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THE JACOBIAN VARIETY

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1. Introduction

We consider a possible approach to the Jacobian conjecture. We define the Jacobian variety of $\mathbb{C}^n$ of degree d, denoted by $J(n, d)$, whose points parametrize the set of all the n-Jacobian tuples of total degree at most d normalized to map $\overline{0} \in \mathbb{C}^n$ onto itself. We use the term "variety" as a not necessarily irreducible algebraic set. The set $\mathrm{Aut}_{0,d}(\mathbb{C}^n)$ of all the polynomial automorphisms of $\mathbb{C}^n$ of total degree at most d that map $\overline{0}$ onto itself corresponds to a subset of the Jacobian variety of degree d. This subset is in fact a subvariety, i.e. it is a Zariski closed subset of the variety. This might be significant to settle the Jacobian conjecture. For instance, if the jacobian variety of degree d is irreducible, then it reduces the problem to computing the dimensions of these two varieties. The conjecture is true if and only if the two dimensions are equal. Otherwise, almost any point on the Jacobian variety will serve as a counterexample to the Jacobian conjecture. Using results of Magnus, Appelgate, Onishi and Nagata, [1, 2, 12, 14, 15, 16], we show that if the Jacobian variety in two variables and of degree $d = p$ a prime integer or $d = pq$ a product of two prime integers, has a dimension which equals that of its subvariety $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$.

We turn to the computation of the dimension of $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$. Here the main tool is the Jung-van der Kulk Theorem that gives us the structure of $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ as the amalgamated product of affine mappings and of de Jonquiéres mappings. It also indicates the useful relation between the degrees of the factors in the amalgamated product and the degree of the resulting automorphism, [11, 4]. The computation was performed in [5, 6].

Thus for future advance on the problem we will have to find a way to compute the dimension of the Jacobian variety, or at least get a very good lower bound for it.

The new results of our paper appear on sections 4 and 6 were we show that for a natural number d which is a prime number or a product of two prime numbers the dimensions of the Jacobian variety $J(2, d)$ and of $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ coincide. Moreover, if d is also at least 4 then $J(2, d)$ must be reducible.
Hence $S(J(2, d))$ the singular locus of the variety $J(2, d)$ is non-empty, and we prove that it contains $\mathrm{SL}_2(\mathbb{C})$.

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The ideas described above have origins in the work of Jean-Philippe Furter. In his beautiful paper [5] he proved that $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ is irreducible if and only if $d \leqslant 3$. He also computed the number of irreducible components of $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ when $d \leqslant 9$.

Going further to the union of all the Jacobian varieties of $\mathbb{C}^n$,

$$J(n) = \bigcup_{d=1}^{\infty} J(n,d),$$

one arrives at an interesting object known as an ind-affine scheme. These structures were discussed by Igor Shafarevich,[18, 19]. In [8], Tatsuji Kambayashi outlined another approach to the Jacobian Conjecture based on these infinite dimensional algebraic varieties. This motivated him to investigate these objects in [9, 10]. One open challenge here is to give a good definition of the regular points of these infinite dimensional varieties. The definition given by Shafarevich in [18, 19], turned out not to be satisfactory. An example of Burt Totaro, [9], demonstrates this.

2. Generic polynomials and the Jacobian varieties

We consider $\mathbb{C}[X_1,\ldots,X_n]$, the $\mathbb{C}$-algebra of polynomials in the n indeterminates $X_1,\ldots,X_n$, $n \in \mathbb{Z}^+$. A typical element (polynomial) in $\mathbb{C}[X_1,\ldots,X_n]$ has the form

$$P(X_1,\ldots,X_n) = \sum_{i_1+\ldots+i_n \leqslant d} \alpha_{i_1,\ldots,i_n} X_1^{i_1}\ldots X_n^{i_n}.$$

Here $\alpha_{i_1,\ldots,i_n} \in \mathbb{C}$ are the coefficients of P. If there exist $i_1,\ldots,i_n$ such that $i_1+\ldots+i_n = d$ and $\alpha_{i_1,\ldots,i_n} \neq 0$ we say that the total degree of P is d and we write $\deg P = d$. We will sometimes denote the sequence $\{X_1,\ldots,X_n\}$ by $\overline{X}$.

