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NOTE ON BOUNDARY OBSTRUCTION TO JACOBIAN CONJECTURE OF TWO VARIABLES

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1. Introduction

We consider a polynomial mapping $\Phi = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ where $f(x, y)$, $g(x, y)$ are polynomials with coefficients in $\mathbb{C}$. (Every argument which follows in this note is also true over any algebraically closed field K of characteristic 0). Put $J(\Phi) = J(f, g)$ the Jacobian, which is given by

$$J(f, g) := f_x(x, y)g_y(x, y) - f_y(x, y)g_x(x, y)$$

where f_x, f_y are partial derivatives with respect to x and y respectively. We can also understand $J(f, g)$ by $(df \wedge dg)(x, y) = J(f, g)\, dx \wedge dy$. Suppose that the mapping $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism in the sense that it has an inverse polynomial mapping $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$. Then by the composition rule of the Jacobian, we have the equality $J(\Phi) \equiv c$ for a non-zero constant c. The *Jacobian conjecture* is concerned with the converse assertion:

(JC): If $J(f, g) \equiv c$, $c \in \mathbb{C}^*$, $(f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism.

The first essential contribution to this conjecture was made by Suzuki [4], Abhyankar [1] and the author did a small work about outside faces of mixed weights using Newton diagram [3]. There are many papers since then about this topic but it seems that there are few further progress on this conjecture of two variable case.

This note is a rewritten version of [3] for a conference talk in Hanoi, October 2006. We tried to make some argument simpler than that of [3] and we gave some new results. Several new examples are given in §5 to illustrate the difficulty of the problem. We believe that some assertions are interesting for themselves. We hope that this note will be of some interest to those who are studying this conjecture.

2. Preliminaries and Key Lemmas

2.1. **A graded ring.** A polynomial $h(x, y)$ is called a *weighted homogeneous polynomial of degree d with respect to the weight vector $P = (a, b)$* if it satisfies the equality: $h(xt^a, yt^b) = t^d h(x, y)$. Here $(a, b) \neq (0, 0)$ and a, b are coprime integers, possibly zero or negative. Thus P is a primitive integer vector. If $a = 0$

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for example, it is necessary to have $b = \pm 1$ for P to be primitive. We call d *the degree of* h *with respect to the weight vector* P and we denote d as $\deg_P h$. We can extend this definition to rational functions. Namely a rational function $h = h_1(x, y)/h_2(x, y)$ is called a *weighted homogeneous rational function of degree* d if h_1, h_2 are weighted homogeneous polynomials of degree d_1, d_2 respectively with respect to a common weight vector $P = (a, b)$ and $d = d_1 - d_2$. It satisfies the Euler equality:

$$d\, h(x, y) = ax\, h_x(x, y) + by\, h_y(x, y).$$

Let $\mathcal{W}_n(P)$ (or simply $\mathcal{W}_n$) be the set of weighted homogeneous rational functions of degree n with weight vector P and let $\mathcal{W}(P)$ (or simply $\mathcal{W}$) be the graded ring defined as $\mathcal{W}(P) = \oplus_{i\in\mathbb{Z}} \mathcal{W}_i(P)$. For a $f \in \mathcal{W}(P)$, we write f_n the degree n component of f. Note that $\mathcal{W}_n \cdot \mathcal{W}_m \subset \mathcal{W}_{n+m}$. Note that the ring of polynomials $\mathbb{C}[x, y]$ is a subring of $\mathcal{W}$.

2.2. **Canonical factorization.** A weight vector $P = (a, b)$ is called a *strictly positive* (respectively *mixed*) weight if $a, b > 0$ (resp. $ab \leqslant 0$). Let $h(x, y)$ be a non-zero weighted homogeneous polynomial of degree d with weight vector P. Then there is a unique factorization

$$h(x,y) = \begin{cases} c x^p\, y^q \prod_{i=1}^m (x^b + c_i y^a)^{\nu_i}, & \text{if } a,\, b > 0 \\ c x^p\, y^{q'} \prod_{i=1}^m (x^b y^{-a} + c_i)^{\nu_i}, & \text{if } a \leqslant 0 < b \end{cases} \tag{2.1}$$

where $c_1, \dots, c_m$ are mutually distinct non-zero complex numbers. Similarly we can factorize any weighted homogeneous rational function $F(x, y)$ as above. The only difference is that the exponents p, q, ν_i might be negative integers.

2.3. **Tchirnhausen approximate roots.** Let $f(x, y) = \sum_{\nu=(\nu_1,\nu_2)} c_\nu x^{\nu_1} y^{\nu_2}$ be a polynomial. *The Newton diagram* $\Delta(f; x, y)$ is the convex hull of integer points $\nu = (\nu_1, \nu_2)$ with $c_\nu \neq 0$ ([2]). A face $\Xi \in \partial\Delta(f; x, y)$ is called *an outside face* if the line supporting Ξ does not pass through the origin and $\Delta(f; x, y)$ and the origin O are in the same half plane. Let $P = (a, b)$ be a fixed weight vector. Let $\deg_P f$ be the maximal degree of the monomials in $f(x, y)$ and put $\Delta(P) = \{(\nu_1, \nu_2) \in \Delta(f; x, y) | a\nu_1 + b\nu_2 = \deg_P f\}$. Obviously $\Delta(P)$ is either a face or a vertex of $\Delta(f; x, y)$. Let $f_P(x, y) = \sum_{\nu\in\Delta(P)} c_\nu x^{\nu_1} y^{\nu_2}$. If $\Delta(P)$ is an outside face, we have $\deg_P f > 0$. By the definition, f_P is a weighted homogeneous polynomial of degree $\deg_P f$. Let $\Xi = \Delta(P)$. We also use the notation $f_\Xi(x, y)$ instead of $f_P(x, y)$. Then we can factorize $f_P(x, y)$ as

$$f_P(x,y) \;=\begin{cases} c x^p\, y^q \prod_{i=1}^m (x^b + c_i y^a)^{\nu_i}, & \text{if } a,\, b > 0 \\ c x^p\, y^{q'} \prod_{i=1}^m (x^b y^{-a} + c_i)^{\nu_i}, & \text{if } a \leqslant 0 < b. \end{cases} \tag{2.2}$$

The *face multiplicity* $\mathrm{m}(f, P)$ of f_P is defined as the greatest common divisor of the integers p, q (or q'), $\nu_1, \dots, \nu_k$. Namely $\mathrm{m}(f, P)$ is the maximum of the positive integers s such that $f_P^{1/s}$ is a polynomial. *The outside boundary multiplicity*

$\mathrm{m}_\infty(f)$ is defined to be the greatest common divisor of all boundary multiplicities $\mathrm{m}(f,P)$ such that P corresponds to an outside face. Put

$$\mathcal{W}_{\leqslant N} := \{F \in \mathcal{W}(P) \mid \deg_P F \leqslant N\}.$$

Lemma 2.1. *Let $r = \mathrm{m}(f,P)$. Then for any negative integer N, there exists a rational function $h(x,y) \in \mathcal{W}$ such that $deg_P\,(f - h^r) < N$.*

We call h an *approximate r-th root of f modulo $\mathcal{W}_{\leqslant N}$*. For simplicity we use the notation $f \equiv_N h^r$ if $\deg_P(f - h^r) \leqslant N$ i.e., $f - h^r \in \mathcal{W}_{\leqslant N}$.

