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ADDITIVE GROUP ACTIONS: QUOTIENTS, INVARIANTS, AND CANCELLATION

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Dedicated to my parents, Edythe Finston and Harmon Finston, on their 80th and 85th birthdays.

1. Proper actions and their quotients

The symbol G_a denotes the additive group of complex numbers, and X denotes a complex quasiaffine variety. By an action of G_a on X we will mean an algebraic action $\sigma : G_a \times X \to X$. Let

$$\bar{\sigma} : G_a \times X \to X \times X$$

denote the graph morphism and

$$\hat{\sigma} : \mathbb{C}[X] \to \mathbb{C}[X, t]$$
$$\tilde{\sigma} : \mathbb{C}[X \times X] \to \mathbf{C}[X, t]$$

the induced map on coordinate rings.

The action is said to be proper if $\bar{\sigma}$ is a proper morphism (i.e. if $\mathbb{C}[X, t]$ is integral over the image of $\tilde{\sigma}$) [1]. For affine X properness is equivalent to surjectivity of $\tilde{\sigma}$. In fact

Theorem 1.1. ([5]) *A G_a action on an affine variety is proper if and only if $\bar{\sigma}$ is a closed immersion.*

If X is normal, since we are working over $\mathbb{C}$, we have an equivalent formulation of properness for an action of a Lie group L on the normal complex space X. The action is proper if and only if for any sequences $\{x_i\}$ in X and $\{g_i\}$ in L:

If x_0 is a limit point of $\{x_i\}$ and y_0 is a limit point of $\{\sigma(g_i, x_i)\}$ then there is a limit point g_0 of $\{g_i\}$ with $\sigma(g_0, x_0) = y_0$.

Theorem 1.2. ([13]) *The space of orbits X/L of a normal complex space X by a proper action of a complex Lie group L admits the structure of a normal complex space. In this case X/L is a quotient in the sense that:*

1. *The open mapping $\pi : X \to X/L$ is holomorphic and*
2. *For each L invariant holomorphic function f on an open subset U of X, there is a holomorphic function f^* on $\pi(U)$ so that $f = f^* \circ \pi$.*

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Theorem 1.3. ([22]) *Given a proper action of an algebraic group G on a normal variety X, there is a variety Z finite over X to which the action lifts, satisfying:*

1. *The extension of quotient fields $qf(O(Z))/qf(O(X))$ is Galois.*
2. *The action of the Galois group Γ of $qf(O(Z))/qf(O(X))$ commutes with the action of G.*
3. *The space of orbits $W = Z/G$ admits the structure of a normal variety,*
4. *W is a (locally trivial) geometric quotient in the sense that*
   (a) $\pi : Z \to W$ *is open*
   (b) $O(W) = O(Z)^G$
   (c) *(Z has a cover by G stable open subsets $Z = \cup_i Z_i$ with*
   $$Z_i \cong \pi(Z_i) \times G).$$
5. *If W is quasiprojective, then W/Γ is the geometric quotient of X by G.*

Note that the quotient of a variety by a finite group need not be a variety. Nevertheless, we have

Corollary 1.1. *If G acts properly on X then the geometric quotient of X exists as an algebraic space, namely W/Γ.*

This leads to

Problem 1.1. *Under what conditions does a proper action of an algebraic group on a variety admit a (locally trivial) quotient which is an algebraic variety?*

Even in the simplest cases this problem is subtle. Following are some positive results for $G = G_a$