Remark 1. The number of multi-indices $i_1,\ldots,i_n$ such that $i_1+\ldots+i_n = j$ equals

$$\left(\begin{array}{c} n-1+j \\ j \end{array}\right) = \left(\begin{array}{c} n-1+j \\ n-1 \end{array}\right).$$

We conclude that if $\deg P = d$, then the total number of its coefficients is

$$N_d = \sum_{j=0}^{d} \left(\begin{array}{c} n-1+j \\ j \end{array}\right) = \sum_{j=0}^{d} \left(\begin{array}{c} n-1+j \\ n-1 \end{array}\right).$$

We should have indicated the dependency of N_d also on n. However, in our applications n will be fixed and so we ignore that.

Definition 2.1. A generic polynomial in the n indeterminates $X_1,\ldots,X_n$ and of a total degree at most d has the form

$$Q = \sum_{i_1+\ldots+i_n \leqslant d} a_{i_1,\ldots,i_n} X_1^{i_1}\ldots X_n^{i_n},$$

where $a_{i_1,\ldots,i_n}$ are N_d new indeterminates.

Remark 2. We formally have

$$Q \in \mathbb{C}[X_1, \ldots, X_n; a_{i_1,\ldots,i_n} \mid i_1 + \ldots + i_n \leqslant d],$$

where the X-degree, $\deg_X Q = d$, and where Q is a linear combination of $a_i = a_{i_1,\ldots,i_n}$ over the X-monomials $X^i = X_1^{i_1} \ldots X_n^{i_n}$.

A polynomial mapping over $\mathbb{C}$ is an n-tuple of polynomials over $\mathbb{C}$. Thus it is $F = (P_1, \ldots, P_n) \in \mathbb{C}[X_1, \ldots, X_n]^n$.

Definition 2.2. A generic polynomial mapping of degree d is an n-tuple of generic polynomials in the indeterminates $X_1, \ldots, X_n$ and of degree (at most) d. Thus it is

$$G = (\sum_{i_1+\ldots+i_n \leqslant d} a^{(1)}_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}, \ldots, \sum_{i_1+\ldots+i_n \leqslant d} a^{(n)}_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}).$$

The total number of (generic) coefficients of G is $n \cdot N_d$. The generic Jacobian matrix is

$$J_G = \left(\frac{\partial G_i}{\partial X_j} \right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$$

where $G_k = \sum_{i_1+\ldots+i_n \leqslant d} a^{(k)}_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$. The determinant of J_G, $\det J_G$, is a polynomial of degree $n(d-1)$ in $\overline{X}$. The coefficients of $\det J_G$ are all multi-linear of degree n in the $a_i^{(k)}$. The number of the coefficients of $\det J_G$ is thus, $N_{n(d-1)}$.

Definition 2.3. The Jacobian condition is the equation $\det J_G \equiv 1$, where J_G is the generic Jacobian matrix and where we think of $\det J_G$ as on a polynomial in $\overline{X}$.

Remark 3. The Jacobian condition gives a system of $N_{n(d-1)}$ equations in the $n(N_d - 1)$ indeterminates $a^{(k)}_{i_1,\ldots,i_n}$, $0 < i_1 + \ldots + i_n \leqslant d$, $1 \leqslant k \leqslant n$. Each equation is homogeneous multi-linear of degree n in the $a_i^{(k)}$ except for the single equation

$$\begin{vmatrix} a^{(1)}_{10\ldots0} & a^{(1)}_{010\ldots0} & \cdots & a^{(1)}_{0\ldots01} \\ a^{(2)}_{10\ldots0} & a^{(2)}_{010\ldots0} & \cdots & a^{(2)}_{0\ldots01} \\ \vdots & \vdots & \ldots & \vdots \\ a^{(n)}_{10\ldots0} & a^{(n)}_{010\ldots0} & \cdots & a^{(n)}_{0\ldots01} \end{vmatrix} = 1.$$

The following sets of polynomial mappings will be useful.