Proof. The proof is easily obtained by a standard approximate root argument. Assume that f_P can be written as $f_P = h_{d_0}^r$ for a weighted homogeneous polynomial $h_{d_0}(x,y)$ of degree d_0 with $d = d_0\,r$. Write f, h in degree components:

$$\begin{aligned} f(x,y) &= f_d(x,y) + f_{d-1}(x,y) + \cdots + f_e(x,y), \quad d = \deg_P f \\ h(x,y) &= h_{d_0} + h_{d_0-1} + \cdots + h_{d_0-m} \end{aligned}$$

We solve the equality

$$\sharp_k : \quad f_k = (h^r)_k, \qquad k = d, d-1, \ldots.$$

inductively from the above, starting from $k = d$. Here $(h^r)_k$ is the degree k component of h^r. The first equality is simply $h_{d_0}^r = f_P$. Assume that $h_{d_0}, \ldots, h_{d_0-s}$ is determined, using the equality of $\sharp_d, \ldots, \sharp_{d-s}$. Then taking $k = d - s - 1$, we have the equality of the degree $d - s - 1$ components:

$$f_{d-s-1} = \sum_{j_1+\cdots+j_r=d-s-1} h_{j_1} \cdots h_{j_r}.$$

In the right hand side, the terms which contains h_{d_0-s-1} is equal to $r h_{d_0}^{r-1} h_{d_0-s-1}$ and the other terms are polynomials of h_j, $j \geqslant d_0 - s$. Thus we can use the equality $\sharp_{d-s-1}$ to determine h_{d_0-s-1}. Now we put $h_{\geqslant\mu} := \sum_{j\geqslant\mu} h_j$. Then we have the inequality by the definition:

$$\deg_P\,(f - h_{\geqslant\mu}^r) < d - (d_0 - \mu) = d - d_0 + \mu.$$

Thus we need simply take $h = h_{\geqslant\mu}$ with $\mu < 0$, $|\mu|$ sufficiently large. □

Lemma 2.2. *(Approximate inverse) Let $\phi \in \mathcal{W}$. Then for any $N < 0$, there exists a rational function $\psi \in \mathcal{W}$ such that $\phi\,\psi \equiv_N 1$.*

The proof is exactly same as that of the approximate root. Assume that $\phi = \phi_m + \phi_{m-1} + \cdots$ with $m = \deg_P = \phi$. We can determine ψ inductively so that $\psi = \psi_{-m} + \psi_{-m-1} + \cdots$ where $\psi_{-m} = \phi_m^{-1}$, $\psi_{-m-1} = -\phi_{m-1}\phi_m^{-2}, \ldots$ and so on. We call ψ *an approximate inverse* of ϕ.

3. Boundary obstruction

3.1. **Similarity lemma.**

Lemma 3.1. *((17.4),[1] and Proposition (2.7),[3]) Suppose $F(x,y)$, $G(x,y)$ are weighted homogeneous rational functions of degree d_1, d_2 with $d_1 \neq 0$ such that $J(F,G)=0$. Then there exists a non-zero constant c so that $F^{d_2} = c\,G^{d_1}$.*

The assertion is immediate from the equality $d\log(F^{d_2}/G^{d_1}) = 0$ which follows by the Euler equality and the assumption $J(F,G)=0$. In case $d_2 = 0$, Lemma 3.1 implies that $G(x,y)$ is a constant.

Assume that f is a given polynomial and we consider the existence problem of a "partner polynomial" g such that $J(f,g)=1$. We consider a fixed weight vector $P=(a,b)$, $\gcd(a,b)=1$ as before. Let us consider the decomposition by $\deg_P$ (=degree with respect to weight P):

$$f = f_n + f_{n-1} + \cdots + f_{n_0}, \qquad g = g_m + g_{m-1} + \cdots + g_{m_0}.$$

Put $\varepsilon = a+b$. Then $J(f,g)$ has the decomposition:

$$J(f,g) = J_\ell + \cdots + J_{\ell_0}$$

where $\ell = n+m-\varepsilon$, $\ell_0 = n_0+m_0-\varepsilon$ and $J_k = \sum_{i+j=k+\varepsilon} J(f_i,g_j)$. The assumption $J(f,g)=1$ implies that $J_k = 0$ for $k \neq 0$ and $J_0 = 1$. In particular we have

Corollary 3.1. *Assume that $n+m-\varepsilon > 0$. Then $J(f_n,g_m)=0$ and there exists a constant $c \neq 0$ such that $g_m^n = c\,f_n^m$.*

Definition 3.1. When the condition $F^{d_2}/G^{d_1} = c \in \mathbb{C}^*$ is satisfied, we say F, G are **similar**. Two polynomials f, g are **similar** if f_P, g_P are similar for any weight vector P such that f_P corresponds to an outside face Ξ.

3.2. **Degree estimation of Jacobians of approximate roots.**

Let us assume $P=(a,b)$ with $b>0$ for simplicity. Then we can factorize $f_P(x,y)$ as in (2.2). Assuming $n+m>\varepsilon$ and putting $r = \mathrm{m}(f,P)$, we write $n = n_0 r$, $m = n_0 s$. Take a sufficiently large negative N. Let h be an approximate r-th root of f (in $\mathcal{W}(P)$) of modulo $\mathcal{W}_{\leqslant 2N}$ so that $f \equiv_{2N} h^r$. We also fix an approximate inverse $k(x,y) \in \mathcal{W}(P)$ of h of modulo $\mathcal{W}_{\leqslant 2N}$.

Lemma 3.2. *Taking N a negative integer with $|N|$ sufficiently large, we have*

$$J(f,h^j) \equiv_N 0, \quad 0 \leqslant j \leqslant [m/n]$$
$$J(f,k^j) \equiv_N 0, \quad 0 < j \leqslant [(n-a-b)/n_0].$$

Here $[p/q]$ is the largest integer which is less than or equal to p/q.