1. ([8]) A proper action on $X = \mathbb{C}^n$ admits a locally trivial quotient if the ring of invariants $\mathbb{C}[X]^{G_a}$ is affine and regular.
2. ([3]) Assume that σ is a proper action on $X = \mathbb{C}^n$ with $\mathbb{C}[X]^{G_a}$ finitely generated. Set
   $$Y = \operatorname{Spec}\mathbb{C}[X]^{G_a}$$
   and $q : X \to Y$ induced by the ring inclusion. Then σ admits a locally trivial quotient if and only if for every $x \in X$, $\widehat{O}_{Y,q(x)}$, the completion of the local ring of $q(x)$ on Y, is a *UFD,* e.g. if Y is smooth.
3. ([9]) A locally trivial action on a factorial affine variety X (i.e. $\mathbb{C}[X]$ is a *UFD*) admits a quasiaffine quotient and $\mathbb{C}[X]^{G_a}$ is finitely generated.
4. ([1]) A proper action on $\mathbb{C}^n$ for which $\mathbb{C}[X]^{G_a}$ defines a contractible variety admits a global trivialization, $\mathbb{C}^n \cong \mathbb{C}^n/G_a \times G_a$. (The case $\mathbb{C}[X]^{G_a}$ isomorphic to a polynomial ring was generalized by Bonnet to actions of G_a^p).
5. ([1]) Every proper action on a quasiaffine surface admits a geometric quotient (which is an affine curve) .

2. Local triviality and invariants

An action of the algebraic group G on the variety X is said to be locally trivial in the Zariski topology if there is a cover of X by affine G stable open subsets $\{U_i : i = 1, \ldots, n\}$ with

$$\pi_i : U_i \cong V_i \times G$$

for some affine varieties V_i and G equivariant isomorphisms π_i. Here G acts trivially on V_i and by translation on G. In this circumstance the geometric quotient X/G is obtained by gluing the U_i via the mappings π_i. If Y is the geometric quotient then the so determined morphism

$$\pi : X \to Y$$

is an algebraic principal G bundle. If $n = 1$, i.e. $X \cong Y \times G$, we say that the action admits a global trivialization.

The following are some useful observations about G_a actions on factorial affine varieties:

1. ([3]) If X is smooth , then $\mathbb{C}[X]^{G_a}$ has discrete divisor class group, i.e. for y a new variable, $\mathbb{C}[X]^{G_a}[[y]]$ is a *UFD*. As a consequence $\mathbb{C}[X]^{G_a}$ satisfies S_3, so for $n \leqslant 5$, $\mathbb{C}[X]^{G_a}$ is Gorenstein when it is finitely generated.
2. Every action arises as
$$\widehat{\sigma} = \exp(t\delta) : \mathbb{C}[X] \to \mathbb{C}[X, t].$$
for some locally nilpotent derivation δ of $\mathbb{C}[X]$.
3. The action is locally trivial iff
$$[\ker(\delta) \cap im(\delta)]\mathbb{C}[X] = \mathbb{C}[X].$$
and admits a global trivialization iff
$$[\ker(\delta) \cap im(\delta)]\mathbb{C}[X]^{G_a} = \mathbb{C}[X]^{G_a}.$$
When $\mathbb{C}[X]^{G_a}$ is finitely generated, these conditions correspond respectively to flatness and faithful flatness of the morphism
$$X \to \mathrm{Spec}\mathbb{C}[X]^{G_a}.$$
4. For a proper action, $\ker(\delta) \cap im(\delta)$ lies in no height one prime ideal of $\mathbb{C}[X]$ (or of $\mathbb{C}[X]^{G_a}$).
5. If $\ker(\delta) \cap im(\delta)$ lies in no height one prime ideal of $\mathbb{C}[X]$, and $\mathbb{C}[X]^{G_a}$ is affine and regular, then the action is locally trivial.
6. ([8]) A proper action admits a global trivialization iff $V = V(\ker(\delta) \cap im(\delta)) \subset \mathbb{C}^n$ has a component of codimension >2.
7. ([8]) Suppose that $Y = \mathrm{Spec}\ \mathbb{C}[X]^{G_a}$ is affine and the action is not locally trivial. Then the image of $q : \mathbb{C}^n \to Y$ has nonempty intersection with the singular locus of Y and nonempty fibers over singular points have dimension >1.

