criteria for $\Lambda$-nilpotency ($\Lambda \in \mathbb{D}_2$) (see Propositions 3.1, 3.2, 3.3 and 3.4). The criteria in this subsection are transformations of the criteria of Hessian nilpotency derived in [25] and [26].

In subsection 3.3, we transform some results in [2], [22] and [23] on **JC**; [5] and [6] on symmetric polynomial maps; [25], [26] and [11] on HN polynomials to certain results on **VC** for $\Lambda \in \mathbb{D}_2$. In subsection 3.4, we consider **VC** for high order differential operators with constant coefficients. We mainly focus on the differential operators $\Lambda = D^{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^n$). Surprisingly, **VC** for these operators is equivalent to a conjecture (see Conjecture 3.1) on Laurent polynomials, which is similar to a non-trivial theorem (see Theorem 3.11) first conjectured by O. Mathieu [16] and later proved by J. Duistermaat and W. van der Kallen [8].

3.1. **Associated polynomial maps and PDEs.** Once and for all in this section, we fix any $n \geqslant 1$ and $A \in SM(n, \mathbb{C})$ of rank $1 \leqslant r \leqslant n$. We use $z$ and $D$, unlike we did before, to denote the $n$-tuples $(z_1, z_2, \dots, z_n)$ and $(D_1, D_2, \dots, D_n)$, respectively. We define a $\mathbb{C}$-bilinear form $\langle \cdot, \cdot \rangle_A$ by setting $\langle u, v \rangle_A := u^\tau A v$ for any $u, v \in \mathbb{C}^n$. Note that, when $A = I_{n\times n}$, the bilinear form defined above is just the standard $\mathbb{C}$-bilinear form of $\mathbb{C}^n$, which we also denote by $\langle \cdot, \cdot \rangle$.

By Lemma 2.2, we may write $A$ as in Eq. (2.8). For any $P(z) \in \bar{\mathcal{A}}_n$, we set

$$\widetilde{P}(z) = \Phi_U^{-1} P(z) = P(Uz). \tag{3.1}$$

Note that, by Lemma 2.1, (b), we have $\Psi_U^{-1}(\Delta_A) = \Delta_r$. By Lemma 2.3, (a), $P(z)$ is $\Delta_A$-nilpotent iff $\widetilde{P}(z)$ is $\Delta_r$-nilpotent.

**Theorem 3.1.** *Let $t$ be a central parameter. For any $P(z) \in \mathcal{A}_n$ with $o(P(z)) \geqslant 2$ and $A \in SGL(n, \mathbb{C})$, set $F_{A,t}(z) := z - tA\nabla P(z)$. Then*

(a) *there exists a unique $Q_{A,t}(z) \in \mathbb{C}[t][[z]]$ such that the formal inverse map $G_{A,t}(z)$ of $F_{A,t}(z)$ is given by*

$$G_{A,t}(z) = z + tA\nabla Q_{A,t}(z). \tag{3.2}$$

(b) *The $Q_{A,t}(z) \in \mathbb{C}[t][[z]]$ in (a) is the unique formal power series solution of the following Cauchy problem:*

$$\begin{cases} \frac{\partial Q_{A,t}}{\partial t}(z) = \frac{1}{2} \langle \nabla Q_{A,t}, \nabla Q_{A,t} \rangle_A, \\ Q_{A,t=0}(z) = P(z). \end{cases} \tag{3.3}$$

*Proof.* Let $\widetilde{P}$ as given in Eq. (3.1) and set

$$\widetilde{F}_{A,t}(z) = z - t\nabla \widetilde{P}(z). \tag{3.4}$$

By Theorem 3.6 in [24], we know the formal inverse map $\widetilde{G}_{A,t}(z)$ of $\widetilde{F}_{A,t}(z)$ is given by

$$\widetilde{G}_{A,t}(z) = z + t\nabla \widetilde{Q}_{A,t}(z), \tag{3.5}$$

where $\widetilde{Q}_{A,t}(z) \in \mathbb{C}[t][[z]]$ is the unique formal power series solution of the following Cauchy problem:

$$\begin{cases} \frac{\partial \widetilde{Q}_{A,t}}{\partial t}(z) = \frac{1}{2} \langle \nabla \widetilde{Q}_{A,t}, \nabla \widetilde{Q}_{A,t} \rangle, \\ \widetilde{Q}_{A,t=0}(z) = \widetilde{P}(z). \end{cases} \tag{3.6}$$

From the fact that $\nabla \widetilde{P}(z) = (U^\tau \nabla P)(Uz)$, it is easy to check that

$$(\Phi_U \circ \widetilde{F}_{A,t} \circ \Phi_U^{-1})(z) = z - tA\nabla P(z) = F_{A,t}(z), \tag{3.7}$$

which is the formal inverse map of

$$(\Phi_U \circ \widetilde{G}_{A,t} \circ \Phi_U^{-1})(z) = z + t(U\nabla \widetilde{Q}_{A,t})(U^{-1}z). \tag{3.8}$$

Set

$$Q_{A,t}(z) := \widetilde{Q}_{A,t}(U^{-1}z). \tag{3.9}$$

Then we have

$$\nabla Q_{A,t}(z) = (U^\tau)^{-1}(\nabla \widetilde{Q}_{A,t})(U^{-1}z),$$
$$U^\tau \nabla Q_{A,t}(z) = (\nabla \widetilde{Q}_{A,t})(U^{-1}z), \tag{3.10}$$

Multiplying $U$ to the both sides of the equation above and noticing that $A = UU^\tau$ by Eq. (2.8) since $A$ is of full rank, we get

$$A\nabla Q_{A,t}(z) = (U\nabla \widetilde{Q}_{A,t})(U^{-1}z). \tag{3.11}$$

Then, combining Eq. (3.8) and the equation above, we see the formal inverse $G_{A,t}(z)$ of $F_{A,t}(z)$ is given by

$$G_{A,t}(z) = (\Phi_U \circ \widetilde{G}_{A,t} \circ \Phi_U^{-1})(z) = z + tA\nabla Q_{A,t}(z). \tag{3.12}$$

Applying $\Phi_U$ to Eq. (3.6) and by Eqs. (3.9), (3.10), we see that $Q_{A,t}(z)$ is the unique formal power series solution of the Cauchy problem Eq. (3.3). ☐

By applying the linear automorphism $\Phi_U$ of $\mathbb{C}[[z]]$ and employing a similar argument as in the proof of Theorem 3.1 above, we can generalize Theorems 3.1 and 3.4 in [25] to the following theorem on $\Delta_A$-nilpotent ($A \in SGL(n, \mathbb{C})$) formal power series.

**Theorem 3.2.** *Let $A$, $P(z)$ and $Q_{A,t}(z)$ be as in Theorem 3.1. We further assume $P(z)$ is $\Delta_A$-nilpotent. Then,*

(a) *$Q_{A,t}(z)$ is the unique formal power series solution of the following Cauchy problem:*

$$\begin{cases} \frac{\partial Q_{A,t}}{\partial t}(z) = \frac{1}{4}\,\Delta_A Q_{A,t}^2, \\ Q_{A,t=0}(z) = P(z). \end{cases} \tag{3.13}$$

(b) *For any $k \geqslant 1$, we have*

$$Q_{A,t}^k(z) = \sum_{m\geqslant 1} \frac{t^{m-1}}{2^m m!(m+k)!}\,\Delta_A^m P^{m+1}(z). \tag{3.14}$$

Applying the same strategy to Theorem 3.2 in [25], we get the following theorem.