Proof. Put $R = f - h^r$. By the definition of an approximate root, $\deg_P R \leqslant 2N$. Thus

$$0 = J(f,f) = J(f,h^r) + J(f,R) = rh^{r-1}J(f,h) + J(f,R).$$

Recall that $\deg_P J(\phi,\psi) \leqslant \deg_P \phi + \deg_P \psi - \varepsilon$ by the definition of Jacobian. Thus we have the inequality

$$\deg_P J(f,h) = \deg_P J(f,R) - (r-1)n_0 \leqslant n + 2N - \varepsilon - (r-1)n_0 = 2N + n_0 - \varepsilon.$$

Now for any $0 \leqslant j \leqslant r$ and N with $|N|$ sufficiently large, we get

$$\deg_P\, J(f, h^j) = \deg_P\, J(f, h) + (j-1)n_0 \leqslant 2N - \varepsilon + jn_0 \leqslant N.$$

For the second inequality, put $S := hk - 1$. Then $\deg_P\, S \leqslant 2N$. Thus

$$\begin{aligned} 2N + n - \varepsilon \quad &\geqslant \deg_P\, J(f, S) = \deg_P\, J(f, hk) \\ &= \deg_P \left(kJ(f,h) + hJ(f,k)\right). \end{aligned}$$

As $\deg_P k = -n_0$, we have

$$\deg_P J(f,k) \leqslant \max\{2N + n - \varepsilon, \deg_P J(f,h) - n_0\} - n_0 \leqslant 2N + n - \varepsilon \leqslant N.$$

The assertion follows immediately. □

3.3. **Normal form of $g(x, y)$.** Now we can describe the normal form of a partner polynomial g in a slightly modified form [3].

Theorem 3.1. *(cf. Proposition 3.3,* [3]*) Let f, g be polynomials with $J(f, g) = 1$ and let h, k be as above. There exist constants c_j, $0 \leqslant j \leqslant s$ and d_k, $0 < k \leqslant \alpha$ with $\alpha = [(n - \varepsilon)/n_0]$ so that*

$$g \;=\; c_s h^s + c_{s-1} h^{s-1} + \cdots + c_1 h + c_0 + d_1 k + \cdots + d_\alpha k^\alpha + R$$

with $\deg_P\, R = -n + \varepsilon$. *Then $J(f_P, R_P) = 1$.*

Proof. We apply Lemma 3.1 repeatedly. Assume that $n + m > \varepsilon$. Then we have that $g_P = c_s h_P^s$ for some $c_s \neq 0$. Put $g = c_s h^s + g_s$. Then by the assumption $J(f, g) = 1$, we have $J(f, g_s) \equiv_N 1$. If $n + \deg_P\, g_s > \varepsilon$, we apply Lemma 3.1 repeatedly so that we arrive at the situation:

$$g = \; c_s h^s + c_{s-1} h^{s-1} + c_0 + d_1 k + \cdots + d_\alpha k^\alpha + R, \quad \deg_P R + n \leqslant \varepsilon$$

and $J(f_P, R_P) = 1$. This implies that $\deg_P\, R = \varepsilon - n$ and the assertion follows immediately. □

This gives the following necessary condition as a corollary.

Theorem 3.2. *(Lemma (3.4),*[3]*) Assume that $J(f, g) = 1$ for a pair of polynomials f, g. Then for any outside face (or a vertex) Ξ with a weight vector P, there exists a weighted rational function $\phi(x, y)$ with weight P such that $J(f_\Xi, \phi) = 1$.*

Definition 3.2. We say that $f(x, y)$ has **no boundary obstruction** if the condition in Theorem 3.2 is satisfied for every outside face $\Xi \in \Delta(f; x, y)$.

Proposition 3.1. *Suppose that* $\mathrm{m}(f, P) = 1$. *Assume also that neither* $(1, 0)$ *nor* $(0, 1)$ *are on the outside face of a positive weight. Then f does not have any polynomial partner g with $J(f, g) = 1$.*

In fact, assume that the face multiplicity $m(f, P) = 1$. Put $\ell = m(g, P)$. Then if $J(f, g) = 1$, we can find a constant c_ℓ so that $\deg_P(g - c_\ell f^\ell) < \deg_P g$. Thus in the normal form discussion in Theorem 3.1, we can simply use f, instead of h and g takes the form

$$g = c_\ell f^\ell + \cdots + c_1 f + g_0, \quad \deg_P g_0 < \deg_P f$$

and the point is that $g_0 \in \mathbb{C}[x, y]$. Then $J(f, g) = 1$ will imply that $J(f_P, g_{0P}) = 1$. This implies that $f_P(x, y)$ (respectively $g_P(x, y)$) contains a linear term y or x (resp. x or y). Then P cannot be positive by the assumption. For a mixed P, this is not possible by Lemma 5.1 (see §5). ☐

Remark 1. If $f(x, y) = c\,y + \sum_{i=1}^{n} c_i x^n$ with some $c \neq 0$, we can take $g(x, y) = x/c$.

4. Vanishing of the obstruction

We consider the Jacobian condition $J(f, g)$, as a polynomial equation for g with $J(f, g) = 1$. More precisely, using Theorem 3.2, we consider the existence problem of a weighted homogeneous rational function solution $\phi(x, y)$ with respect to P satisfying the equality $J(f_P, \phi) = 1$.

4.1. **Strictly positive faces.** We first consider the case where $P = (a, b)$ is a strictly positive weight vector (i.e., $a, b > 0$). This case has been studied by Abhyankar [1]:

Let σ be one of x, y, $x^b + cy^a$, $c \neq 0$. Let $F(x, y)$ be a non-zero weighted homogeneous rational function with a factorization

$$F(x, y) = cx^p\, y^q \prod_{i=1}^{m} (x^b + c_i y^a)^{\nu_i}, \quad c,\, c_i \neq 0,\, \nu_i \neq 0 \tag{4.1}$$

We define $\mathrm{val}_\sigma F = p,\, q,\, \nu_i,\, 0$ for $\sigma = x, y, x^b + c_i y^a$ and $x^b + cy^a$, $c \neq c_i$ respectively. An important key lemma is

Lemma 4.1. *(Lemma (2.8),[3]) Assume that $a, b > 0$ and $F(x, y)$ is a non-zero weighted homogeneous rational function of degree $d \neq 0$ and let σ be as above. Then $val_\sigma F = 0$ implies $val_\sigma J(\sigma, F) = 0$.*

The following is fundamental for the cancellation of strictly positive faces.

Theorem 4.1. *([1, 3]) Let $h(x, y)$ be a weighted homogeneous polynomial of degree $d > 0$ with respect to a strictly positive weight vector $P = (a, b)$. Assume that there exists a weighted homogeneous rational function $\phi(x, y)$ with respect to P such that $J(h, \phi) = c$, where $c \neq 0$ is a constant. Then h and ϕ are one of the following list up to a multiplication by a constant.*

(1) *$h(x, y) = x^p y^q$ and $\phi(x, y) = xy\, h^{-1}$.*
(2) *$a = 1$, $h(x, y) = x^p(y + c\,x^b)^q$ and $\phi = x(y + c\,x^b)\, h^{-1}$.*
(3) *$b = 1$, $h(x, y) = y^p(x + cy^a)^q$ and $\phi = y(x + c\,y^a)\, h^{-1}$.*
(4) *$a = b = 1$ and $h(x, y) = (x + c_1 y)^p(x + c_2 y)^q$ and $\phi = (x + c_1 y)^{1-p}(x + c_2 y)^{1-q}$.*

where $p \neq q$ and c, c_1, c_2 are non-zero complex numbers.

