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DYNAMICS OF PREDATOR-PREY POPULATION WITH MODIFIED LESLIE-GOWER AND HOLLING-TYPE II SCHEMES

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Abstract. In this paper, we investigate a predator-prey population modeled by a system of differential equations modified Leslie-Gower and Holling-Type II schemes with time-dependent parameters. We establish a sufficient criterion posed on the behavior at infinity of coefficients for the permanence of systems, globally asymptotic stability of solutions. In the case where the coefficients of equations are periodic functions with a same period, it is proved that there exists a unique periodic orbit which attracts every solution starting in $\operatorname{int}\mathbb{R}^2_+$.

1. Introduction

In mathematical ecology, one of the popular models is a model consisting of two difference species where one of them provides food to the other. The interaction between population in this type is very universal in nature and is called "Prey-Predator" relation. The predator-prey system plays an important role both in theory and practice and has been studied by many authors. Recently, there are many works revealing the dynamics of prey-predator systems for a so-called semi-ratio-dependent class with functional responses. This class consists of systems which are described by the equation

$$
\begin{aligned}
x' &= x[a - bx] - c(x)y \\
y' &= y\big[d - e\frac{y}{x}\big],
\end{aligned}
\tag{1.1}
$$

where x and y stand for the quantity (or density) of the prey and the predator, respectively. The function $c(x)$ is called predator functional response. The biological signification of a, d, e and a/b has been explained in [4]. The predator consumes the prey according to the functional response $c(x)$ and carrying capacity $x(t)/e$ proportional to the population size of prey (or prey abundance).

The prey-predator equation (1.1) with functional response was first proposed by Leslie (1960). Since then, there has been much interest both in theory and application of this model. Based on experiments, Holling [13] suggested some

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kinds of functional responses to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. Depending on the form of the functional responses, these models are classified into five types (see [4]). If $c(x) = mx$, we have type 1. These models had been researched by P.H. Leslie [16], C. S. Holling [13], S. B. Hsu, T. W. Huang [14]. The functional response $c(x)$ is of type 2 if $c(x) = \frac{m}{A+x}$. There are some papers studying about stability, furcation behavior and so on of these models. For example, we can refer to J. B. Collings [9], A. A. Berryman [7], K. S. Cheng, S. B. Hsu, S. S. Lin [8], S. B. Hsu, S. P. Hubbell, P. Waltman [18] etc. When $c(x) = \frac{mx^n}{A+x^n}\quad n \geqslant 2$, we called it type 3. It was suggested by the biologist Holling [13]. The general form of functional response of this type was introduced by Kazarinov and van den Driessche in [15]. If $c(x) = \frac{mx^2}{(A+x)(B+x)}$, it is concerned with type 4. This model can be seen in J. B. Collings [9], J. Tanner [17] and so forth. The type 5 is also called Ivlev's functional response. In this case, $c(x) = m(1 - e^{-Ax})$.

We begin by analyzing the model that M. A. Aziz. Alaoui and M. D. Okiye have dealt with in [1], that is the model of type 2

$$\begin{aligned} \dot{x} &= x\Big[r_1 - b_1 x - \frac{a_1}{x+k_1} y\Big], \\ \dot{y} &= y\Big[r_2 - \frac{a_2}{x+k_2} y\Big], \end{aligned} \tag{1.2}$$

with $x(0) \geqslant 0$ and $y(0) \geqslant 0$, where $r_1,\ a_1,\ b_1,\ k_1,\ r_2,\ a_2$ and k_2 are the model's parameters, assuming to be positive. It is proved in [1] that the system (1.2) is ultimately bounded with respect to $\mathbb{R}^2_+$. In considering globally asymptotical stability, the authors had observed

$$L_1 = \frac{1}{4a_2 b_1}\big(a_2 r_1 (r_1 + 4) + (r_2 + 1)^2 (r_1 + b_1 k_2)\big)$$