There are normal affine varieties with locally trivial G_a actions but nonfinitely generated invariants [6]. To construct one, take any of the known locally nilpotent derivations on $\mathbb{C}^n$ with nonfinitely generated kernel K. From Nagata (e.g. [19]) we know that K can be realized as the transform $\mathcal{T}_I(N)$ of an affine normal algebra N (by necessity not a *UFD*) with respect to a height 1 ideal I. The algebra $\mathcal{T}_I(N)$ is the subring of the quotient field $Q(N)$ of N given by

$$\mathcal{T}_I(N) = \cup_{n\geqslant 0}\{\alpha \in Q(N) | I^n a \subseteq N\}$$

and consists of the rational functions on Spec N that are regular on the complement of the variety defined by I. From [16] we can assume that I is generated by 2 elements x, y. Then

$$A(m,n) \equiv N[u,v]/(x^m v - y^n u - 1)$$

admits the locally nilpotent derivation δ :

$$\delta(N) = 0, \delta(u) = x^m, \delta(v) = y^n,$$

and

$$\ker(\delta) = T_{(x,y)}N = K.$$

(Note that by 3 in the previous section $A(m,n)$ is not a *UFD*.)

In low dimensions local triviality of G_a actions can be fairly well understood.

Surfaces

1. ([10]) Every fixed point free action on a factorial quasiaffine surface X ($O(X)$ is a *UFD*) admits a global trivialization $X \cong X/G_a \times G_a$.
2. ([1]) Every proper action on a normal quasiaffine surface admits a global trivialization $X \cong X/G_a \times G_a$.
3. It is well known that the famous regular but nonfactorial Danielewski surfaces $X_n : x^n z - y^2 = 1$ can be realized as total spaces for principal G_a bundles over the affine line with two origins. Indeed the defining polynomial $y^2 - 2x^n z - 1$ is a G_a invariant for the action on $\mathbb{C}^3$ given by

$$\begin{aligned} \sigma \quad &: \quad G_a \times \mathbb{C}^3 \to \mathbb{C}^3 \\ \sigma(t)(x,y,z) \quad &= \quad (x, y + tx^n, z + ty + \frac{t^2}{2}x^n). \end{aligned}$$

   The induced G_a action on X_n is improper.

Threefolds

From Zariski, for any G_a action on a normal affine 3-fold X, $\mathbb{C}[X]^{G_a}$ is finitely generated.

1. ([12]) If G_a acts properly on a smooth complex affine 3-fold X, then the action admits a locally trivial quasiprojective quotient. Here we use the fact that a two dimensional smooth algebraic space is a scheme [21] and facts about flat families of affine lines [20]. (Counterexamples for smooth, rational, factorial 4-folds are given in [1]).

2. ([12]) If in addition X is factorial and $\mathbb{C}[X]^{G_a}$ is regular, then for each maximal ideal $m \lhd \mathbb{C}[X]^{G_a}$, with $S = \mathbb{C}[X]^{G_a} - m$, we have either

$$\begin{aligned} S^{-1}\mathbb{C}[X] &= \mathbb{C}[X]^{G_a}_m[u] \text{ or} \\ S^{-1}\mathbb{C}[X] &= \mathbb{C}[X]^{G_a}_m[u,v]/(au-bv-1) \text{ with } \sqrt{(a,b)} = m. \end{aligned}$$

3. ([18]) If $X = \mathbb{C}^3$, then $\mathbb{C}[X]^{G_a} = \mathbb{C}[f,g]$ for two algebraically independent invariants.
4. ([17]) If X is contractible and G_a acts without fixed points, then $X \cong X/G_a \times G_a$.

We defer a discussion of dimension 4. In dimensions 5 and higher, very little is known. Two notable examples in this dimension are:

1. ([24]) The action

$$t \cdot (x_1, y_1, x_2, y_2, z\ ) = (x_1, y_1, x_2 + tx_1, y_2 + ty_1, z + t(x_1y_2 - y_1x_2 + 1))$$

on $\mathbb{C}^5$ is locally trivial, but not a product, with quotient the non affine complement of a surface in a smooth affine 4-fold.
2. ([7]) The action on $\mathbb{C}^5$ generated by the derivation

$$x_1\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_2} + (1 + x_1y_2^2)\frac{\partial}{\partial z}$$

is proper, but not locally trivial, i.e. the quotient exists as an algebraic space but not as a scheme. In this case, $\mathbb{C}[X]^{G_a}$ is affine, the singular locus in the associated variety is pure of codimension 3 and lies in the image of the map from $\mathbb{C}^5$. None of the completed local rings of the singular points retains factoriality.

The situation in dimension 4 is completely unclear.