Proof. First we factorize h, ϕ (up to a multiplication by a constant) as

$$h(x, y) = x^{\alpha_{-1}} y^{\alpha_0} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k}, \quad \phi(x, y) = x^{\beta_{-1}} y^{\beta_0} \sigma_1^{\beta_1} \cdots \sigma_k^{\beta_k} \tau_1^{\gamma_1} \cdots \tau_\ell^{\gamma_\ell}$$

where $\alpha_{-1},\,\alpha_0 \geqslant 0$ and $\alpha_i > 0,\, i \geqslant 1$. Here $\sigma_i\,(i = 1,\dots,k)$, $\tau_j\,(= 1,\dots,\ell)$ are mutually distinct divisors of the type $x^b + c_i y^a$ or $x^b + d_j y^a$. A key observation is:

Assertion 4.1. *(Assertion 1,* [3]*) There are inequalities:*

(1) *For* $i = -1, 0$,

$$\alpha_i + \beta_i \geqslant \begin{cases} 1, & \alpha_i > 0 \\ 0, & \alpha_i = 0 \end{cases}$$

(2) $\alpha_j + \beta_j \geqslant 1$ *for* $j = 1,\dots,k$.
(3) $\gamma_s \geqslant 1$, *for* $s = 1,\dots,\ell$.

Theorem 4.1 follows from an easy degree estimation, using Assertion 4.1. □

4.2. **Reduction of Newton diagram.** We consider a polynomial mapping $\Phi = (f,g) : \mathbb{C}^2 \to \mathbb{C}^2$ with a non-zero constant Jacobian $J(f,g) \equiv c$. Assume that $f(x,y) = \sum_\nu c_\nu\, x^{\nu_1} y^{\nu_2}$. For any outside face Ξ, we can attach a unique weight vector $P = (a,b)$ such that $\gcd(a,b) = 1$ and $f_\Xi(x,y)$ is a weighted homogeneous polynomial with weight P and $\deg_P(f_\Xi) > 0$. A vertex $R = (\alpha,\beta) \in \Delta(f;x,y)$ is called *an outside vertex* if it belongs to an outside face. We can find a weight vector (not unique) $P = (a,b)$ for a given R such that $\deg_P f$ is given by $a\alpha + b\beta$. Observe that there is no vertex (α,α) on the diagonal $\nu_1 = \nu_2$ by Theorem 4.1.

Now suppose that $\Delta(f;x,y)$ has an outside face Ξ with a strictly positive weight vector $P = (a,b)$. First, by a linear change of coordinates if necessary, we may assume that $f(x,y)$ does not have a face described in (4) of Theorem 4.1. Then by Theorem 4.1, we can write $f_P(x,y) = ex^p(y + cx^b)^q$ with $e \neq 0, p \neq q$, permuting x and y if necessary. We take the triangular change of coordinates $x_1 = x, \quad y_1 = y + cx^b$. Then the face Ξ shrinks to a vertex (p,q) in $\Delta(f;x_1,y_1)$ and we can easily see that $\Delta(f+t;x_1,y_1) \subsetneq \Delta(f+t;x,y)$ for a generic $t \in \mathbb{C}$. Note that $\Delta(f+t;x,y)$ is the convex hull of $\Delta(f;x,y)$ and the origin. Repeating this process, we may assume that $\Delta(f+t;x,y)$ for t generic has a minimum area. This implies there exists no strictly positive outside faces. After this reduction, if f reduces to a linear function, Φ is certainly an automorphism. So in the next section, we consider the case when f is not a linear function and $\Delta(f;x,y)$ has only non-strictly positive faces.

There is another face we can eliminate. A face Ξ is called *horizontal* (resp. *vertical*) if the weight is given by $P = (0,1)$ (resp. $P = (1,0)$). We say that Ξ is an *elementary horizontal* face of f is f_P takes the form

$$f_P(x,y) = e\,y^q(x+c)^p, \quad c \neq 0,\ p \neq q,\, e \neq 0. \tag{4.2}$$

An elementary vertical face is defined similarly. If f has such a face, we can take the parallel change of coordinates: $x_1 = x + c, \quad y_1 = y$. By this change of coordinates, the face Ξ of $\Delta(f;x,y)$ shrinks to a vertex (p,q) but any other face Ξ remains as it was. This implies that $\Delta(f+t;x_1,y_1) \subsetneq \Delta(f+t;x,y)$ for a generic t.

Assumption. Thus from now on, we assume that f is not a linear function, $\Delta(f;x,y)$ has neither strictly positive face nor elementary horizontal nor elementary vertical faces. If this is the case, we say that f *is strictly reduced.* The Jacobian conjecture is equivalent to the *non-existence of a Jacobian partner* g for any non-linear strictly reduced polynomial f.

Assume that $f(x,y)$ is a non-linear strictly reduced polynomial. Then there exists a unique vertex $(\alpha,\beta)\in\Delta(f;x,y)$ so that

$$\Delta(f;x,y)\subset\{(\nu_1,\nu_2)\,|\,\nu_1\leqslant\alpha,\ \nu_2\leqslant\beta\}.$$

We call (α,β) *the top vertex* of f.

4.3. **Faces with mixed weight vectors.** We assume that Ξ is an outside face of $\Delta(f;x,y)$ with a mixed weight vector $P=(a,b)$, where P is chosen so that $\deg_P f_\Xi>0$. We call Ξ an *upper face* (resp. a *lower face*) if $b>0\geqslant a$ (resp. $a>0\geqslant b$). In Figure 1, Ξ_1, Ξ_2,Ξ_3 are upper faces and Γ_1, Γ_2 are lower faces.