Definition 2.4.

$$et_{0,d}(\mathbb{C}^n) := \{G = (G_1, \ldots, G_n) \in \mathbb{C}[X] \mid \det J_G(\overline{X}) \equiv 1,\ G(\overline{0}) = \overline{0},\ \deg G \leqslant d\}.$$

$\mathrm{Aut}_{0,d}(\mathbb{C}^n)$ is the set of all the invertible polynomial mappings on $\mathbb{C}^n$ that lie in $et_{0,d}(\mathbb{C}^n)$.

$$et_0(\mathbb{C}^n) := \bigcup_{d=1}^{\infty} et_{0,d}(\mathbb{C}^n),\ \mathrm{Aut}_0(\mathbb{C}^n) = \bigcup_{d=1}^{\infty} \mathrm{Aut}_{0,d}(\mathbb{C}^n).$$

The étale non-proper mappings are the following

$$np_0(\mathbb{C}^n) = et_0(\mathbb{C}^n) - \mathrm{Aut}_0(\mathbb{C}^n),$$

$$np_{0,d}(\mathbb{C}^n) = et_{0,d}(\mathbb{C}^n) - \mathrm{Aut}_{0,d}(\mathbb{C}^n).$$

Remark 4. The number of defining equations in the Jacobian condition equals $N_{n(d-1)}$. The number of unknowns in these equations is $n \cdot (N_d - 1)$. Since we have $n \cdot (N_d - 1) << N_{n(d-1)}$ the system is overdetermined and so initially we expect few solutions. However we clearly have the following,

Proposition 2.1. *For any values $n, d \in \mathbb{Z}^+$ the solution set of the Jacobian condition contains the sets of the coefficients of all the mappings in* $\mathrm{Aut}_{0,d}(\mathbb{C}^n)$.

Definition 2.5. The set of all the solutions of the Jacobian condition, $\det J_G \equiv 1$ for generic mappings that map $\overline{0}$ onto itself and of degree at most d will be denoted by $J(n, d)$. It will be called the Jacobian variety of degree d. We will denote $J(n) = \cup_{d=1}^{\infty} J(n, d)$ and will call $J(n)$ the Jacobian variety.

Remark 5. Calling $J(n, d)$ an algebraic variety is fully justified (it is the zero set of finitely many polynomial equations). Not so with $J(n)$!

Remark 6. As usual we can endow $J(n, d)$ with two different topologies, the complex topology and the Zariski topology. Also we should note that our description of $J(n, d)$ uses an embedding in $\mathbb{C}^{n(N_d-1)}$.

Remark 7. The Jacobian conjecture states that $et_{0,d}(\mathbb{C}^n) = \mathrm{Aut}_{0,d}(\mathbb{C}^n)$ for all $n, d \in \mathbb{Z}^+$. Alternatively, it states that $np_{0,d}(\mathbb{C}^n) = \emptyset$.

Remark 8. We have monotonicity of the following two d-sequences

$$J(n, d) \subset J(n, d + 1), \;\; \mathrm{Aut}_{0,d}(\mathbb{C}^n) \subset \mathrm{Aut}_{0,d+1}(\mathbb{C}^n).$$

In this paper $\subset$ means strict inclusion.

Here are a few natural questions about Jacobian varieties:

1. What are $\dim J(n, d)$ and $\dim \mathrm{Aut}_{0,d}(\mathbb{C}^n)$? (We soon will show that $\mathrm{Aut}_{0,d}(\mathbb{C}^n)$ can be represented as a sub-variety of $J(n, d)$).

2. Are the $J(n, d)$ connected in both topologies?

3. Are the $J(n, d)$ non-singular?

4. Are the $J(n, d)$ irreducible?

Regarding question 2 we have

Proposition 2.2. ([9]) *For any $n, d \in \mathbb{Z}^+$ the algebraic varieties $J(n, d)$ are connected both in the complex and in the Zariski topologies.*