and have claimed that under the assumptions

(1.a) $L_1 < \frac{r_1 k_1}{2a_1}$,
(1.b) $k_1 < 2k_2$,
(1.c) $4(r_1 + b_1 k_1) < a_1$,

the interior equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable (see [1, Theorem 6]). The idea to make this assumption is interesting because it is rather simple and it can be verified by direct calculation. However, in our opinion, there is no set of parameters of (1.2) which satisfies the conditions (1.a)$-$(1.c). Indeed, from (1.a) and (1.c) we have

$$\frac{a_2 r_1}{a_2 b_1} = \frac{r_1}{b_1} < L_1 < \frac{r_1 k_1}{2a_1} < \frac{r_1 k_1}{8b_1 k_1} < \frac{r_1}{8b_1}.$$

This relation is impossible, hence the set of parameters satisfying (1.a)$-$(1.c) is empty.

Therefore, in this paper, we want to improve the above conditions to study the dynamic of differential equations modeling a predator-prey system (1.2) in the case where its coefficients vary in time. We give some reasonable restriction on the model and determine conditions that ensure the permanence and the globally asymptotic stability of the solutions of system. This conditions are similar to what is done in [1] and [4] but more realistic and it is easy to verify by simple direct calculation.

The paper is organized as follows: In section 2, we give a condition to ensure the permanence of solutions of (1.2). The condition is imposed by the behavior of the coefficients at infinity meanwhile most previous works put likely the conditions on whole trajectory of coefficients. The section 3 deals with the asymptotic stability of the solutions of (1.2). The last section, section 4, studies the existence of a periodic solution when all parameters are periodic functions. Since we are unable to construct an invariant set with our assumption, we have to use a difference technique to show the existence of such a solution.

2. Permanence of the solutions

Consider a time-varying predator-prey system:

$$\begin{aligned} \dot{x} &= x\Big[r_1(t) - b_1(t)x - \frac{a_1(t)}{x + k_1(t)}y\Big] \\ \dot{y} &= y\Big[r_2(t) - \frac{a_2(t)}{x + k_2(t)}y\Big], \end{aligned} \tag{2.1}$$

where $r_i(t),\ a_i(t),\ i = 1,2,\ b_1(t),\ k_1(t)$ are continuous functions, defined on $\mathbb{R}$, bounded above and below by positive constants; $k_2(t)$ is supposed to be a nonnegative function, bounded above by a positive constant.

By the uniqueness of the solution of (2.1), it is easy to see that both the nonnegative cone $\mathbb{R}^2_+ = \{(x,y) : x \geqslant 0, y \geqslant 0\}$ and positive cone $\mathrm{int}\mathbb{R}^2_+ = \{(x,y) : x > 0, y > 0\}$ are invariant with respect to System (2.1). This means that if $x(t_0) \geqslant 0,\ y(t_0) \geqslant 0$ (resp. $x(t_0) > 0,\ y(t_0) > 0$) then $x(t) \geqslant 0,\ y(t) \geqslant 0$ (resp. $x(t_0) > 0,\ y(t_0) > 0$ for all $t > t_0$).

Definition 2.1. System (2.1) is said to be permanent if there exists a compact set $\mathcal{A} \subset \mathrm{int}\mathbb{R}^2_+$ such that for every solution $(x(t), y(t))$ of (2.1) with positive initial value $(x(t_0), y(t_0)) \in \mathrm{int}\,\mathbb{R}^2_+$, there exists a $T > t_0$ such that $(x(t), y(t)) \in \mathcal{A}$ for all $t \geqslant T$.

In order to construct a set $\mathcal{A}$ as in the definition 2.1 we denote

$$\begin{aligned} M_1^* &:= \limsup_{t\to\infty} \frac{r_1(t)}{b_1(t)},\quad M_2^* := \limsup_{t\to\infty} \frac{r_2(t)(M_1^* + k_2(t))}{a_2(t)}, \\ m_1^* &:= \liminf_{t\to\infty} \frac{1}{b_1(t)}[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2^*], \\ m_2^* &:= \liminf_{t\to\infty} \frac{r_2(t)(m_1^* + k_2(t))}{a_2(t)}. \end{aligned}$$

Since all coefficients are bounded above and $b_1,\ a_2$ are bounded below by positive constants, it follows that $M_1^* < \infty;\ M_2^* < \infty$.