**Theorem 3.3.** *Let $A$, $P(z)$ and $Q_{A,t}(z)$ be as in Theorem 3.2. For any non-zero $s \in \mathbb{C}$, set*

$$V_{t,s}(z) := \exp(sQ_t(z)) = \sum_{k=0}^{\infty} \frac{s^k Q_t^k(z)}{k!}.$$

*Then, $V_{t,s}(z)$ is the unique formal power series solution of the following Cauchy problem of the heat-like equation:*

$$\begin{cases} \frac{\partial V_{t,s}}{\partial t}(z) = \frac{1}{2s}\,\Delta_A V_{t,s}(z), \\ U_{t=0,s}(z) = \exp(sP(z)). \end{cases} \tag{3.15}$$

3.2. **Some criteria of $\Delta_A$-nilpotency.** In this subsection, with the notation and remarks fixed in the previous subsection in mind, we apply the linear automorphism $\Phi_U$ to transform some criteria of Hessian nilpotency derived in [25] and [26] to criteria of $\Delta_A$-nilpotency ($A \in SM(n, \mathbb{C})$) (see the Propositions 3.1, 3.2, 3.3 and 3.4 below).

**Proposition 3.1.** *Let $A$ be given as in Eq. (2.8). Then, for any $P(z) \in \mathcal{A}_n$, it is $\Delta_A$-nilpotent iff the submatrix of $U^\tau (Hes\, P)\, U$ consisting of the first $r$ rows and $r$ columns is nilpotent.*

*In particular, when $r = n$, i.e. $\Delta_A$ is full rank, any $P(z) \in \mathbb{D}_2[n]$ is $\Delta_A$-nilpotent iff $U^\tau (Hes\, P)\, U$ is nilpotent.*

*Proof.* Let $\widetilde{P}(z)$ be as in Eq. (3.1). Then, as pointed earlier, $P(z)$ is $\Delta_A$-nilpotent iff $\widetilde{P}(z)$ is $\Delta_r$-nilpotent.

If $r = n$, then by Theorem 1.1 , $\widetilde{P}(z)$ is $\Delta_r$-nilpotent iff $\operatorname{Hes} \widetilde{P}(z)$ is nilpotent. But note that in general we have

$$\operatorname{Hes} \widetilde{P}(z) = \operatorname{Hes} P(Uz) = U^\tau [(\operatorname{Hes} P)(Uz)]\, U. \tag{3.16}$$

Therefore, $\operatorname{Hes} \widetilde{P}(z)$ is nilpotent iff $U^\tau [(\operatorname{Hes} P)(Uz)]\, U$ is nilpotent iff, with $z$ replaced by $U^{-1}z$, $U^\tau [(\operatorname{Hes} P)(z)]\, U$ is nilpotent. Hence the proposition follows in this case.

Assume $r < n$. We view $\mathcal{A}_r$ as a subalgebra of the polynomial algebra $K[z_1, \dots, z_r]$, where $K$ is the rational field $\mathbb{C}(z_{r+1}, \dots, z_n)$. By Theorem 1.1 and Lefschetz's principle, we know that $\widetilde{P}$ is $\Delta_r$-nilpotent iff the matrix $\left(\frac{\partial^2 \widetilde{P}}{\partial z_i \partial z_j}\right)_{1 \leqslant i,j \leqslant r}$ is nilpotent.

Note that the matrix $\left(\frac{\partial^2 \widetilde{P}}{\partial z_i \partial z_j}\right)_{1 \leqslant i,j \leqslant r}$ is the submatrix of $\operatorname{Hes} \widetilde{P}(z)$ consisting of the first $r$ rows and $r$ columns. By Eq. (3.16), it is also the submatrix of $U^\tau [\operatorname{Hes} P(Uz)]\, U$ consisting of the first $r$ rows and $r$ columns. Replacing $z$ by $U^{-1}z$ in the submatrix above, we see $\left(\frac{\partial^2 \widetilde{P}}{\partial z_i \partial z_j}\right)_{1 \leqslant i,j \leqslant r}$ is nilpotent iff the submatrix of $U^\tau [\operatorname{Hes} P(z)]\, U$ consisting of the first $r$ rows and $r$ columns is nilpotent. Hence the proposition follows. $\square$

Note that, for any homogeneous quadratic polynomial $P(z) = z^\tau B z$ with $B \in SM(n, \mathbb{C})$, we have $\operatorname{Hes} P(z) = 2B$. Then, by Proposition 3.1, we immediately have the following corollary.

**Corollary 3.1.** *For any homogeneous quadratic polynomial $P(z) = z^\tau B z$ with $B \in SM(n, \mathbb{C})$, it is $\Delta_A$-nilpotent iff the submatrix of $U^\tau B\, U$ consisting of the first $r$ rows and $r$ columns is nilpotent.*

**Proposition 3.2.** *Let $A$ be given as in Eq. (2.8). Then, for any $P(z) \in \bar{\mathcal{A}}_n$ with $o(P(z)) \geqslant 2$, $P(z)$ is $\Delta_A$-nilpotent iff $\Delta_A^m P^m = 0$ for any $1 \leqslant m \leqslant r$.*

*Proof.* Again, we let $\widetilde{P}(z)$ be as in Eq. (3.1) and note that $P(z)$ is $\Delta_A$-nilpotent iff $\widetilde{P}(z)$ is $\Delta_r$-nilpotent.

Since $r \leqslant n$, we view $\mathcal{A}_r$ as a subalgebra of the polynomial algebra $K[z_1, \ldots, z_r]$, where $K$ is the rational field $\mathbb{C}(z_{r+1}, \ldots, z_n)$. By Theorem 1.1 and Lefschetz's principle (if $r < n$), we have $\widetilde{P}(z)$ is $\Delta_r$-nilpotent iff $\Delta_r^m \widetilde{P}^m = 0$ for any $1 \leqslant m \leqslant r$. On the other hand, by Eqs. (2.6) and (2.7), we have $\Phi_U \left( \Delta_r^m \widetilde{P}^m \right) = \Delta_A^m P^m$ for any $m \geqslant 1$. Since $\Phi_U$ is an automorphism of $\mathcal{A}_n$, we have that, $\Delta_r^m \widetilde{P}^m = 0$ for any $1 \leqslant m \leqslant r$ iff $\Delta_A^m P^m = 0$ for any $1 \leqslant m \leqslant r$. Therefore, $\widetilde{P}(z)$ is $\Delta_A$-nilpotent iff $\Delta_A^m P^m = 0$ for any $1 \leqslant m \leqslant r$. Hence the proposition follows. $\square$

**Proposition 3.3.** *For any $A \in SGL(n, \mathbb{C})$ and any homogeneous $P(z) \in \mathcal{A}_n$ of degree $d \geqslant 2$, we have, $P(z)$ is $\Delta_A$-nilpotent iff, for any $\beta \in \mathbb{C}$, $(\beta_D)^{d-2} P(z)$ is $\Lambda$-nilpotent, where $\beta_D := \langle \beta, D \rangle$.*

*Proof.* Let $A$ be given as in Eq. (2.8) and $\widetilde{P}(z)$ as in Eq. (3.1). Note that, $\Psi_U^{-1}(\Delta_A) = \Delta_n$ (for $\Delta_A$ is of full rank), and $P(z)$ is $\Delta_A$-nilpotent iff $\widetilde{P}(z)$ is $\Delta_n$-nilpotent.