Proof. The space $SL_n(\mathbb{C})$ is connected in the complex topology. It can be regarded as a subspace of $J(n,d)$ for any d. Let $P \in J(n,d)$ represent $F \in et_{0,d}(\mathbb{C}^n)$. Then $J_F(\overline{0}) \in SL_n(\mathbb{C})$. Let P_0 be the point in $J(n,d)$ that represents $J_F(\overline{0})$. Let $F = (F_1, \ldots, F_n) \in \mathbb{C}[\overline{X}]^n$. We have $\det J_F(\overline{X}) \equiv 1$. We take a parameter $t \in \mathbb{C}^\times$ and define

$$F_t(\overline{X}) = \frac{1}{t} F(t\overline{X}).$$

Then by the chain rule $\det J_{F_t}(\overline{X}) \equiv 1$. So for each t we have $F_t \in et_{0,d}(\mathbb{C}^n)$. Let L_F be the linear part of F. Then P_0 represents L_F. The following two identities are clear:

$$F_{t=1} = F, \;\; F_{t=0} = \lim_{t \to 0} F_t = L_F.$$

Hence we found a path inside $et_{0,d}(\mathbb{C}^n)$ that joins F to L_F. Thus there is a path inside $J(n,d)$ that joins P to P_0. This proves that $J(n,d)$ is connected in the complex topology.
Finally, we recall that if X is an algebraic variety over $\mathbb{C}$ and if X_{an} is the associated complex space, then X is connected in the Zariski topology if and only if X_{an} is connected. This is a result of Serre, [17]. It concludes the proof. $\square$

Remark 9. As indicated in the introduction answering questions 1 and 3 is related to settling the Jacobian conjecture. See [3]. Thus it makes sense to recall few more results of Serre from [17]:

$\dim X = \dim X_{an}$.

X is irreducible if and only if X_{an} is irreducible.

X is non-singular if and only if X_{an} is a complex manifold.

By fixing an order on the monomials in $\overline{X}$ we created a canonical identification between the mappings in $et_{0,d}(\mathbb{C}^n)$ and the points of the Jacobian variety of degree d, $J(n,d)$. Let us denote this canonical idenification by

$$C(n,d) \,:\, et_{0,d}(\mathbb{C}^n) \to J(n,d).$$

3. Closedness of the sub-variety of the automorphisms

Theorem 3.1. ([3, 5, 8]) *For any $n, d \in \mathbb{Z}^+$, the set $C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ is Zariski closed in the algebraic variety $J(n,d)$.*

Proof. If $F \in et_{0,d}(\mathbb{C}^n)$ then the local inverse F^{-1} at $\overline{0}$ is given by a well known formal power series expansion, the so called Abhayankar formula. Clearly $F \in \mathrm{Aut}_{0,d}(\mathbb{C}^n)$ if and only if this expansion of the inverse has only finitely many non-zero coefficients. If this is the case then $\deg F^{-1} \leqslant (\deg F)^{n-1}$ by Gabber, [3]. There exists a number $w(n,d)$ depending on n and d only such that if all the coefficients of F^{-1} that correspond to monomials in F^{-1} of degrees greater than $(\deg F)^{n-1}$ and smaller than $w(n,d)$ vanish, then in fact F^{-1} is polynomial. This follows by results of D. Wright on his tree expansions.

Hence we can characterize all the points of $C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ by finitely many polynomial equations. $\square$

A few remarks are in order.

Remark 10. We could have avoided the use of Wright's result as follows: by Gabber's degree bound, $\mathrm{Aut}_{0,d}(\mathbb{C}^n)$ is a subset of the ideal that is generated by the coefficients of F^{-1} that correspond to monomials of degree greater than d^{n-1}. By Hilbert's finite basis theorem this ideal is finitely generated and hence we are done.

Remark 11. A different approach, is to write down a finite set of defining equations for $C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ in $J(n,d)$ using the resultant reformulation of the Jacobian conjecture.
Another way to see the closedness of $C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ originates in elimination theory, i.e. by considering systems of polynomial equations of the form $F\circ G(\overline{X}) = \overline{X}$ for the coefficients of the polynomials in F. The fact that there exists such a G of a bounded degree says that the coefficients of F are the solutions of some system of polynomial equations.

One conclusion of Theorem 3.1 is that $C(n,d)(np_{0,d}(\mathbb{C}^n))$ is Zariski open in $J(n,d)$. So if $np_{0,d}(\mathbb{C}^n) \neq \emptyset$, and if the variety $J(n,d)$ is an irreducible variety, then $np_{0,d}(\mathbb{C}^n)$ is a huge subset of $et_{0,d}(\mathbb{C}^n)$. In other words almost any étale mapping will serve as a counter example to the Jacobian conjecture.