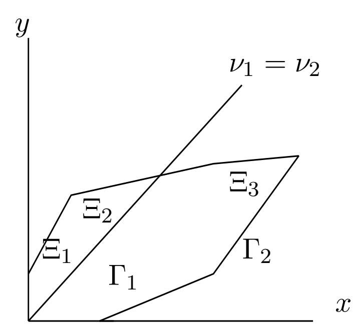

FIGURE 1. Newton diagram

Now we consider the boundary obstruction for f_Ξ. We consider a weighted homogeneous polynomial $h(x,y)$ with weight vector $P=(a,b)$, $b>0\geqslant a$ and $\deg_P h>0$. We assume that

$$h(x,y)=ex^py^q\prod_{i=1}^{k}(x^by^{-a}+c_i)^{\alpha_i},\quad e\neq 0 \tag{4.3}$$

The assumption $\deg_P h>0$ is equivalent to $qb+ap>0$. The condition of the existence of a weighted homogeneous rational function solution ϕ for the equation $J(h,\phi)=1$ is more difficult to be given explicitly. The weight vector $P=(-1,1)$ is called to be *exceptional.* This is the only case for which we can give the complete answer. In fact, we have

Lemma 4.2. (Theorem (5.2),[3]) *Assume that* $P=(-1,1)$. *Then the necessary and sufficient condition for the existence of the rational solution* ϕ *for the equation* $J(h,\phi)=1$ *is* $\deg_P h\neq 0$.

Proof. If $p \neq q$, we can take $\phi = c'xy\,h^{-1}$, $c'e(p-q) = 1$. If $p = q$, $\deg_P f = 0$ and thus $\deg_P \phi = 0$ and ϕ is a rational function of one variable xy. Thus $J(h,\phi) = 0$ and there exists no solution. $\square$

Thus we consider hereafter non-exceptional weight vector $P = (a,b)$, $a \leqslant 0 < b$. Assume that Ξ is the line segment $\overline{AB}$ where $A = (p,q)$ is the left end and $B = (p+bN, q-aN)$ is the right end where $N = \sum_{i=1}^k \nu_i$. We say Ξ is a *crossing face* if $q - aN < p + bN$. This means that B is in the lower side of the diagonal $\nu_1 = \nu_2$ and the left half line of L_Ξ starting from B intersects with the diagonal $y = x$. Similarly we define crossing faces for lower faces. For example, in Figure 1, Ξ_2, Ξ_3 are crossing but $\Xi_1, \Gamma_1, \Gamma_2$ are not crossing faces. Note that if the top vertex (α,β) is below the diagonal, i.e. $\alpha > \beta$ (resp. above the diagonal i.e., $\alpha < \beta$), there exists an upper crossing (resp. a lower crossing) face but no lower crossing (resp. no upper crossing) face. Now, we modify Lemma 4.1 for the mixed weight situation:

Lemma 4.3. ([3])(Lemma (2.8),[3]) *Assume that $F(x,y)$ is a non-zero weighted homogeneous rational function of degree $d \neq 0$ with a mixed weight vector $P = (a,b)$, $b > 0$ and let σ be either x, y or $x^b y^{-a} + c$, $c \neq 0$. Then $\mathrm{val}_\sigma F = 0$ implies $\mathrm{val}_\sigma J(\sigma, F) = 0$.*

Proof. We present a simpler proof than that of Lemma (2.8) in [3], as the original proof treats both cases where P being strictly positive or mixed weight simultaneously and it was not so clear. Assume $F(x,y) = x^p y^q \prod_{i=1}^k (x^b y^{-a} + c_i)^{\nu_i}$.
(1) For $\sigma = x$ (or y), $\mathrm{val}_\sigma F = 0$ implies $p = 0$ and $q \neq 0$ as $\deg_P F = qb$. (If $\sigma = y$, $q = 0$ and $d = pa \neq 0$. Thus $a, p \neq 0$ in this case.) Put $\sigma_i = x^b y^{-a} + c_i$. We use the equality:

$$J(x,F) = \frac{\partial F}{\partial y} = q\, y^{q-1} \prod_i \sigma_i^{\nu_i} - \sum_j \nu_j a x^b y^{-a-1} {\sigma_j}^{-1} \prod_i \sigma_i^{\nu_i}.$$

Thus the above equality implies the assertion, as $J(x,F)|_{x=0} \neq 0$.
(2) Next we consider the case $\sigma = x^b y^{-a} + c$ with $c \neq c_1, \dots, c_k$. Consider a unimodular matrix:

$$A = \begin{pmatrix} a & a' \\ b & -b' \end{pmatrix}, \qquad ba' + ab' = 1, \quad A^{-1} = \begin{pmatrix} b' & a' \\ b & -a \end{pmatrix}.$$

We use the toric coordinates (u,v) where

$$\begin{cases} x = u^a v^{a'} \\ y = u^b v^{-b'} \end{cases}, \qquad \begin{cases} u = x^{b'} y^{a'} \\ v = x^b y^{-a} \end{cases}.$$

Note that $\deg_P u = 1$ and $\deg_P v = 0$. Thus

$$df \wedge dg = J(f,g;x,y)dx \wedge dy = J(f,g;u,v) du \wedge dv$$

where we used the notations $J(f,g;x,y)$ and $J(f,g;u,v)$ to distinguish Jacobians in two coordinate systems (x,y) and (u,v). In the toric coordinates (u,v), we have simply

$$\sigma = v + c, \quad F = u^{\alpha} v^{\beta} \prod_i (v + c_i)^{\nu_i}, \text{ where } \alpha = bq + ap,\ \beta = pa' - qb'.$$

Thus $\alpha = \deg_P F \neq 0$ and

$$J(v + c, F; u, v) = -\frac{\partial F}{\partial u} = -\alpha u^{\alpha-1} v^{\beta} \prod_i (v + c_i)^{\nu_i} \neq 0.$$

Now recall that $dx \wedge dy/xy = du \wedge dv/(uv)$. Thus we see immediately $\mathrm{val}_\sigma F = 0$ implies $\mathrm{val}_\sigma J(\sigma, F; x, y) = 0$. □

Now we state a modification of Assertion 4.1.

Assertion 4.2. *Assume that $h(x,y)$ is a weighted homogeneous polynomial of degree $d \neq 0$ with a non-exceptional weight vector $P = (a,b)$, $b > 0 \geqslant a$ and assume that h has a factorization:*

$$h(x,y) = e x^{\alpha_{-1}} y^{\alpha_0} \prod_{i=1}^{k} (x^b y^{-a} + c_i)^{\alpha_i}, \quad e \neq 0.$$

Assume that $J(h, \phi) = 1$ for some weighted homogeneous rational function ϕ and consider the factorization of ϕ:

$$\phi(x,y) \;=\; e' x^{\beta_{-1}} y^{\beta_0} \prod_{i=1}^{k} (x^b y^{-a} + c_i)^{\beta_i} \prod_{j=1}^{\ell} (x^b y^{-a} + d_j)^{\gamma_j},\ e' \neq 0.$$

Then either $\beta_i = \min(\alpha_i, 1 - \alpha_i)$ or $\beta_i = \alpha_i(a + b - d)/d$ and $\gamma_j \geqslant 0$.

In particular, if $d\,(a+b) \geqslant 0$, we have $\beta_i \geqslant 1 - \alpha_i$ if $\alpha_i > 0$, $\beta_i \geqslant 0$ if $\alpha_i = 0$ and $\gamma_j > 0$ for any j.