Since $\widetilde{P}$ is also homogeneous of degree $d \geqslant 2$, by Theorem 1.1 in [26], we know that, $\widetilde{P}(z)$ is $\Delta_n$-nilpotent iff, for any $\beta \in \mathbb{C}^n$, $\beta_D^{d-2} \widetilde{P}$ is $\Delta_n$-nilpotent. Note that, from Lemma 2.1, (b), we have

$$
\begin{aligned}
\Psi_U(\beta_D) &= \langle \beta, U^\tau D \rangle \\
&= \langle U\beta, D \rangle \\
&= (U\beta)_D,
\end{aligned}
$$

and

$$
\Phi_U(\beta_D^{d-2} \widetilde{P}) = \Psi_U(\beta_D)^{d-2} \Phi_U(\widetilde{P}) = (U\beta)_D^{d-2} P. \tag{3.17}
$$

Therefore, by Lemma 2.3, (a), $\beta_D^{d-2} \widetilde{P}$ is $\Delta_n$-nilpotent iff $(U\beta)_D^{d-2} P$ is $\Delta_A$-nilpotent since $\Psi_U(\Delta_n) = \Delta_A$. Combining all equivalences above, we have $P(z)$ is $\Delta_n$-nilpotent iff, for any $\beta \in \mathbb{C}^n$, $(U\beta)_D^{d-2} P$ is $\Delta_A$-nilpotent. Since $U$ is invertible, when $\beta$ runs over $\mathbb{C}^n$ so does $U\beta$. Therefore the proposition follows. $\square$

Let $\{e_i \,|\, 1 \leqslant i \leqslant n\}$ be the standard basis of $\mathbb{C}^n$. Applying the proposition above to $\beta = e_i$ $(1 \leqslant i \leqslant n)$, we have the following corollary.

**Corollary 3.2.** *For any homogeneous $\Delta_A$-nilpotent polynomial $P(z) \in \mathcal{A}_n$ of degree $d \geqslant 2$, $D_i^{d-2} P(z)$ $(1 \leqslant i \leqslant n)$ are also $\Delta_A$-nilpotent.*

We think that Proposition 3.3 and Corollary 3.2 are interesting because, due to Corollary 3.1, it is much easier to decide whether a quadratic form is $\Delta_A$-nilpotent or not.

To state the next criterion, we need to fix the following notation.

For any $A \in SGL(n, \mathbb{C})$, we let $X_A(\mathbb{C}^n)$ be the set of isotropic vectors $u \in \mathbb{C}^n$ with respect to the $\mathbb{C}$-bilinear form $\langle \cdot, \cdot \rangle_A$. When $A = I_{n\times n}$, we also denote $X_A(\mathbb{C}^n)$ simply by of $X(\mathbb{C}^n)$.

For any $\beta \in \mathbb{C}^n$, we set $h_\alpha(z) := \langle \alpha, z\rangle$. Then, by applying $\Phi_U$ to a well-known theorem on classical harmonic polynomials, which is the following theorem for $A = I_{n\times n}$ (see, for example, [12] and [21]), we have the following result on homogeneous $\Delta_A$-nilpotent polynomials.

**Theorem 3.4.** *Let $P$ be any homogeneous polynomial of degree $d \geqslant 2$ such that $\Delta_A P = 0$. We have*

$$P(z) = \sum_{i=1}^{k} h_{\alpha_i}^d(z) \tag{3.18}$$

*for some $k \geqslant 1$ and $\alpha_i \in X_A(\mathbb{C}^n)$ $(1 \leqslant i \leqslant k)$.*

Next, for any homogeneous polynomial $P(z)$ of degree $d \geqslant 2$, we introduce the following matrices:

$$\Xi_P := (\langle \alpha_i, \alpha_j\rangle_A)_{k\times k}\,, \tag{3.19}$$

$$\Omega_P := \Big(\langle \alpha_i, \alpha_j\rangle_A\, h_{\alpha_j}^{d-2}(z)\Big)_{k\times k}\,. \tag{3.20}$$

Then, by applying $\Phi_U$ to Proposition 5.3 in [25] (the details will be omitted here), we have the following criterion of $\Delta_A$-nilpotency for homogeneous polynomials.

**Proposition 3.4.** *Let $P(z)$ be as given in Eq. (3.18). Then $P(z)$ is $\Delta_A$-nilpotent iff the matrix $\Omega_P$ is nilpotent.*

One simple remark on the criterion above is as follows.

Let $B$ be the $k \times k$ diagonal matrix with $h_{\alpha_i}(z)$ $(1 \leqslant i \leqslant k)$ as the $i^{th}$ diagonal entry. For any $1 \leqslant j \leqslant k$, set

$$\Omega_{P;j} := B^j \Xi_P B^{d-2-j} = (h_{\alpha_i}^j \langle \alpha_i, \alpha_j\rangle h_{\alpha_j}^{d-2-j}). \tag{3.21}$$

Then, by repeatedly applying the fact that, for any $C, D \in M(k, \mathbb{C})$, $CD$ is nilpotent iff so is $DC$, it is easy to see that Proposition 3.4 can also be re-stated as follows.

**Corollary 3.3.** *Let $P(z)$ be given by Eq. (3.18) with $d \geqslant 2$. Then, for any $1 \leqslant j \leqslant d-2$ and $m \geqslant 1$, $P(z)$ is $\Delta_A$-nilpotent iff the matrix $\Omega_{P;j}$ is nilpotent.*

Note that, when $d$ is even, we may choose $j = (d-2)/2$. So $P$ is $\Delta_A$-nilpotent iff the symmetric matrix

$$\Omega_{P;(d-2)/2} = (h_{\alpha_i}^{(d-2)/2} \langle \alpha_i, \alpha_j\rangle_A h_{\alpha_j}^{(d-2)/2}) \tag{3.22}$$

is nilpotent.

3.3. **Some results on the vanishing conjecture of the 2nd order homogeneous differential operators with constants coefficients.** In this subsection, we transform some known results of **VC** for the Laplace operators $\Delta_n$ ($n \geqslant 1$) to certain results on **VC** for $\Delta_A$ ($A \in SGL(n, \mathbb{C})$).

First, by Wang's theorem [22], we know that **JC** holds for any polynomial maps $F(z)$ with $\deg F \leqslant 2$. Hence, **JC** also holds for symmetric polynomials $F(z) = z - \nabla P(z)$ with $P(z) \in \mathbb{C}[z]$ of degree $d \leqslant 3$. By the equivalence of **JC** and **VC** for the Laplace operators established in [25], we know **VC** holds if $\Lambda = \Delta_n$ and $P(z)$ is a HN polynomials of degree $d \leqslant 3$. Then, applying the linear automorphism $\Phi_U$, we have the following proposition.

**Theorem 3.5.** *For any $A \in SGL(n, \mathbb{C})$ and $\Delta_A$-nilpotent $P(z) \in \mathcal{A}_n$ (not necessarily homogeneous) of degree $d \leqslant 3$, we have $\Lambda^m P^{m+1} = 0$ when $m >> 0$, i.e.* **VC**$[n]$ *holds for $\Lambda$ and $P(z)$.*

Applying the classical homogeneous reduction on **JC** (see [2], [23]) to associated symmetric maps, we know that, to show **VC** for $\Delta_n$ ($n \geqslant 1$), it will be enough to consider only homogeneous HN polynomials of degree 4. Therefore, by applying the linear automorphism $\Phi_U$ of $\mathcal{A}_n$, we have the same reduction for **HVC** too.

**Theorem 3.6.** *To study* **HVC** *in general, it will be enough to consider only homogeneous $P(z) \in \mathcal{A}$ of degree 4.*

In [5] and [6] it has been shown that **JC** holds for symmetric maps $F(z) = z - \nabla P(z)$ ($P(z) \in \mathcal{A}_n$) if the number of variables $n$ is less or equal to 4, or $n = 5$ and $P(z)$ is homogeneous. By the equivalence of **JC** for symmetric polynomial maps and **VC** for the Laplace operators established in [25], and Proposition 2.2 and Corollary 2.2, we have the following results on **VC** and **HVC**.