We can now state a few claims that imply the Jacobian conjecture. For example, to show that the Jacobian conjecture holds true for the parameters (n,d) it is sufficient to show that $C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ contains an interior point in $J(n,d)$ relative to the complex topology and that $J(n,d)$ is an irreducible variety. The first half of the condition could be phrased in terms of mappings as follows:

Find $F \in \mathrm{Aut}_{0,d}(\mathbb{C}^n)$ such that $\exists\, \epsilon > 0$ so that if $||G-F||_2 < \epsilon$ and $G \in et_{0,d}(\mathbb{C}^n)$, i.e. $\det J_G \equiv 1$, then $G \in \mathrm{Aut}_{0,d}(\mathbb{C}^n)$.

This formulation is of the type of the Hurwitz Theorem on schlicht functions of one complex variable. Though interesting, there is no reason to expect that it will not be hard to prove that formulation.

Thus we are naturally led to consider a different approach. Namely, try to answer question 1 in the list of questions that comes after Remark 8. In view of Theorem 3.1 we conclude that the Jacobian conjecture is true for the parameters (n,d) if $J(n,d)$ is irreducible and

$$\dim C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n)) = \dim J(n,d).$$

The Jacobian conjecture is false for the parameters (n,d) if

$$\dim C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n)) < \dim J(n,d).$$

So that even if exact computations of these two dimensions are difficult, it might suffice to compute a good upper bound for $\dim C(n,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^n))$ and a good lower bound for $\dim J(n,d)$. If the first is smaller than the second it will show that the Jacobian conjecture is false for the parameters (n, d). One might hope to get a strict inequality using asymptotic estimates on these bounds by letting n or d tend to ∞.

4. The variety $J(2,d)$ for d a product of at most two prime integers

Theorem 4.1. (1) *If $d = p$ is a prime integer or $d = pq$ is a product of two prime integers, then* $\dim J(2,d) = \dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$.
(2) *If $d = p$ is a prime integer or $d = pq$ is a product of two prime integers, and $d \geqslant 4$, then $J(2,d)$ is reducible.*

Proof. First we note that (1) $\Rightarrow$ (2). For $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$ is an affine subvariety of $J(2,d)$ (by Theorem 3.1) and by (1) the two varieties have equal dimension. If the super-variety $J(2,d)$ is irreducible then the two varieties coincide. Thus $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ is irreducible. Hence by a result of Jean-Philippe Furter, [5], $d \leqslant 3$.
Thus we only need to prove (1). The key is to use a classical result of Magnus, Appelgate, Onishi and Nagata, [1, 12, 14, 15, 16] also [2] is related. This result is stated in the book [4] on page 255, as follows:

Corollary. *If $F = (P, Q)$ satisfies* $\det J_F \in \mathbb{C}^*$ *and* $\deg P$ *or* $\deg Q$ *is a product of at most two prime numbers, then F is invertible over k.*

We now complete the proof of (1) as follows. Let us denote the following sets

$$\mathrm{Aut}_{0,=j}(\mathbb{C}^2) = \{F \in \mathrm{Aut}_0(\mathbb{C}^2) \mid \deg F = j\},$$

$$J(2,=j) = \{\alpha \in J(2) \mid \alpha \text{ parameterizes } F \in \mathbb{C}[X,Y]^2,\ \deg F = j\}.$$

The set $J(2,=d)$ is Zariski open in $J(2,d)$. The reason is that $J(2,=d)$ is determined in $J(2,d)$ by $2(d+1)$ polynomial non-equalities, namely the statements that at least one of the coefficients of the monomials of degree d in P or in Q is not zero (where $F = (P,Q)$). Similarly, the set $C(2,d)(\mathrm{Aut}_{0,=d}(\mathbb{C}^2))$ is Zariski open in $C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. Hence, we get the (topological) dimension equalities

$$\dim J(2,=d) = \dim J(2,d),$$

and

$$\dim C(2,d)(\mathrm{Aut}_{0,=d}(\mathbb{C}^2)) = \dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)).$$

If $d = p$ or $d = pq$ (p, q are primes), then the corollary quoted above from [4] implies that

$$J(2,=d) = C(2,d)(\mathrm{Aut}_{0,=d}(\mathbb{C}^2)),$$

and we conclude that $\dim J(2,d) = \dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2))$. $\square$