Conjecture 2.1. *Every proper G_a action on $\mathbb{C}^4$ admits a locally trivial quotient.*

In [4] this conjecture was proved for certain very special actions, namely those for which the generating derivation δ is "twin triangular, i.e. has the form:

$$\delta = p(x_1, x_2)\frac{\partial}{\partial x_4} + q(x_1, x_2)\frac{\partial}{\partial x_3} + r(x_1)\frac{\partial}{\partial x_2}.$$

More generally, a triangulable action on $\mathbb{C}^n$ is conjugate to one generated by a locally nilpotent derivation

$$\delta = \sum_{i=2}^{n} p_i(x_1, \ldots, x_{i-1})\frac{\partial}{\partial x_i}.$$

By virtue of [2] we know that if G_a acts triangulably on $\mathbb{C}^4$ then $\mathbb{C}[X]^{G_a}$ is finitely generated. On the other hand, finite generation of $\mathbb{C}[X]^{G_a}$ is not the issue in example 2 of the above section. Moreover, by restricting that action to certain G_a stable hypersurfaces, we obtain proper but not locally trivial actions on smooth, rational, factorial affine fourfolds. These fourfolds are not isomorphic (nor even homeomorphic) to $\mathbb{C}^4$.

In fact a locally trivial triangulable action on $\mathbb{C}^4$ admits a global trivialization [3] with quotient isomorphic to $\mathbb{C}^3$[25]. In particular, no example like the first one of the above section can arise in lower dimensions. There is no known example of a fixed point free action on any $\mathbb{C}^n$ which is not triangulable. In light of these obervations the following weaker conjecture is also interesting:

Conjecture 2.2. *Every proper triangular G_a action on $\mathbb{C}^4$ admits a global trivialization.*

The following remarks may prove useful in attacking these conjectures

1. Every nontrivial triangular derivation δ on $\mathbb{C}[x_1, \ldots x_n]$ commutes with $n-2$ nontrivial triangular derivations. If $\delta(x_1) = 1$ this is clear. Otherwise one can work over $\mathbb{C}(x_1)$ where it is again clear. Thus
$$\ker(\delta) \text{ admits an action of } \underbrace{G_a \times \ldots \times G_a}_{n-2}.$$
2. As noted in section 1, the quotient of a proper triangular G_a action on $\mathbb{C}^4$ admits a geometric quotient $\mathbb{C}^4/G_a$ which is a smooth algebraic space, and contractible in the complex topology. The invariant coordinate function x_1 induces a function
$$\theta : \mathbb{C}^4/G_a \to \mathbb{C}$$
all of whose fibers are isomorphic to $\mathbb{C}^2$. Indeed each x_1 fiber in $\mathbb{C}^4$ is isomorphic to a G_a stable $\mathbb{C}^3$ whose action admits a global trivialization with quotient $\mathbb{C}^2$. For affine varieties the conditions satisfied by $\mathbb{C}^4/G_a$ would guarantee that it is isomorphic to $\mathbb{C}^3$ by a result of Kaliman [15]. Incidentally, the aforementioned result in [25] can be deduced from this result since there we have
$$\begin{aligned} \mathbb{C}^4 &\cong {}_{G_a} G_a \times \text{Spec } \mathbb{C}[X]^{G_a} \\ \mathbb{C}^4/G_a &\cong \text{Spec } \mathbb{C}[X]^{G_a} \end{aligned}$$
3. Since $\mathbb{C}[X]^{G_a}$ is a *UFD*, the singular locus of
$$Y = \text{Spec } \mathbb{C}[X]^{G_a}$$
has codimension at least 2 and lies in the closed subset V defined as the zero locus of $\ker(\delta) \cap im(\delta)$. We know of no example of a proper action where the singular locus is of maximal dimension. In case $n = 4$ only finitely many singular points can lie in the image of $q : X \to Y$. Assuming that Y has isolated singularities we have, by a result of Flenner and Zaidenberg [27], that the singularities are rational (Y for a triangular action admits an action of G_a^2) hence canonical (since they are Gorenstein). On the other hand, these singularities cannot be:
   (a) *Quotient singularities*: For a quotient singularity $y \in Y$, The map $q : \mathbb{C}^n - q^{-1}(y) \to Y - \{y\}$ restricted to $N - \{y\}$, for a good neighborhood N of y, would factor through the topological universal covering space of $N - \{y\}$ with disconnected fibers. But fibers over the open set $Y - V$ are G_a orbits, hence connected.