**Theorem 3.7.** (a) *For any $n \geqslant 1$,* **VC**$[n]$ *holds for any $\Lambda \in \mathbb{D}_2$ of rank $1 \leqslant r \leqslant 4$.*

(b) **HVC**$[5]$ *holds for any $\Lambda \in \mathbb{D}_2[5]$.*

Note that the following vanishing properties of HN formal power series have been proved in Theorem 6.2 in [25] for the Laplace operators $\Delta_n$ ($n \geqslant 1$). By applying the linear automorphism $\Phi_U$, one can show it also holds for any $\Lambda$-nilpotent ($\Lambda \in \mathbb{D}_2$) formal power series.

**Theorem 3.8.** *Let $\Lambda \in \mathbb{D}_2[n]$ and $P(z) \in \bar{\mathcal{A}}_n$ be $\Lambda$-nilpotent with $o(P) \geqslant 2$. The following statements are equivalent.*

(1) *$\Lambda^m P^{m+1} = 0$ when $m >> 0$.*
(2) *There exists $k_0 \geqslant 1$ such that $\Lambda^m P^{m+k_0} = 0$ when $m >> 0$.*
(3) *For any fixed $k \geqslant 1$, $\Lambda^m P^{m+k} = 0$ when $m >> 0$.*

By applying the linear automorphism $\Phi_U$, one can transform Theorem 1.5 in [11] on **VC** of the Laplace operators to the following result on **VC** of $\Lambda \in \mathbb{D}_2$.

**Theorem 3.9.** *Let $\Lambda \in \mathbb{D}_2[n]$ and $P(z) \in \bar{\mathcal{A}}_n$ be any $\Lambda$-nilpotent polynomial with $o(P) \geqslant 2$. Then* **VC** *holds for $\Lambda$ and $P(z)$ iff, for any $g(z) \in \mathcal{A}_n$, we have $\Lambda^m(g(z)P^m) = 0$ when $m >> 0$.*

In [11], the following theorem has also been proved for $\Lambda = \Delta_n$. Next we show it is also true in general.

**Theorem 3.10.** *Let $A \in SGL(n, \mathbb{C})$ and $P(z) \in \mathcal{A}_n$ be a homogeneous $\Delta_A$-nilpotent polynomial with $\deg P \geqslant 2$. Assume that $\sigma_{A^{-1}}(z) := z^\tau A^{-1} z$ and the partial derivatives $\frac{\partial P}{\partial z_i}$ $(1 \leqslant i \leqslant n)$ have no non-zero common zeros. Then* **HVC**$[n]$ *holds for $\Delta_A$ and $P(z)$.*

*In particular, if the projective subvariety determined by the ideal $\langle P(z) \rangle$ of $\mathcal{A}_n$ is regular,* **HVC**$[n]$ *holds for $\Delta_A$ and $P(z)$.*

*Proof.* Let $\widetilde{P}$ be as given in Eq. (3.1). By Theorem 1.2 in [11], we know that, when $\sigma_2(z) := \sum_{i=1}^n z_i^2$ and the partial derivatives $\frac{\partial \widetilde{P}}{\partial z_i}$ $(1 \leqslant i \leqslant n)$ have no non-zero common zeros, **HVC**$[n]$ holds for $\Delta_n$ and $\widetilde{P}$. Then, by Lemma 2.3, (b), **HVC**$[n]$ also holds for $\Delta_A$ and $P$.

But, on the other hand, since $U$ is invertible and, for any $1 \leqslant i \leqslant n$,

$$\frac{\partial \widetilde{P}}{\partial z_i} = \sum_{j=1}^n u_{ji} \frac{\partial P}{\partial z_j}(Uz),$$

$\sigma_2(z)$ and $\frac{\partial \widetilde{P}}{\partial z_i}$ $(1 \leqslant i \leqslant n)$ have no non-zero common zeros iff $\sigma_2(z)$ and $\frac{\partial P}{\partial z_i}(Uz)$ $(1 \leqslant i \leqslant n)$ have no non-zero common zeros, and iff, with $z$ replaced by $U^{-1}z$, $\sigma_2(U^{-1}z) = \sigma_{A^{-1}}(z)$ and $\frac{\partial P}{\partial z_i}(z)$ $(1 \leqslant i \leqslant n)$ have no non-zero common zeros. Therefore, the theorem holds. ☐

3.4. **The vanishing conjecture for higher order differential operators with constant coefficients.** Even though the most interesting case of **VC** is for $\Lambda \in \mathbb{D}_2$, at least when **JC** is concerned, the case of **VC** for higher order differential operators with constant coefficients is also interesting and non-trivial. In this subsection, we mainly discuss **VC** for the differential operators $D^{\mathbf{a}}$ $(\mathbf{a} \in \mathbb{N}^n)$. At the end of this subsection, we also recall a result proved in [26] which says that, when the base field has characteristic $p > 0$, **VC**, even under a weaker condition, actually holds for any differential operator $\Lambda$ (not necessarily with constant coefficients).

Let $\beta_j \in \mathbb{C}^n$ $(1 \leqslant j \leqslant \ell)$ be linearly independent and set $\delta_j := \langle \beta_j, D \rangle$. Let $\Lambda = \prod_{j=1}^{\ell} \delta_j^{a_i}$ with $a_j \geqslant 1$ $(1 \leqslant j \leqslant \ell)$.

When $\ell = 1$, **VC** for $\Lambda$ can be proved easily as follows.

**Proposition 3.5.** *Let $\delta \in \mathbb{D}_1[z]$ and $\Lambda = \delta^k$ for some $k \geqslant 1$. Then*

(a) *A polynomial $P(z)$ is $\Lambda$-nilpotent if (and only if) $\Lambda P = 0$.*
(b) **VC** *holds for $\Lambda$.*

*Proof.* Applying a change of variables, if necessary, we may assume $\delta = D_1$ and $\Lambda = D_1^k$.

Let $P(z) \in \mathbb{C}[z]$ such that $\Lambda P(z) = D_1^k P(z) = 0$. Let $d$ be the degree of $P(z)$ in $z_1$. From the equation above, we have $k > d$. Therefore, for any $m \geqslant 1$, we have $km > dm$ which implies $\Lambda^m P(z)^m = D_1^{km} P^m(z) = 0$. Hence, we have (a).

To show (b), let $P(z)$ be a $\Lambda$-nilpotent polynomial. By the same notation and argument above, we have $k > d$. Choose a positive integer $N > \frac{d}{k-d}$. Then, for any $m \geqslant N$, we have $m > \frac{d}{k-d}$, which is equivalent to $(m+1)d < km$. Hence we have $\Lambda^m P(z)^{m+1} = D_1^{km} P^{m+1}(z) = 0$. $\square$

In particular, when $k = 1$ in the proposition above, we have the following corollary.

**Corollary 3.4.** **VC** *holds for any differential operator* $\Lambda \in \mathbb{D}_1$.

Next we consider the case $\ell \geqslant 2$. Note that, when $\ell = 2$ and $a_1 = a_2 = 1$. $\Lambda \in \mathbb{D}_2$ and has rank 2. Then, by Theorem 3.7, we know **VC** holds for $\Lambda$.

Beside the case above, **VC** for $\Lambda = \prod_{j=1}^{\ell} \delta_j^{a_i}$ with $\ell \geqslant 2$ seems to be non-trivial at all. Actually, we will show below, it is equivalent to a conjecture (see Conjecture 3.1) on Laurent polynomials.