5. Dimension computations

The computation for $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ was done by Jean-Philippe Furter, [5] (see also Fridland-Milnor, [6]). The result is

Theorem 5.1. ([5]) *If $d > 1$, then*

$$\dim C(2,d)(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) = d + 4.$$

Proof. We only need to recall that our $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ is normalized so that each map takes $\overline{0}$ to itself and has a determinant of its Jacobian equals 1. Thus we get the result from Proposition 10, on page 619 of [5]. □

Thus we are left with the task of evaluating $\dim J(2,d)$ for values of d that are not prime integers or products of two such primes. We record the fact that

Corollary 5.1. (1) *If $d \leqslant 100$ then $\dim J(2,d) = d + 4$.*

(2) *If $\exists\, d \in \mathbb{Z}^+$ such that $\dim J(2,d) > d + 4$, then the Jacobian conjecture is false and there exist counterexamples of degree d.*

Proof. (1) Follows by Theorem 5.1 and by [13] and (2) is a direct consequence of Theorem 5.1 and Theorem 3.1. □

6. Normality, singularity and irreducibility of $J(2,d)$

Let $d \geqslant 4$ be a prime integer or a product of two prime integers. Then by part (2) of Theorem 4.1 it follows that $J(2,d)$ must be reducible. If $J(2,d)$ is normal, i.e. $\hat{J}(2,d) = J(2,d)$ then by the theorem on page 168 of [7] it follows that $J(2,d)$ is not connected (part (vii) of that theorem). This, however, contradicts Proposition 2.15. We conclude that $\hat{J}(2,d) \neq J(2,d)$, i.e. that $J(2,d)$ is not normal.

By the Theorem of Oka, on page 128 of [7]: The set $N(X)$ of all non normal points of a reduced complex space X is an analytic subset of X which is contained in the singular locus $S(X)$ of X.

Taking $X = J(2,d)$, it follows that $J(2,d)$ must have singular points, i.e. $S(J(2,d)) \neq \emptyset$ because $N(J(2,d)) \neq \emptyset$. This, of course follows directly also from the reducibility of $J(2,d)$.

Let us summarize what we have proved so far.

Theorem 6.1. *Let $d \geqslant 4$ be a prime integer or a product of two prime integers. Then $J(2,d)$ is not a normal complex space and in particular it has a non empty singular set of points. That is*

$$\emptyset \neq N(J(2,d)) \subseteq S(J(2,d)).$$

Moreover, $J(2,d)$ is reducible.

Let us assume that $F \in et_{0,d}(\mathbb{C}^2)$ is such that $C(2,d)(F) \in S(J(2,d))$. Let $M \in \mathrm{GL}_2(\mathbb{C})$. Conjugation of F by M is to be understood in the usual way, that is, if $F(X,Y) = (P(X,Y), Q(X,Y)) \in \mathbb{C}[X,Y]^2$ then

$$(M^{-1} \circ F \circ M)(X,Y) = (M^{-1}(F((M \begin{pmatrix} X \\ Y \end{pmatrix})^t)^t))^t.$$

In other words, if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then

$$\begin{aligned}(M^{-1} \circ F \circ M)(X,Y) = (&\alpha P(aX+bY, cX+dY) + \beta Q(aX+bY, cX+dY),\\ &\gamma P(aX+bY, cX+dY) + \delta Q(aX+bY, cX+dY)).\end{aligned}$$

Lemma 6.1. *Let $F \in et_{0,d}(\mathbb{C}^2)$ and $M \in \mathrm{GL}_2(\mathbb{C})$. Then the following are equivalent:*

(1) $C(2,d)(F) \in S(J(2,d))$.
(2) $C(2,d)(M^{-1} \circ F \circ M) \in S(J(2,d))$.

Proof. The following mapping

$$\phi_M : J(2,d) \to J(2,d),$$
$$\phi_M(C(2,d)(G)) = C(2,d)(M^{-1} \circ G \circ M) \quad (G \in et_{0,d}(\mathbb{C}^2)),$$

is clearly a diffeomorphism and hence preserves the singular locus of $J(2,d)$. ☐

An immediate consequence is the following.