Proof. Put $\sigma_{-1} = x$, $\sigma_0 = y$, $\sigma_i = x^b y^{-a} + c_i$ for $i \geqslant 1$. The proof of this assertion works in the same way as that of Assertion 4.1 using Lemma 4.3. The proof of the assertion about γ_j is immediate. Fix i, $-1 \leqslant i \leqslant k$ with $\alpha_i \neq 0$ and consider $\psi = \phi^{\alpha_i} h^{-\beta_i}$. Then $\mathrm{val}_{\sigma_i} \psi = 0$. A slight difference is in Case 1.

Case 1. Assume that $\deg_P \psi = 0$. Then we have the equalities

$$d + \deg_P \phi = a + b, \quad \alpha_i \deg_P \phi - \beta_i\, d = 0.$$

From this, we get the second possibility: $\beta_i = \alpha_i(a+b-d)/d$. If further $d(a+b) > 0$, we get

$$\frac{\beta_i}{\alpha_i} = \frac{\deg_P \phi}{d} = -1 + \frac{a+b}{d} > -1$$

which implies $\beta_i > -\alpha_i$ or $\alpha_i + \beta_i \geqslant 1$.

The case $\deg_P \psi \neq 0$ can be treated exactly as in the proof of Assertion 1 in [3] as follows. Consider $J(h, \psi)$ in two ways. On one hand, $J(h, \psi) = \alpha_i \phi^{\alpha_i - 1} h^{-\beta_i}$. On the other hand,

$$J(\psi, h) = J(\psi, h/\sigma_i^{\alpha_i}) \sigma_i^{\alpha_i} + J(\psi, \sigma_i) \alpha_i \sigma_i^{\alpha_i - 1} h/\sigma_i^{\alpha_i}.$$

Comparing the multiplicity of σ_i, we get

$$(\alpha_i - 1)\beta_i - \alpha_i\beta_i = \alpha_i - 1, \quad \text{or } \beta_i = 1 - \alpha_i.$$

□

If $d\,(a+b) < 0$, the inequality $\beta_i \geqslant 1 - \alpha_i$ may not be true. We thank to P. Cassou-Noguès for informing us an counter-example of this case. We call such a polynomial *a negatively crossing*. Geometrically, the supporting line of h intersects at a negative part of the diagonal $\nu_1 = \nu_2$.

4.4. **Nice crossing faces.** Assume that $f(x,y)$ is strictly reduced and Ξ is an upper non-exceptional face of $\Delta(f;x,y)$ with weight vector $P = (a,b)$, $b > 0 \geqslant a$ and $f_\Xi(x,y)$ is given as

$$f_\Xi(x,y) = e\,x^p y^q \prod_{i=1}^{k} (x^b y^{-a} + c_i)^{\alpha_i}, \quad e \neq 0.$$

Assume that A, B are the left and right end of Ξ. Let us consider the line $L_P = \{(\nu_1,\nu_2) \,|\, a\nu_1 + b\nu_2 = a+b\}$. We call L_P *the* $(1,1)$*-line* of weight P. Note that $(1,1) \in L_P$ and $\deg_P x^\alpha y^\beta = a+b$ for $(\alpha,\beta) \in L_P \cap \mathbb{N}^2$. Let E be the intersection of OB and L_P (see Figure 2). We call a face Ξ **a nice** face if E (the intersection $L_P \cap \overline{OB}$) is an integer point and $(a+b)\deg_P f_\Xi > 0$.

Lemma 4.4. (Theorem (5.5),[3]) *Assume that there exists a solution ϕ for $J(f_\Xi, \phi) = 1$ and $(a+b)\deg_P f_\Xi > 0$. Then Ξ must be a nice face. In particular, if Ξ is a non-crossing face, Ξ cannot be nice and there does not exist any rational solution ϕ, $J(f_\Xi,\phi) \equiv 1$.*

The same assertion holds for a lower face.

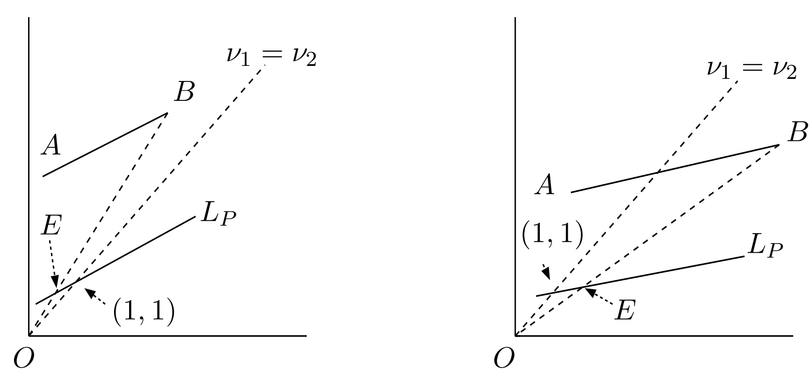

FIGURE 2. non-crossing(left) and crossing(right)

Proof. Put $h = f_\Xi$ for simplicity. Assume that Ξ is not nice but there exists a solution ϕ, $J(h,\phi) = 1$ and put $\psi = h\phi$. Then by Assertion 4.2, ψ is a polynomial which can be factorized as

$$\psi = e' x^r y^s \prod_{i=1}^{k+\ell} (x^b y^{-a} + c_i)^{\gamma_i},\ \gamma_i \geqslant 1,\ e' \neq 0.$$

Let Θ be the support segment of ψ. It has two end points C, D where $C = (r,s)$ and $D = (r + bN, s - aN)$ with $N = \sum_i \gamma_i$. As $\deg_P \psi = a + b$, the supporting line L_Θ contains the point $(1,1)$ (which implies $L_\Theta \subset L$). Thus D is either $(1,1)$ or situated in the lower part of the diagonal line $\nu_1 = \nu_2$. Now we examine the equality : $J(h,\psi) = h$. Recall that

$$J(x^a y^b, x^c y^d) = (ad - bc) x^{a+c-1} y^{b+d-1}. \tag{4.4}$$

Thus this Jacobian vanishes iff three points (a,b), (c,d), $(0,0)$ are colinear. Consider the right end of the support of $J(h,\psi)$. This must be B by the assumption $J(h,\psi) = h$. As D is an integer point and E is not an integer point by the assumption, $D \neq E$, and this implies O, D, B are not colinear. Thus for B to be the right end of the support of $J(h,\psi)$, we need to have $D = (1,1)$. Now we consider the left end of the support of $J(h,\psi)$. We have two possibilities:

(a1) $C = D$, $\psi = xy$, or (a2) O, C, A are colinear.