First, by applying a change of variables, if necessary, we may (and will) assume $\Lambda = D^{\mathbf{a}}$ with $\mathbf{a} \in (\mathbb{N}^+)^\ell$. Secondly, note that, for any $\mathbf{b} \in \mathbb{N}^n$ and $h(z) \in \mathbb{C}[z]$, $D^{\mathbf{b}} h(z) = 0$ iff the holomorphic part of the Laurent polynomial $z^{-\mathbf{b}} h(z)$ is zero.

Now we fix a $P(z) \in \mathbb{C}[z]$ and set $f(z) := z^{-\mathbf{a}} P(z)$. With the observation above, it is easy to see that, $P(z)$ is $D^{\mathbf{a}}$-nilpotent iff the holomorphic parts of the Laurent polynomials $f^m(z)$ $(m \geqslant 1)$ are all zero; and **VC** holds for $\Lambda$ and $P(z)$ iff the holomorphic part of $P(z) f^m(z)$ is zero when $m >> 0$. Therefore, **VC** for $D^{\mathbf{a}}$ can be restated as follows:

**Re-stated VC for** $\Lambda = D^{\mathbf{a}}$: *Let* $P(z) \in \mathcal{A}_n$ *and* $f(z)$ *be as above. Suppose that, for any* $m \geqslant 1$*, the holomorphic part of the Laurent polynomial* $f^m(z)$ *is zero, then the holomorphic part of* $P(z) f^m(z)$ *equals to zero when* $m >> 0$.

Note that the re-stated **VC** above is very similar to the following non-trivial theorem which was first conjectured by O. Mathieu [16] and later proved by J. Duistermaat and W. van der Kallen [8].

**Theorem 3.11.** *Let* $f$ *and* $g$ *be Laurent polynomials in* $z$*. Assume that, for any* $m \geqslant 1$*, the constant term of* $f^m$ *is zero. Then the constant term* $g f^m$ *equals zero when* $m >> 0$.

Note that, Mathieu's conjecture [16] is a conjecture on all real compact Lie groups $G$, which is also mainly motivated by **JC**. The theorem above is the special case of Mathieu's conjecture when $G$ the $n$-dimensional real torus. For other compact real Lie groups, Mathieu's conjecture seems to be still wide open.

Motivated by Theorem 3.11, the above re-stated **VC** for $\Lambda = D^{\mathbf{a}}$ and also the result on **VC** in Theorem 3.9, we would like to propose the following conjecture on Laurent polynomials.

**Conjecture 3.1.** *Let $f$ and $g$ be Laurent polynomials in $z$. Assume that, for any $m \geqslant 1$, the holomorphic part of $f^m$ is zero. Then the holomorphic part $gf^m$ equals zero when $m >> 0$.*

Note that, a positive answer to the conjecture above will imply **VC** for $\Lambda = D^{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^n$) by simply choosing $g(z)$ to be $P(z)$.

Finally let us to point out that, it is well-known that **JC** does not hold over fields of finite characteristic (see [2], for example), but, by Proposition 5.3 in [26], the situation for VC over fields of finite characteristic is dramatically different even though it is equivalent to **JC** over the complex field $\mathbb{C}$.

**Proposition 3.6.** *Let $k$ be a field of char. $p > 0$ and $\Lambda$ any differential operator of $k[z]$. Let $f \in k[[z]]$. Assume that, for any $1 \leqslant m \leqslant p-1$, there exists $N_m > 0$ such that $\Lambda^{N_m} f^m = 0$. Then, $\Lambda^m f^{m+1} = 0$ when $m >> 0$.*

From the proposition above, we immediately have the following corollary.

**Corollary 3.5.** *Let $k$ be a field of char. $p > 0$. Then*

(a) **VC** *holds for any differential operator $\Lambda$ of $k[z]$.*

(b) *If $\Lambda$ strictly decreases the degree of polynomials, then, for any polynomial $f \in k[z]$ (not necessarily $\Lambda$-nilpotent), we have $\Lambda^m f^{m+1} = 0$ when $m >> 0$.*

## 4. A remark on $\Lambda$-nilpotent polynomials and classical orthogonal polynomials

In this section, we first in subsection 4.1 consider the "formal" connection between $\Lambda$-nilpotent polynomials or formal power series and classical orthogonal polynomials, which has been discussed in Section 1 (see page 261). We then in subsection 4.2 transform the isotropic properties of homogeneous HN polynomials proved in [25] to isotropic properties of homogeneous $\Delta_A$-nilpotent ($A \in SGL(n, \mathbb{C})$) polynomials (see Theorem 4.1 and Corollary 4.1). Note that, as pointed in Section 1, the isotropic results in subsection 4.2 can be understood as some natural consequences of the connection of $\Lambda$-nilpotent polynomials and classical orthogonal polynomials discussed in subsection 4.1.

4.1. **Some classical orthogonal polynomials.** First, let us recall the definition of classical orthogonal polynomials. Note that, to be consistent with the tradition for orthogonal polynomials, we will in this subsection use $x = (x_1, x_2, \ldots, x_n)$ instead of $z = (z_1, z_2, \ldots, z_n)$ to denote free commutative variables.

**Definition 4.1.** Let $B$ be an open set of $\mathbb{R}^n$ and $w(x)$ a real valued function defined over $B$ such that $w(x) \geqslant 0$ for any $x \in B$ and $0 < \int_B w(x)dx < \infty$. A sequence of polynomials $\{f_m(x) \,|\, m \geqslant 0\}$ is said to be *orthogonal* over $B$ if

(a) $\deg f_m = m$ for any $m \geqslant 0$.
(b) $\int_B f_m(x) f_k(x) w(x)\, dx = 0$ for any $m \neq k$.

The function $w(x)$ is called the *weight function*. When the open set $B \subset \mathbb{R}^n$ and $w(x)$ are clear in the context, we simply call the polynomials $f_m(x)$ $(m \geqslant 0)$ in the definition above *orthogonal polynomials*. If the orthogonal polynomials $f_m(x)$ $(m \geqslant 0)$ also satisfy $\int_B f_m^2(x) w(x) dx = 1$ for any $m \geqslant 0$, we call $f_m(x)$ $(m \geqslant 0)$ *orthonormal polynomials*. Note that, in the one dimensional case $w(x)$ determines orthogonal polynomials over $B$ up to multiplicative constants, i.e. if $f_m(x)$ $(m \geqslant 0)$ are orthogonal polynomials as in Definition 4.1, then, for any $a_m \in \mathbb{R}^\times$ $(m \geqslant 0)$, $a_m f_m$ $(m \geqslant 0)$ are also orthogonal over $B$ with respect to the weight function $w(x)$.

The most natural way to construct orthogonal or orthonormal sequences is: first to list all monomials in an order such that the degrees of monomials are non-decreasing; and then to apply Gram-Schmidt procedure to orthogonalize or orthonormalize the sequence of monomials. But, surprisingly, most of classical orthogonal polynomials can also be obtained by the so-called Rodrigues' formulas.

We first consider orthogonal polynomials in one variable.