Theorem 6.2. *Let $d \geqslant 4$ be a prime integer or a product of two prime integers. Then $S(J(2,d)) \cap \mathrm{SL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$.*

Proof. By Theorem 6.1 we have $S(J(2,d)) \neq \emptyset$, say $G \in et_{0,d}(\mathbb{C}^2)$ satisfies $C(2,d)(G) \in S(J(2,d))$. Let $t \in \mathbb{C}^\times$, and let

$$G_t(X,Y) = \frac{1}{t} G(tX, tY),$$

as usual. By Lemma 6.1 we have $C(2,d)(G_t) \in S(J(2,d))$. By Theorem 2 on page 117 of [7] $S(J(2,d))$ is an analytic set in $J(2,d)$. In particular $S(J(2,d))$ is closed in $J(2,d)$. If $L(G)$ is the linear part of G then $L(G) \in \mathrm{SL}_2(\mathbb{C})$. Also $\lim_{t\to 0} G_t = L(G)$. Hence $L(G) \in S(J(2,d)) \cap \mathrm{SL}_2(\mathbb{C})$. Thus we see that any element of $\mathrm{SL}_2(\mathbb{C})$ which equals the linear part of a $G \in et_{0,d}(\mathbb{C}^2)$ which is singular (i.e. $C(2,d)(G) \in S(J(2,d))$) must be a singular point of $J(2,d)$. However, we note that $\forall\, M \in \mathrm{SL}_2(\mathbb{C})$ the mapping

$$\psi_M : J(2,d) \to J(2,d),$$
$$\psi_M(C(2,d)(H)) = C(2,d)(H \circ M) \quad (H \in et_{0,d}(\mathbb{C}^2)),$$

is a diffeomorphism and hence preserves $S(J(2,d))$. Thus any element of $\mathrm{SL}_2(\mathbb{C})$ is the linear part of some singular $G \in et_{0,d}(\mathbb{C}^2)$. Hence, indeed $\mathrm{SL}_2(\mathbb{C}) \subseteq S(J(2,d))$. □

We point out that the mappings ψ_M and ϕ_M that were used above are special cases of the following family of mappings

$$\phi_{M_1,M_2} : J(n,d) \to J(n,d),$$
$$\phi_{M_1,M_2}(C(n,d)(G)) = C(n,d)(M_1 \circ G \circ M_2), G \in et_{0,d}(\mathbb{C}^n),$$

where M_1, M_2 are fixed elements of $\mathrm{SL}_n(\mathbb{C})$. Just like Lemma 6.1 we have the following

Lemma 6.2. *Let $F \in et_{0,d}(\mathbb{C}^n)$ and $M_1, M_2 \in \mathrm{SL}_n(\mathbb{C})$. Then the following are equivalent:*

(1) $C(n,d)(F) \in S(J(n,d))$.
(2) $\phi_{M_1,M_2}(C(n,d)(F)) \in S(J(n,d))$.

Proof. The mapping $\phi_{M_1,M_2} : J(n,d) \to J(n,d)$ is clearly a diffeomorphism and hence preserves the singular loci of $J(n,d)$. □

We would like to point out at a different type of a mapping which is a diffeomorphism and hence preserves the singular loci. There are two differences between the new mapping and the mappings ϕ_{M_1,M_2}. The first is that we will now work again in dimension $n = 2$. The second difference is that now $\mathrm{Aut}_{0,d}(\mathbb{C}^2)$ will replace $et_{0,d}(\mathbb{C}^2)$.

Lemma 6.3. *The mapping*

$$\mathrm{inv} : \mathrm{Aut}_{0,d}(\mathbb{C}^2) \to \mathrm{Aut}_{0,d}(\mathbb{C}^2),$$
$$\mathrm{inv}(F) = F^{-1},$$

is a diffeomorphism.

Proof. The only thing we need to show is that $\mathrm{inv}(\mathrm{Aut}_{0,d}(\mathbb{C}^2)) = \mathrm{Aut}_{0,d}(\mathbb{C}^2)$. In general, if $G \in \mathrm{Aut}_0(\mathbb{C}^n)$ then we have the degree inequality $\deg G^{-1} \leqslant (\deg G)^{n-1}$. For $n = 2$ this implies that $\deg G^{-1} = \deg G$ from which the result follows. □

We end by posing the following obvious question:

What is $S(J(2,d))$?

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