Throughout this paper, we assume that

Hypothesis 2.1.

$$\text{(A1)} \qquad \liminf_{t\to+\infty}[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2^*] > 0.$$

With this assumption, we see that $m_1^* > 0$ and $m_2^* > 0$.

This assumption is slightly improved from the condition $(A4)$ in [4] in the case where $c(t,x) = \frac{a_1(t)}{x+k_1(t)}x$ and $k_2(t) = 0$. Indeed, let $r_{iM} = \sup_{t\in\mathbb{R}} r_i(t)$, $r_{iL} = \inf_{t\in\mathbb{R}} r_i(t)$ for $i = 1, 2$. The conditions $(A1)$, $(A2)$ in [4] are obviously satisfied. The condition $(A3)$ ($\exists\, C_0 > 0$, such that $c(t,x) \leqslant C_0 x$ for any $t \in \mathbb{R}, x > 0$) becomes $c(t,x) = \frac{a_1(t)}{x+k_1(t)}x \leqslant C_0 x \;\forall\, t \geqslant t_0 \Leftrightarrow C_0 \geqslant \sup_{t\geqslant t_0} \frac{a_1(t)}{k_1(t)}$. Let the condition $(A4)$ in [4] hold, i.e., $r_{1L} - C_0 M_2 > 0$ where $M_1 > \frac{r_{1M}}{b_{1L}}$ and $M_2 > \frac{r_{2M}}{e_M}M_1$. It is easy to see that

$$\begin{aligned}\liminf_{t\to\infty}\Big[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2\Big] &\geqslant \liminf_{|t|\to\infty} r_1(t) - \sup_{t\geqslant t_0}\frac{a_1(t)}{k_1(t)}M_2\\ &\geqslant r_{1L} - C_0M_2.\end{aligned}$$

Thus, $r_{1L} - C_0M_2 > 0$ implies $\liminf_{t\to\infty}\Big[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2\Big] > 0$. Further, $M_2 > M_2^*$. Hence, Hypothesis 2.1 holds.

On the other hand, the following example shows that condition $(A1)$ is in fact weaker than $(A4)$ in [4]. Indeed,

Example 1. *Consider the following system*

$$\begin{aligned}\dot{x}(t) &= x(t)\Big[\frac{3}{2}(2+\cos t) - (2+\cos t)x(t) - \frac{(2+\cos t)}{6(x(t)+1)}y(t)\Big],\\ \dot{y}(t) &= y(t)\Big[2+\cos t - \frac{\cos t+2}{x}y(t)\Big].\end{aligned}$$

Here we have

$$\begin{aligned}&r_1(t) = \frac{3}{2}(2+\cos t), \quad r_2(t) = 2+\cos t, \quad a_1(t) = \frac{1}{6}(2+\cos t),\\ &b_1(t) = 2+\cos t, \quad a_2(t) = 2+\cos t, \quad k_1(t) = 1, \quad k_2(t) = 0.\end{aligned}$$

Therefore, $M_1^* = \limsup_{t\to\infty}\frac{r_1(t)}{b_1(t)} = 1.5$, $M_2^* = \limsup_{t\to\infty}\frac{r_2(t)(M_1^*+k_2(t))}{a_2(t)} = 1.5$, and $\liminf_{t\to\infty}\frac{1}{b_1(t)}[\, r_1(t) - \frac{a_1(t)}{k_1(t)}M_2^*] = \frac{5}{4}$. Hence, $(A1)$ is satisfied. But neither condition $(A4)$ nor condition $(A7)$ in [4] hold for the function $c(t,x) = \frac{(2+\cos t)x}{6(x+1)}$.