**Rodrigues' formula**. *Let $f_m(x)$ $(m \geqslant 0)$ be the orthogonal polynomials as in Definition 4.1. Then, there exist a function $g(x)$ defined over $B$ and non-zero constants $c_m \in \mathbb{R}$ $(m \geqslant 0)$ such that*

$$f_m(x) = c_m w(x)^{-1} \frac{d^m}{dx^m} (w(x) g^m(x)). \tag{4.1}$$

Let $P(x) := g(x)$ and $\Lambda := w(x)^{-1} \left(\frac{d}{dx}\right) w(x)$, where, throughout this paper any polynomial or function appearing in a (differential) operator always means the multiplication operator by the polynomial or function itself. Then, by Rodrigues' formula above, we see that the orthogonal polynomials $\{f_m(x) \mid m \geqslant 0\}$ have the form

$$f_m(x) = c_m \Lambda^m P^m(x), \tag{4.2}$$

for any $m \geqslant 0$.

In other words, all orthogonal polynomials in one variable, up to multiplicative constants, has the form $\{\Lambda^m P^m \mid m \geqslant 0\}$ for a single differential operator $\Lambda$ and a single function $P(x)$.

Next we consider some of the most well-known classical orthonormal polynomials in one variable. For more details on these orthogonal polynomials, see [20], [1], [9].

**Example 4.1. (Hermite polynomials)**

(a) $B = \mathbb{R}$ and the weight function $w(x) = e^{-x^2}$.

(b) Rodrigues' formula:

$$H_m(x) = (-1)^m e^{x^2} (\frac{d}{dx})^m e^{-x^2}.$$

($c$) Differential operator $\Lambda$ and polynomial $P(x)$:

$$\begin{cases} \Lambda = e^{x^2}(\frac{d}{dx})e^{-x^2} = \frac{d}{dx} - 2x, \\ P(x) = 1, \end{cases}$$

(d) Hermite polynomials in terms of $\Lambda$ and $P(x)$:

$$H_m(x) = (-1)^m \Lambda^m P^m(x).$$

**Example 4.2. (Laguerre polynomials)**

(a) $B = \mathbb{R}^+$ and $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$.

(b) Rodrigues' formula:

$$L^\alpha_m(x) = \frac{1}{m!} x^{-\alpha} e^x (\frac{d}{dx})^m (x^{m+\alpha} e^{-x}).$$

($c$) Differential operator $\Lambda$ and polynomial $P(x)$:

$$\begin{cases} \Lambda_\alpha = x^{-\alpha} e^x (\frac{d}{dx})(e^{-x} x^\alpha) = \frac{d}{dx} + (\alpha x^{-1} - 1), \\ P(x) = x, \end{cases}$$

(d) Laguerre polynomials in terms of $\Lambda$ and $P(x)$:

$$L_m(x) = \frac{1}{m!} \Lambda^m P^m(x).$$

**Example 4.3. (Jacobi polynomials)**

(a) $B = (-1, 1)$ and $w(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha, \beta > -1$.

(b) Rodrigues' formula:

$$P^{\alpha,\beta}_m(x) = \frac{(-1)^m}{2^m m!} (1-x)^{-\alpha} (1+x)^{-\beta} (\frac{d}{dx})^m (1-x)^{\alpha+m} (1+x)^{\beta+m}.$$

(c) Differential operator $\Lambda$ and polynomial $P(x)$:

$$\begin{aligned} \Lambda &= (1-x)^{-\alpha}(1+x)^{-\beta}(\frac{d}{dx})(1-x)^\alpha(1+x)^\beta \\ &= \frac{d}{dx} - \alpha(1-x)^{-1} + \beta(1+x)^{-1}, \end{aligned}$$

and

$$P(x) = 1 - x^2.$$

(d) Laguerre polynomials in terms of $\Lambda$ and $P(x)$:

$$P^{\alpha,\beta}_m(x) = \frac{(-1)^m}{2^m m!} \Lambda^m P^m(x).$$

A very important special family of Jacobi polynomials are the *Gegenbauer polynomials* which are obtained by setting $\alpha = \beta = \lambda - 1/2$ for some $\lambda > -1/2$. Gegenbauer polynomials are also called *ultraspherical polynomials* in the literature.

**Example 4.4. (Gegenbauer polynomials)**

(a) $B = (-1, 1)$ and $w(x) = (1 - x^2)^{\lambda - 1/2}$, where $\lambda > -1/2$.

(b) Rodrigues' formula:

$$P_m^\lambda(x) = \frac{(-1)^m}{2^m(\lambda + 1/2)_m}(1 - x^2)^{1/2-\lambda}(\frac{d}{dx})^m(1 - x^2)^{m+\lambda-1/2}.$$

where, for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$, $(c)_k = c(c + 1)\cdots(c + k - 1)$.

(c) Differential operator $\Lambda$ and polynomial $P(x)$:

$$\begin{aligned} \Lambda &= (1 - x^2)^{1/2-\lambda}(\frac{d}{dx})(1 - x^2)^{\lambda-1/2} \\ &= \frac{d}{dx} - \frac{(2\lambda - 1)\, x}{(1 - x^2)}, \end{aligned} \tag{4.3}$$

and

$$P(x) = 1 - x^2.$$

(d) Laguerre polynomials in terms of $\Lambda$ and $P(x)$:

$$P_m^\lambda(x) = \frac{(-1)^m}{2^m(\lambda + 1/2)_m}\,\Lambda^m P^m(x).$$

Note that, for the special cases with $\lambda = 0, 1, 1/2$, the Gegenbauer polynomials $P_m^\lambda(x)$ are called the *Chebyshev polynomial of the first kind, the second kind* and *Legendre polynomials*, respectively. Hence all these classical orthogonal polynomials also have the form of $\Lambda^m P^m$ $(m \geqslant 0)$ up to some scalar multiple constants $c_m$ with $P(x) = 1 - x^2$ and the corresponding special forms of the differential operator $\Lambda$ in Eq. (4.3).

**Remark 1.** Actually, the Gegenbauer polynomials are more closely and directly related with **VC** in some different ways. See [27] for more discussions on connections of the Gegenbauer polynomials with **VC**.

Next, we consider some classical orthogonal polynomials in several variables. We will see that they can also be obtained from certain sequences of the form $\{\Lambda^m P^m \mid m \geqslant 0\}$ in a slightly modified way. One remark is that, unlike the one-variable case, orthogonal polynomials in several variables up to multiplicative constants are not uniquely determined by weight functions.

The first family of classical orthogonal polynomials in several variables can be constructed by taking Cartesian products of orthogonal polynomials in one variable as follows.

Suppose $\{f_m \mid m \geqslant 0\}$ is a sequence of orthogonal polynomials in one variable, say as given in Definition 4.1. We fix any $n \geqslant 2$ and set

$$W(x) := \prod_{i=1}^{n} w(x_i), \tag{4.4}$$

$$f_{\mathbf{m}}(x) := \prod_{i=1}^{n} f_{m_i}(x_i), \tag{4.5}$$

for any $x \in B^{\times n}$ and $\mathbf{m} \in \mathbb{N}^n$.

Then it is easy to see that the sequence $\{f_{\mathbf{m}}(x) \mid \mathbf{m} \in \mathbb{N}^n\}$ are orthogonal polynomials over $B^{\times n}$ with respect to the weight function $W(x)$ defined above.

Note that, by applying the construction above to the classical one-variable orthogonal polynomials discussed in the previous examples, one gets the classical *multiple Hermite polynomials*, *multiple Laguerre polynomials*, *multiple Jacobi polynomials* and *multiple Gegenbauer polynomials*, respectively.