By Assertion 4.2, ψ is divisible by $x^b y^{-a} + c_1$. Thus (a1) is impossible. For (a2), C must be $(0,1)$ and $p = 0$ (thus $\psi = e' y(x + c_1)$) but this is only possible for $P = (0,1)$. Thus h is horizontal. However by the assumption that f is strictly reduced, h is not elementary and $k \geqslant 2$. This implies that $(x + c_1)(x + c_2) | \psi$ which is obviously impossible. If Ξ is non-crossing, E cannot be an integer point (see Figure 2, the left side). This completes the proof. □

The following Lemma shows the difficulty of nice crossing faces.

Lemma 4.5. *Assume that f_P is crossing where $P = (a,b)$ and assume that*

$$f_P(x,y) = x^p\, y^q \prod_{i=1}^{s} (x^{b'} y^{-a'} + c_i)^r,$$

where $c_1, \dots, c_s$ are mutually distinct and $a'/b' = a/b$ and

$$\frac{1 + sb'}{1 - sa'} = \frac{p + rsb'}{q - rsa'}. \tag{4.5}$$

We assume that $(a+b)\,(pa + qb) > 0$. Then f_P is nice and $J(f_P,\phi) = f_P$ has a polynomial solution if and only if there exists a non-zero constant α such that

$$\prod_{i=1}^{s} (x^{b'} y^{-a'} + c_i) = x^{sb'} y^{-sa'} + \alpha.$$

If this is the case, $\phi = c\,(x^{sb'} y^{-sa'} + \alpha)$, $c^{-1} = \alpha(p - q)$ and $\psi = \phi f_P^{-1}$ satisfies $J(f_P,\psi) = 1$.

Proof. By Assertion 4.2, ϕ is divisible by $xy\prod_{i=1}^{s}(x^{b'}y^{-a'}+c_i)$. Note that $O,(1+sb',1-sa')$, $(p+rsb',q-rsa')$ are colinear by the assumption. Thus ϕ cannot have further divisor by (4.4) and $\phi = cxy\prod_{i=1}^{s}(x^{b'}y^{-a'}+c_i)$. Now we examine the condition $J(f_P,\phi)=f_P$. Write $f_P = x^{p-r}y^{q-r}\phi^r$. Then $J(f_P,\phi)=J(x^{p-r}y^{q-r},\phi)\phi^r$. Thus we must have $J(x^{p-r}y^{q-r},\phi)=x^{p-r}y^{q-r}$. As

$$J(x^{p-r}y^{q-r},x^{1+jb'}y^{1-ja'})\neq 0$$

for $j=1,\dots,s-1$ by the assumption (4.5), we must have

$$c_1+\cdots+c_s=\sum_{i\neq j}c_ic_j=\cdots=c_1c_2\cdots c_s\sum_{i=1}^{s}\frac{1}{c_i}=0.$$

This implies that $c_1,\dots,c_s$ are solutions of $t^s+\alpha=0$ for some $\alpha\in\mathbb{C}^*$ and $c^{-1}=\alpha(p-q)$. □

Example 4.1. Let $h(x,y)=y^{2n}(1+x^{3n}y^n)^{1+3n}$. This has a Jacobian solution $\psi = xy(1+x^{3n}y^n)/(-2nh(x,y))$.

5. Reformulation of results and examples

Now we consider the Jacobian problem from the viewpoint of existence problem of Jacobian partner polynomial $g(x,y)$ such that $J(f,g)=1$ when $f(x,y)$ is given. *We assume always that $f(x,y)$ is a non-linear strictly reduced polynomial.* We collect various non-existence assertions, translating the assertions for weighted homogeneous rational functions.

5.1. **Various non-existence assertions.**

Proposition 5.1. *If $f:\mathbb{C}^2\to\mathbb{C}$ has a critical point, there does not exist any partner g.*

This follows from condition $J(f,g)=f_xg_y-f_yg_x=1$ which implies that $f_x=f_y=0$ has no solutions. This condition cannot be read from the outside boundary.

For the weighted homogeneous polynomials, the non-existence for a partner g becomes trivial.

Lemma 5.1. *For any non-monomial weighted homogeneous polynomial $h(x,y)$ with a mixed weight vector $P=(a,b)$, there does not exist any weighted homogeneous polynomial $\phi(x,y)$ such that $J(h,\phi)=1$.*

Proof. Put $h(x,y)=f_P(x,y)$ and factorize it as

$$h(x,y)=cx^py^q\prod_{i=1}^{k}(x^by^{-a}+c_i)^{\nu_i},\quad k\geqslant 1.$$

Note that $\deg_P h = bq+ap$. First for the condition $J(h,\phi)=1$, it is necessary that h and ϕ must have a linear term x or y. Thus we may assume that $q=$

1, $p = 0$. Suppose that $\deg_P h \neq 0$. Then $b > 0$ and h, ϕ take the form

$$h(x,y) = y\prod_{i=1}^{k}(x^b y^{-a} + c_i)^{\nu_i}, \quad \phi = x\prod_{j=1}^{\ell}(x^b y^{-a} + d_j)^{\mu_j}.$$

Put $N = \sum_{i=1}^{k}\nu_i$, $M = \sum_{j=1}^{\ell}\mu_j$. Then the right ends of the supports of h and ϕ are given by $B = (bN, 1 - aN)$, $D = (1 + bM, -aM)$ and we see that B, D, O cannot be colinear as $\det(B, D) = -1 - bM + aN < 0$. This implies $J(h, \phi)$ cannot be a constant, as it contains the term $x^{b(M+N)}y^{-a(M+N)}$.

If $\deg_P h = 0$, this implies $b = 0$, $P = (1, 0)$ and $h(x, y) = h(y)$ and $\phi = x\,\phi_1(y)$. But the condition $J(h, \phi) = 1$ implies that $\phi_1(y)h'(y) = -1$ which is only possible if $h(x, y) = cy$, $\phi = c'x$. This is also not possible as $h(x, y)$ is not a monomial by the assumption. Thus $J(h, \phi) = 1$ is possible only if $k = 0$ and $\ell = 0$ which is a contradiction to the assumption that $h(x, y)$ is not a monomial.

□

Recall that $f(x, y)$ is *convenient* if $f(x, 0)$, $f(0, y)$ are non-constant polynomials.