Since the functions $a_1(t)$ and $k_1(t)$ are bounded above and below by positive constants, from the condition $(A1)$, we can choose positive numbers $M_1 > M_1^*$ and $M_2 > M_2^*$ such that

$$\text{(2.2)} \qquad M_2 > \limsup_{t\to\infty}\frac{r_2(t)(M_1+k_2(t))}{a_2(t)}$$

and

$$\liminf_{t\to\infty}[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2] > 0. \tag{2.3}$$

Let m_1 and m_2 be real numbers satisfying

$$0 < m_1 < \liminf_{t\to\infty} \frac{1}{b_1(t)}[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2], \tag{2.4}$$

$$0 < m_2 < \liminf_{t\to\infty} \frac{r_2(t)(m_1 + k_2(t)}{a_2(t)} \cdot \tag{2.5}$$

We need some following lemmas.

Lemma 2.1. *Let $G(t)$ and $F(t)$ be two differentiable functions defined on $(0,\infty)$ such that $\lim_{t\to+\infty} G(t) = \lim_{t\to+\infty} F(t) = +\infty$, then*

$$\limsup_{t\to+\infty} \frac{G(t)}{F(t)} \leqslant \limsup_{t\to+\infty} \frac{G'(t)}{F'(t)}; \quad \liminf_{t\to+\infty} \frac{G(t)}{F(t)} \geqslant \liminf_{t\to+\infty} \frac{G'(t)}{F'(t)}.$$

Proof. See [2, Lemma 2]. □

Lemma 2.2. *Let h be a real number and f be a nonnegative function defined on $[h, +\infty)$, uniformly continuous on $[h, +\infty)$ and f is integrable on $[h, +\infty)$ in Riemann sense. Then it holds $\lim_{t\to+\infty} f(t) = 0$.*

Proof. We think that this lemma has been proved somewhere but in order to complete our paper, we introduce the following proof. Assume that

$$\limsup_{t\to+\infty} f(t) = 4\alpha > 0.$$

By definition, there exists a sequence $(t_n)_{n\in\mathbb{N}} \uparrow \infty$ such that $f(t_n) > 2\alpha$. Since f is uniformly continuous on $[h, +\infty)$, there is a $\delta > 0$ such that $|f(t) - f(t')| < \alpha$ if $|t - t'| < \delta$. Therefore, $f(t) > \alpha$ for any $t \in [t_n, t_n + \delta]$ and $n \in \mathbb{N}$. Without loss of generality, we can suppose that $t_n + \delta < t_{n+1}$ for all $n \in \mathbb{N}$. Hence

$$\int_h^{+\infty} f(t)\,dt \geqslant \sum_{n=1}^{\infty} \int_{t_n}^{t_n+\delta} f(t)\,dt \geqslant \sum_{n=1}^{\infty} \int_{t_n}^{t_n+\delta} \alpha\,dt = \infty.$$

This contradicts the hypothesis. Lemma 2.2 is proved. □

Theorem 2.1. *Suppose the assumption (A1) holds. Then System (2.1) is permanent.*

Proof. We consider the set

$$\mathcal{A} := \{(x, y) \in \mathbb{R}_+^2 : m_1 \leqslant x \leqslant M_1,\ m_2 \leqslant y \leqslant M_2\} \tag{2.6}$$

Let $(x(t), y(t))$ be the solution of (2.1) with a positive initial value $(x(t_0), y(t_0)) \in \operatorname{int}\mathbb{R}_+^2$. From the first equation of (2.1) we have $\dot{x} < x[r_1(t) - b_1(t)x]$. Hence, by comparison theorem, it follows that

$$x(t) \leqslant \frac{x(t_0)e^{A(t)}}{1 + x(t_0)\int\limits_{t_0}^{t} b_1(s)e^{A(s)}ds},$$

where $A(t) = \int\limits_{t_0}^{t} r_1(s)ds$. Since $r_1(t),\ \ b_1(t)$ are bounded below by positive constants,