To see that the multi-variable orthogonal polynomials constructed above can be obtained from a sequence of the form $\{\Lambda^m P^m(x) \mid m \geqslant 0\}$, we suppose $f_m$ ($m \geqslant 0$) have Rodrigues' formula Eq. (4.1). Let $s = (s_1, \ldots, s_n)$ be $n$ central formal parameters and set

$$\Lambda_s := W(x)^{-1} \left( \sum_{i=1}^{n} s_i \frac{\partial}{\partial x_i} \right) W(x), \tag{4.6}$$

$$P(x) := \prod_{i=1}^{n} g(x_i). \tag{4.7}$$

Let $V_{\mathbf{m}}(x)$ ($\mathbf{m} \in \mathbb{N}^n$) be the coefficient of $s^{\mathbf{m}}$ in $\Lambda_s^{|\mathbf{m}|} P^{|\mathbf{m}|}(x)$. Then, from Eqs. (4.1), (4.4)–(4.7), it is easy to check that, for any $\mathbf{m} \in \mathbb{N}^n$, we have

$$f_{\mathbf{m}}(x) = c_{\mathbf{m}} \frac{\mathbf{m}!}{|\mathbf{m}|!} V_{\mathbf{m}}(x), \tag{4.8}$$

where $c_{\mathbf{m}} = \prod_{i=1}^{n} c_{m_i}$.

Therefore, we see that any multi-variable orthogonal polynomials constructed as above from Cartesian products of one-variable orthogonal polynomials can also be obtained from a single differential operator $\Lambda_s$ and a single function $P(x)$ via the sequence $\{\Lambda_s^m P^m \mid m \geqslant 0\}$.

**Remark 2.** Note that, one can also take Cartesian products of different kinds of one-variable orthogonal polynomials to create more orthogonal polynomials in several variables. By a similar argument as above, we see that all these multi-variable orthogonal polynomials can also be obtained similarly from a single sequence $\{\Lambda_s^m P^m \mid m \geqslant 0\}$.

Next, we consider the following two examples of classical multi-variable orthogonal polynomials which are not Cartesian products of one-variable orthogonal polynomials.

**Example 4.5. (Classical orthogonal polynomials over unit balls)**

(a) Choose $B$ to be the open unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ and the weight function

$$W_\mu(x) = (1 - ||x||^2)^{\mu - 1/2},$$

where $||x|| = \sum_{i=1}^n x_i^2$ and $\mu > 1/2$.

(b) Rodrigues' formula: For any $\mathbf{m} \in \mathbb{N}^n$, set

$$U_{\mathbf{m}}(x) := \frac{(-1)^{\mathbf{m}}(2\mu)_{|\mathbf{m}|}}{2^{|\mathbf{m}|}\mathbf{m}!(\mu+1/2)_{|\mathbf{m}|}} \frac{\partial^{|\mathbf{m}|}}{\partial x_1^{m_1}\cdots \partial x_n^{m_n}} (1 - ||x||^2)^{|\mathbf{m}|+\mu-\mathbf{1/2}}.$$

Then, by Proposition 2.2.5 in [9], $\{U_{\mathbf{m}}(x) \,|\, \mathbf{m} \in \mathbb{N}^n\}$ are orthonormal over $\mathbb{B}^n$ with respect to the weight function $W_\mu(x)$.

(c) Differential operator $\Lambda_s$ and polynomial $P(x)$: Let $s = (s_1, \dots, s_n)$ be $n$ central formal parameters and set

$$\begin{aligned} \Lambda_s :=& W_\mu(x)^{-1} \left( \sum_{i=1}^n s_i \frac{\partial}{\partial x_i} \right) W_\mu(x), \\ P(x) :=& 1 - ||x||^2. \end{aligned}$$

Let $V_{\mathbf{m}}(x)$ ($\mathbf{m} \in \mathbb{N}^n$) be the coefficient of $s^{\mathbf{m}}$ in $\Lambda_s^{|\mathbf{m}|} P^{|\mathbf{m}|}(x)$. Then from the Rodrigues type formula above, we have, for any $\mathbf{m} \in \mathbb{N}^n$,

$$U_{\mathbf{m}}(x) = \frac{(-1)^{|\mathbf{m}|}(2\mu)_{|\mathbf{m}|}}{2^{|\mathbf{m}|}|\mathbf{m}|!(\mu+1/2)_{|\mathbf{m}|}} V_{\mathbf{m}}(x).$$

Therefore, the classical orthonormal polynomials $\{U_{\mathbf{m}}(x) \,|\, \mathbf{m} \in \mathbb{N}^n\}$ over $\mathbb{B}^n$ can be obtained from a single differential operator $\Lambda_s$ and $P(x)$ via the sequence $\{\Lambda_s^m P^m \,|\, m \geqslant 0\}$.

**Example 4.6. (Classical orthogonal polynomials over simplices)**

(a) Choose $B$ to be the simplex

$$T^n = \{x \in \mathbb{R}^n \,|\, \sum_{i=1}^n x_i < 1;\; x_1, ..., x_n > 0\}$$

in $\mathbb{R}^n$ and the weight function

$$W_\kappa(x) = x_1^{\kappa_1} \cdots x_n^{\kappa_n} (1 - |x|_1)^{\kappa_{n+1} - 1/2}, \tag{4.9}$$

where $\kappa_i > -1/2$ ($1 \leqslant i \leqslant n+1$) and $|x|_1 = \sum_{i=1}^n x_i$.

(b) Rodrigues' formula: For any $\mathbf{m} \in \mathbb{N}^n$, set

$$U_{\mathbf{m}}(x) := W_\kappa(x)^{-1} \frac{\partial^{|\mathbf{m}|}}{\partial x_1^{m_1}\cdots \partial x_n^{m_n}} \left( W_\kappa(x)(1 - |x|_1)^{|\mathbf{m}|} \right).$$

Then, $\{U_{\mathbf{m}}(x) \,|\, \mathbf{m} \in \mathbb{N}^n\}$ are orthonormal over $T^n$ with respect to the weight function $W_\kappa(x)$. See Section 2.3.3 of [9] for a proof of this claim.

(c) Differential operator $\Lambda$ and polynomial $P(x)$: Let $s = (s_1, \ldots, s_n)$ be $n$ central formal parameters and set

$$\Lambda_s := W_\kappa(x)^{-1} \left( \sum_{i=1}^{n} s_i \frac{\partial}{\partial x_i} \right) W_\kappa(x),$$
$$P(x) := 1 - |x|_1.$$

Let $V_{\mathbf{m}}(x)$ ($\mathbf{m} \in \mathbb{N}^n$) be the coefficient of $s^{\mathbf{m}}$ in $\Lambda_s^{|\mathbf{m}|} P^{|\mathbf{m}|}(x)$. Then from the Rodrigues type formula in (b), we have, for any $\mathbf{m} \in \mathbb{N}^n$,

$$U_{\mathbf{m}}(x) = \frac{\mathbf{m}!}{|\mathbf{m}|!} V_{\mathbf{m}}(x).$$

Therefore, the classical orthonormal polynomials $\{U_{\mathbf{m}}(x) \,|\, \mathbf{m} \in \mathbb{N}^n\}$ over $T^n$ can be obtained from a single differential operator $\Lambda_s$ and a function $P(x)$ via the sequence $\{\Lambda_s^m P^m \,|\, m \geqslant 0\}$.

4.2. **The isotropic property of $\Delta_A$-nilpotent polynomials.** As discussed in Section 1, the "formal" connection of $\Lambda$-nilpotent polynomials with classical orthogonal polynomials predicts that $\Lambda$-nilpotent polynomials should be isotropic with respect to a certain $\mathbb{C}$-bilinear form of $\mathcal{A}_n$. In this subsection, we show that, for differential operators $\Lambda = \Delta_A$ ($A \in SGL(n, \mathbb{C})$), this is indeed the case for any homogeneous $\Lambda$-nilpotent polynomials (see Theorem 4.1 and Corollary 4.1).