(b) *Weighted homogeneous complete intersection*: Such a singularity would imply that $\widehat{O}_{Y,y}$ is a *UFD* since $O_{Y,y}$ is [26], and therefore a locally trivial action. But such an action admits a global trivialization with quotient isomorphic to $\mathbb{C}^3$.

3. Cancellation

In light of remark 2 in the section on threefolds, let m and n denote postive integers and consider the affine hyperfurfaces in $\mathbb{C}^4$ given by

$$X_{m,n} : x^m u - y^n v = 1.$$

The $X_{m,n}$ are affine total spaces for principal G_a bundles over $Y = \mathbb{C}^2 - \{(0,0)\}$. In fact, the assignment

$$X_{m,n} \mapsto \frac{1}{x^m y^n} \in H^1(Y, O_Y)$$

identifies the $X_{m,n}$ with a vector space basis for the space of algebraic principal G_a bundles over Y. Since the $X_{m,n}$ are affine, indeed all total spaces of nontrivial algebraic principal G_a bundles over Y are affine [12], the vanishing of $H^1(X, O_X)$ for X affine shows for all $(m,n), (m',n')$ that

$$X_{m,n} \times \mathbb{C} \cong X_{m,n} \times_Y X_{m',n'} \cong X_{m',n'} \times \mathbb{C}.$$

Problem 3.1. *Do the $X_{m,n}$ provide counterexamples to the Generalized Cancellation Problem, i.e. . for affine varieties X and Z does $X \times \mathbb{C} \cong Z \times \mathbb{C}$ imply $X \cong Z$? In particular, are the distinct $X_{m,n}$ which are not obviously isomorphic (e.g. $X_{m,n} \cong X_{n,m}$) isomorphic as varieties?*

A partial solution to this problem was given by A. Dobouloz (personal communication):

Proposition 3.1. *If $m + n = m' + n'$ then $X_{m,n} \cong X_{m'n'}$.*

Unlike Danielewski surfaces and related varieties, the total spaces $X_{m,n}$ are factorial and the base quasiaffine, in particular separated. As a consequence, topologically all $X_{m,n}$ are *homeomorphic* to $Y \times \mathbb{C}$ in the complex topology [23]. Moreover, it is easy to see that they all have trivial Makar-Limanov invariant, i.e.

$$ML(\mathbb{C}[X_{m,n}]) = \mathbb{C}.$$

Topological tools and the Makar-Limanov have been the primary means by which the Danielewski surfaces were exhibited as counterexamples to cancellation.

Note that $Aut(Y)$ can be identified with the group of origin preserving automorphisms of $\mathbb{C}^2$, a large group. Varieties closely related to the $X_{m,n}$, also total spaces for algebraic principal G_a bundles, but over less symmetric bases, are also interesting. For an example with a long and famous history, replace $Y = \mathbb{C}^2 - \{(0,0)\}$ by $Y = S - \{(0,0,0)\}$ where $S \subset \mathbb{C}^3$ is defined by

$$x^2 + y^3 + z^5 = 0.$$

$S \cong \mathbb{C}^2/G$ where $|G| = 120$, G is the binary icosahedral group viewed as a subgroup of $SL_2(\mathbb{C})$. Define $X_{m,n}(G)$ by

$$\begin{aligned} x^2 + y^3 + z^5 &= 0 \\ x^m u - y^n v &= 1 \end{aligned}$$

Problem 3.2. *Are the distinct $X_{m,n}(G)$ isomorphic? What is $ML(\mathbb{C}[X_{m,n}(G)])$?*

Note that:

1. $X_{m,n}$ and $X_{m,n}(G)$ are factorial.
2. $X^G_{m,n} \cong X_{m,n}/G$ via $\mathbb{C}^2 - \{0\} \times_Y X_{m,n} \to X_{m,n}(G)$, i.e. G acts trivially on the right factor, and this map is an étale covering.
3. $2, 3, 5$ are pairwise relatively prime, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$.