Theorem 5.1. *Let $f(x, y)$ be a polynomial which is non-linear strictly reduced polynomial and assume that f is not convenient. Then there does not exist any polynomial $g(x, y)$ such that $J(f, g) = 1$.*

Proof. Suppose there exists $g(x, y)$ such that $J(f, g) = 1$. We may assume that $f(0, y) \equiv 0$, taking $f(x, y) - f(0, 0)$ if necessary. This implies that $f(x, y)$ does not have the monomial y. Thus $g(x, y)$ must have y with non-zero coefficient and $g(0, y)$ is a non-constant polynomial. Thus $\Delta(g; x, y)$ has a face Ξ touching ν_2-axis outside of the origin. Let P be the corresponding weight vector. Then either $J(f_P, g_P) = 0$ or $J(f_P, g_P) = 1$. The first equality is impossible. In fact $\deg_P f > 0$ is not possible, as $\Delta(f)$ cannot have a face which is similar to the support of $g_P(x, y)$, by the assumption on $f(0, y) \equiv 0$. Thus $\deg_P f = 0$. This implies f_P is a constant by Lemma 3.1, which is also not possible. The second equality $J(f_P, g_P) = 1$ is impossible by Lemma 5.1. □

Example 5.1. Let $f(x, y) = x + \psi(x^p y^q)$ where $\psi(t)$ is an arbitrary polynomial and $p \geqslant 2$. Then f has no critical point but f is not convenient. Thus there does not exist any polynomial $g(x, y)$ such that $J(f, g) = 1$.

The following is a generalization of Proposition 3.1.

Theorem 5.2. *Assume that f is non-linear, strictly reduced polynomial with $m_\infty(f) = 1$. Then there is no polynomial g such that $J(f, g) = 1$.*

Proof. If f, g are not similar, there exists an outside face Ξ with a weight vector P such that $f_P(x, y)$ and $g_P(x, y)$ are not similar. Then we must have $J(f_P, g_P) = 1$ which is impossible by Lemma 5.1. Therefore $\Delta(f; x, y)$ and $\Delta(g; x, y)$ are similar. By Lemma 3.1, we see that $\deg_P g/\deg_P f$ is independent of P. We fix a weight vector P which corresponds to a face Ξ of $\Delta(f; x, y)$. Put $s = m_\infty(g)$. Thus the assumption $m_\infty(f) = 1$ implies that $\deg_P g = s\deg_P f$. Thus we can find a constant $c_s \neq 0$ so that $\deg_P(g - c_s f^s) < \deg_P g$. As $J(f, g - cf^s) = 1$, we can apply an inductive argument so that $\exists c_s, \cdots, \exists c_1$ so that

$$g_0 := g - (c_s f^s + \cdots + c_1 f), \quad J(f, g_0) = 1.$$

Now $\deg_P g_0 < \deg_P f$. Thus it is impossible that g_0 and f are similar. □

Combining the criteria we have shown in previous sections, we get a following partial result (essentially stated in [3]) for the Jacobian conjecture.

Main Theorem. *Assume that $f(x,y)$ is a non-linear strictly reduced polynomial which has a Jacobian partner polynomial $g(x,y)$, $J(f,g) = 1$. Then the following conditions are necessary.*

(1) *$f : \mathbb{C}^2 \to \mathbb{C}$ has no critical point.*
(2) *$\Delta(f; x, y)$ is convenient.*
(3) *$\Delta(f; x, y)$ have no boundary obstructions.*
(4) *The outside boundary multiplicity* $\mathrm{mult}_\infty(f)$ *of f is strictly greater than 1.*

Proof. The third condition is due to Theorem 3.2. The last condition follows from Theorem 5.2. □

Remark 2. Under the above condition (3), $\Delta(f; x, y)$ has only exceptional faces and nice crossing faces or negatively crossing faces without boundary obstruction.

5.2. **Concluding remark and examples.** Our strategy to prove (or disprove) the Jacobian conjecture of two variable case is to show that there is no polynomial f which satisfies four conditions in Main Theorem.If there exist such polynomials, we have to look for "inside obstructions." So far, we do not have any examples of polynomials which satisfy four conditions but there are many polynomials which satisfy all conditions but (4).

Example 5.2. Consider

$$\begin{aligned} f_1(x,y) &:= y^2\,(1+xy)^4 + 3\,y\,(1+xy)^3 + (3 - 8/3\,y)\,(1+xy)^2 - 4\,xy + x \\ f_3(x,y) &:= (xy+1)^3\,y^3 + 2\,y^2\,(xy+1)^2 + 2\,y\,(xy+1)^2 + 2\,xy + x \\ f_4(x,y) &:= y^2\left(1+xy^2\right)^3 + x\left(y^4x+1\right)^2 - x^3y^8 + 3\,xy^2 \end{aligned}$$

$f_4(x,y)$ does not satisfy the condition (3). It has a non-crossing, non-exceptional face AB.

$f_1(x,y)$ and $f_3(x,y)$ satisfy (1), (2) and (3) of Main Theorem but they do not satisfy (4). In both cases, the faces AB are exceptional and the faces BC are crossing without obstruction. We can see that there are birational solutions:

$$J(x(1+xy^2)^3, y(1+xy^2)^{-2}) = 1, \quad J(x(1+xy^3)^2, y(1+xy^3)^{-1}) = 1.$$

P. Cassou-Noguès informed us that she has an example of a polynomial satisfying four conditions (1) $\sim$ (4).

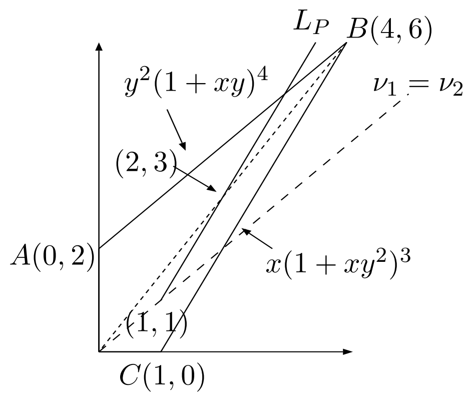

FIGURE 3. Newton diagrams $\Delta(f_1)$

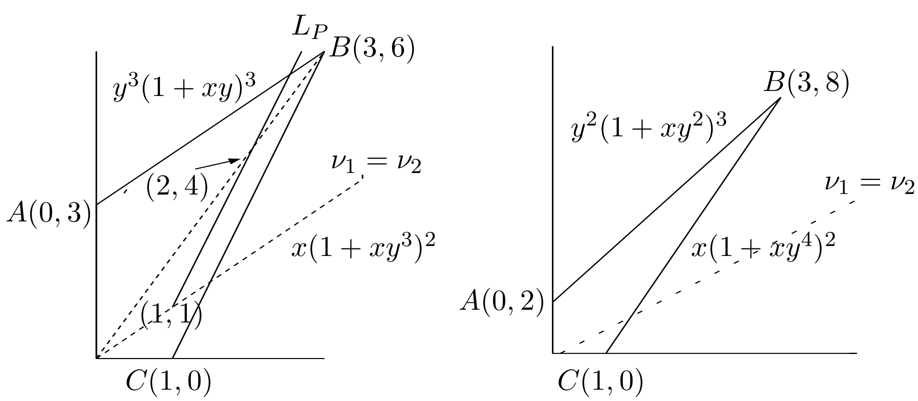

FIGURE 4. Newton diagrams $\Delta(f_3)$ (left), $\Delta(f_4)$(right)

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