We fix any $n \geqslant 1$ and let $z$ and $D$ denote the $n$-tuples $(z_1, \ldots, z_n)$ and $(D_1, D_2, \ldots, D_n)$, respectively. Let $A \in SGL(n, \mathbb{C})$ and define the $\mathbb{C}$-bilinear map

$$\begin{aligned} \{\cdot, \cdot\}_A : \mathcal{A}_n \times \mathcal{A}_n \to &\quad \mathcal{A}_n \\ (f,\, g) \quad &\to f(AD)g(z), \end{aligned} \tag{4.10}$$

Furthermore, we also define a $\mathbb{C}$-bilinear form

$$\begin{aligned} (\cdot, \cdot)_A : \mathcal{A}_n \times \mathcal{A}_n \to &\quad \mathbb{C} \\ (f,\, g) \quad &\to \{f, g\}_A|_{z=0}, \end{aligned} \tag{4.11}$$

It is straightforward to check that the $\mathbb{C}$-bilinear form defined above is symmetric and its restriction on the subspace of homogeneous polynomials of any fixed degree is non-singular. Note also that, for any homogeneous polynomials $f, g \in \mathcal{A}_n$ of the same degree, we have $\{f, g\}_A = (f, g)_A$.

The main result of this subsection is the following theorem.

**Theorem 4.1.** *Let $A \in SGL(n, \mathbb{C})$ and $P(z) \in \mathcal{A}_n$ be a homogeneous $\Delta_A$-nilpotent polynomial of degree $d \geqslant 3$. Let $\mathcal{I}(P)$ be the ideal of $\mathcal{A}_n$ generated by $\sigma_{A^{-1}}(z) := z^\tau A^{-1} z$ and $\frac{\partial P}{\partial z_i}$ ($1 \leqslant i \leqslant n$). Then, for any $f(z) \in \mathcal{I}(P)$ and $m \geqslant 0$, we have*

$$\{f, \Delta_A^m P^{m+1}\}_A = f(AD)\, \Delta_A^m P^{m+1} = 0. \tag{4.12}$$

Note that, by Theorem 6.3 in [25], we know that the theorem does hold when $A = I_n$ and $\Delta_A = \Delta_n$.

*Proof.* Note first that, elements of $\mathcal{A}_n$ satisfying Eq. (4.12) do form an ideal. Therefore, it will be enough to show $\sigma_{A^{-1}}(z)$ and $\frac{\partial P}{\partial z_i}$ $(1 \leqslant i \leqslant n)$ satisfy Eq. (4.12). But Eq. (4.12) for $\sigma_{A^{-1}}(z)$ simply follows from the facts that $\sigma_{A^{-1}}(Az) = z^\tau Az$ and $\sigma_{A^{-1}}(AD) = \Delta_A$.

Secondly, by Lemma 2.2, we can write $A = UU^\tau$ for some $U = (u_{ij}) \in GL(n, \mathbb{C})$. Then, by Eq. (2.7), we have $\Psi_U(\Delta_n) = \Delta_A$ or $\Psi_U^{-1}(\Delta_A) = \Delta_n$. Let $\widetilde{P}(z) := \Phi_U^{-1}(P) = P(Uz)$. Then by Lemma 2.3, (a), $\widetilde{P}$ is a homogeneous $\Delta_n$-nilpotent polynomial, and by Eq. (2.6), we also have

$$\Phi_U^{-1}(\Delta_A^m P^{m+1}) = \Delta_n^m \widetilde{P}^{m+1}. \tag{4.13}$$

By Theorem 6.3 in [25], for any $1 \leqslant i \leqslant n$ and $m \geqslant 0$, we have

$$\frac{\partial \widetilde{P}}{\partial z_i}(D)\left(\Delta_n^m \widetilde{P}^{m+1}\right) = 0$$

Since

$$\frac{\partial \widetilde{P}}{\partial z_i}(z) = \sum_{k=1}^{n} u_{ki} \frac{\partial P}{\partial z_k}(Uz),$$

we further have

$$\sum_{k=1}^{n} u_{ki} \frac{\partial P}{\partial z_k}(UD)\left(\Delta_n^m \widetilde{P}^{m+1}\right) = 0.$$

Since $U$ is invertible, for any $1 \leqslant i \leqslant n$, we have

$$\frac{\partial P}{\partial z_i}(UD)\left(\Delta_n^m \widetilde{P}^{m+1}\right) = 0. \tag{4.14}$$

Combining the equation above with Eq. (4.13), we get

$$\frac{\partial P}{\partial z_i}(UD)\Phi_U^{-1}\left(\Delta_A^m P^{m+1}\right) = 0,$$

$$\Phi_U^{-1}(\Phi_U \frac{\partial P}{\partial z_i}(UD)\Phi_U^{-1})\left(\Delta_A^m P^{m+1}\right) = 0,$$

$$(\Phi_U \frac{\partial P}{\partial z_i}(UD)\Phi_U^{-1})\left(\Delta_A^m P^{m+1}\right) = 0. \tag{4.15}$$

By Lemma 2.1, (b), Eq. (4.15) and the fact that $A = UU^\tau$, we get

$$\frac{\partial P}{\partial z_i}(UU^\tau D)\left(\Delta_A^m P^{m+1}\right) = \frac{\partial P}{\partial z_i}(AD)\left(\Delta_A^m P^{m+1}\right) = 0,$$

which is Eq. (4.12) for $\frac{\partial P}{\partial z_i}$ $(1 \leqslant i \leqslant n)$. $\square$

**Corollary 4.1.** *Let $A$ be as in Theorem 4.1 and $P(z)$ be a homogeneous $\Delta_A$-nilpotent polynomial of degree $d \geqslant 3$. Then, for any $m \geqslant 1$, $\Delta_A^m P^{m+1}$ is isotropic with respect to the $\mathbb{C}$-bilinear form $(\cdot,\cdot)_A$, i.e.*

$$(\Delta_A^m P^{m+1}, \Delta_A^m P^{m+1})_A = 0. \tag{4.16}$$

*In particular, we have $(P,P)_A = 0$.*

*Proof.* By the definition Eq. (4.11) of the $\mathbb{C}$-bilinear form $(\cdot,\cdot)_A$ and Theorem 4.1, it will be enough to show that $P$ and $\Delta_A^m P^{m+1}$ $(m \geqslant 1)$ belong to the ideal generated by the polynomials $\frac{\partial P}{\partial z_i}$ $(1 \leqslant i \leqslant n)$ (here we do not need to consider the polynomial $\sigma_{A^{-1}}(z)$). But this statement has been proved in the proof of Corollary 6.7 in [25]. So we refer the reader to [25] for a proof of the statement above. $\square$

Theorem 4.1 and Corollary 4.1 do not hold for homogeneous HN polynomials $P(z)$ of degree $d = 2$. But, by applying similar arguments as in the proof of Theorem 4.1 above to Proposition 6.8 in [25], one can show that the following proposition holds.

**Proposition 4.1.** *Let $A$ be as in Theorem 4.1 and $P(z)$ a homogeneous $\Delta_A$-nilpotent polynomial of degree $d = 2$. Let $\mathfrak{J}(P)$ the ideal of $\mathbb{C}[z]$ generated by $P(z)$ and $\sigma_{A^{-1}}(z)$. Then, for any $f(z) \in \mathfrak{J}(P)$ and $m \geqslant 0$, we have*

$$\{f, \Delta_A^m P^{m+1}\}_A = f(AD)\, \Delta_A^m P^{m+1} = 0. \tag{4.17}$$

*In particular, we still have $(P,P)_A = 0$.*

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