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int\limits_{t_0}^{t} b_1(s)e^{A(s)}ds = \infty.$$

Therefore, by using Lemma 2.1 we have

$$\limsup_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} \frac{r_1(t)}{b_1(t)}.$$

Noting that $M_1 > M_1^* = \limsup\limits_{t\to\infty} \frac{r_1(t)}{b_1(t)}$, this implies the existence of a constant $t_1 \geqslant t_0$ such that

$$x(t) < M_1 \ \text{ for all } \ t \geqslant t_1. \tag{2.7}$$

Substituting the estimate (2.7) into the second equation of (2.1) we obtain

$$\dot{y} \leqslant y\big[r_2(t) - \frac{a_2(t)}{M_1 + k_2(t)}y\big] \quad \text{for any } t \geqslant t_1.$$

Therefore

$$y(t) \leqslant \frac{y(t_1)e^{B(t)}}{1 + y(t_1)\int\limits_{t_1}^{t} \frac{a_2(s)}{M_1+k_2(s)}e^{B(s)}ds} \quad \text{where } \ B(t) = \int\limits_{t_1}^{t} r_2(s)ds.$$

By virtue of the boundedness of $a_2(t), r_2(t), k_2(t)$ by positive constants we get

$$\lim_{t\to\infty} B(t) = \lim_{t\to\infty} \int\limits_{t_1}^{t} \frac{a_2(s)}{M_1 + k_2(s)}e^{B(s)}ds = \infty.$$

Using again Lemma 2.1 we get

$$\limsup_{t\to\infty} y(t) \leqslant \limsup_{t\to\infty} \frac{r_2(t)(M_1 + k_2(t))}{a_2(t)}.$$

Since $M_2 > \limsup\limits_{t\to\infty} \frac{r_2(t)(M_1+k_2(t))}{a_2(t)}$, there exists $t_2 \geqslant t_1$ such that

$$y(t) < M_2 \quad \text{for all} \quad t \geqslant t_2. \tag{2.8}$$

Substituting (2.8) into the first equation of (2.1) we obtain

$$\dot{x} \geqslant \big[r_1(t) - \frac{a_1(t)}{k_1(t)}M_2 - b_1(t)x\big]x \quad \text{for any } t \geqslant t_2.$$

Or

$$x(t) \geqslant \frac{x(t_2)e^{C(t)}}{1 + x(t_2)\int\limits_{t_2}^{t} b_1(s)e^{C(s)}ds}, \ t \geqslant t_2,$$

where $C(t) = \int\limits_{t_2}^{t}[r_1(s) - \frac{a_1(s)}{k_1(s)}M_2]ds$. Hence

$$\liminf_{t\to\infty} x(t) \geqslant \liminf_{t\to\infty} \frac{(r_1(t) - \frac{a_1(t)}{k_1(t)}M_2)}{b_1(t)}.$$

Thus from (2.4) there exists $t_3 \geqslant t_2$ such that

$$x(t) > m_1 \quad \text{for all} \quad t \geqslant t_3. \tag{2.9}$$

On the other hand, from the second equation of (2.1) we have

$$\dot{y} \geqslant \big[r_2(t) - \frac{a_2(t)}{m_1 + k_2(t)}y\big]y, \quad \text{for any } t \geqslant t_3,$$

which implies

$$y(t) \geqslant \frac{y(t_3)\cdot e^{D(t)}}{1 + y(t_3)\int\limits_{t_3}^{t} \frac{a_2(s)}{m_1+k_2(s)}ds}, \ \text{ where } \ D(t) = \int\limits_{t_3}^{t} r_2(s)ds.$$

Therefore

$$\liminf_{t\to\infty} y(t) \geqslant \liminf_{t\to\infty} \frac{r_2(t)(m_1 + k_2(t))}{a_2(t)}.$$

Hence by (2.5) there exists $t_4 \geqslant t_3$ such that

$$y(t) > m_2 \quad \text{for all} \quad t \geqslant t_4. \tag{2.10}$$

Put $T = \max\{t_1, t_2, t_3, t_4\}$, and by combining (2.7)$-$(2.10) we see that

$$(x(t), y(t)) \in \operatorname{int}\mathcal{A} \tag{2.11}$$

for all $t \geqslant T$. This means that System (2.1) is permanent. The theorem is proved. □

3. Asymptotic stability

We now study the stability of the solutions of (2.1).