In [11] it was shown that for pairwise relatively prime a, b, c satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ (i.e. all other triples of relatively prime integers each greater than 1) the varieties defined by

$$\begin{aligned} x^a + y^b + z^c &= 0 \\ x^m u - y^n v &= 1 \end{aligned}$$

are factorial counterexamples to generalized cancellation.

Counterexamples to generalized cancellation can easily be obtained from other quotients of $\mathbb{C}^2 - \{(0,0)\}$ by finite subgroups of $SL_2(\mathbb{C})$. Let $Y = S - \{(0,0,0)\}$ where $S \subset \mathbb{C}^3$ is defined by

$$x^2 + y^2 z + z^c = 0, \;\; c \geqslant 3.$$

$S \cong \mathbb{C}^2/G$ where G is the binary dihedral group of order $4c$. Define $X_{m,n}(G)$ analogously. Then:

1. Y and $X_{m,n}(G)$ are not factorial, but $ML(\mathbb{C}[X_{m,n}(G)]) = \mathbb{C}[Y]$. To see this, let δ be locally nilpotent with kernel R, and $K = qf(R)$. Then
$$K \otimes_R \mathbb{C}[X_{m,n}(G)] = K[s], \;\; \delta(s) = 1.$$
If $\deg_s(z) > 0$, then $z(s)$ divides both $x(s), y(s)$ contradicting $x^m u - y^n v = 1$. Thus $z \in R$ from which it follows that both $x, y \in R$.
2. $X_{m,n}(G) \cong X_{m',n'}(G)$ if and only if $(m,n) = (m',n')$. This follows from arguments analogous to those in [11] and the fact that any automorphism of $\mathbb{C}[Y]$ sends
$$(x, y, z) \longmapsto (\mu_1 x, \mu_2 y, \mu_3 z)$$
for certain roots of unity μ_i.
3. $\mathbb{C}^2 - \{0\} \times_Y X_{m,n} \to X_{m,n}(G)$ is an étale covering.

As a consequence of these examples we see that not all total spaces of algebraic principal G_a bundles over Y can be isomorphic as varieties. This follows from the fact that a finite subgroup of $SL_2(\mathbb{C})$ can act in essentially only one way on $\mathbb{C}^2 - \{(0,0)\}$.

The following question was raised by J-P. Furter at the International Conference on Polynomial Automorphisms in Hanoi:

Problem 3.3. *Let A and B be $\mathbb{C}$ algebras finite dimensional as $\mathbb{C}$ vector spaces satisfying $A[x] \cong B[x]$. Is $A \cong B$?*

In fact just assuming A and B to be artinian they must be isomorphic if $A[x] \cong B[x]$. This follows from a "unique factorization" theorem for noetherian algebras proved by Horst [14] and some straightforward localization arguments. On the other hand, the following refinement of the question raised by P. Russell has a negative answer:

Problem 3.4. *With A and B artinian, in any isomorphism $\theta : A[x] \cong B[x]$ must $\theta(A) = B$?*

Indeed, let A be any algebra over the rational numbers admitting a non trivial locally nilpotent derivation δ. (For example

$$\begin{aligned} A &= \mathbb{C}[x,y]/(x,y)^2 \\ \delta &= x\frac{\partial}{\partial y}.) \end{aligned}$$

Then, as in the first section,

$$\begin{aligned} \widehat{\sigma}(a) &\equiv \exp(t\delta)(a) \\ &= \sum_{i \geqslant 0} \frac{t^i}{i!}\delta^i(a) \end{aligned}$$

defines an injective ring homomorphism $\widehat{\sigma} : A \to A[t]$. Arguing by induction on the smallest power of δ that annihilates an element $a \in A$, it is easy to see that $A \subset \widehat{\sigma}(A)[t]$. We therefore obtain an automorphism θ of $A[t]$ via

$$\begin{aligned} \theta|_A &= \widehat{\sigma} \\ \theta(t) &= t, \end{aligned}$$

but $\theta(A) \neq A$.

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