Definition 3.1. System (2.1) is said to be globally asymptotically stable if any two solutions $(x_i(t), y_i(t))$, $i = 1, 2$ of (2.1) with positive initial values $(x_i(t_0), y_i(t_0)) \in \mathbb{R}^2_+$ have the property

$$\lim_{t\to+\infty} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) = 0.$$

Theorem 3.1. *Assume that (A1) holds and the following conditions hold*

$$\liminf_{t\to\infty}\Big[b_1(t)-\Big(\frac{a_1(t)}{(m_1^*+k_1(t))^2}+\frac{a_2(t)}{(m_1^*+k_2(t))^2}\Big)M_2^*\Big]>0, \tag{A2}$$

$$\liminf_{t\to\infty}\Big[\frac{a_2(t)}{M_1^*+k_2(t)}-\frac{a_1(t)}{m_1^*+k_1(t)}\Big]>0. \tag{A3}$$

Then System (2.1) *is globally asymptotically stable.*

Proof. Let $(x_i(t), y_i(t))$, $i = 1, 2$, be two arbitrary solutions of (2.1) starting respectively from $(x_i(t_0), y_i(t_0)) \in \mathbb{R}_+^2$ at t_0. For any $\epsilon > 0$, let

$$M_1 = M_1^* + \varepsilon; M_2 = \limsup_{t\to\infty}\frac{r_2(t)(M_1+k_2(t))}{a_2(t)} > M_2^*,$$
$$m_1 = \liminf_{t\to\infty}\frac{1}{b_1(t)}[r_1(t)-\frac{a_1(t)}{k_1(t)}M_2] < m_1^*; m_2 = \liminf_{t\to\infty}\frac{r_2(t)(m_1+k_2(t))}{a_2(t)} < m_2^*.$$

Since the parameters of System (2.1) are bounded above and below by positive constants, we see that $\lim_{\varepsilon\to 0} M_1 = M_1^*; \lim_{\varepsilon\to 0} M_2 = M_2^*; \lim_{\varepsilon\to 0} m_1 = m_1^*; \lim_{\varepsilon\to 0} m_2 = m_2^*$. Therefore, from the assumptions (A2)-(A3), we can choose $\varepsilon > 0$ small enough such that

$$\alpha := \liminf_{t\to\infty}[b_1(t)-(\frac{a_1(t)}{(m_1+k_1(t))^2}+\frac{a_2(t)}{(m_1+k_2(t))^2})M_2] > 0, \tag{3.1}$$

$$\beta := \liminf_{t\to\infty}[\frac{a_2(t)}{M_1+k_2(t)}-\frac{a_1(t)}{m_1+k_1(t)}] > 0. \tag{3.2}$$

Thus we have shown that it is possible to choose the numbers M_1, M_2, m_1, m_2 such that (2.2), (2.3),(2.4), (2.5) and (3.1), (3.2) hold simultaneously. From Theorem 2.1, with the set $\mathcal{A}$ defined by (2.6), there exists a $T \geqslant t_0$ such that the following relations hold

- $(x_i(t), y_i(t)) \in \operatorname{int}\mathcal{A}, i = 1, 2$ for all $t \geqslant T$ (see (2.11),

(3.3)
- $b_1(t)-(\frac{a_1(t)}{(m_1+k_1(t))^2}+\frac{a_2(t)}{(m_1+k_2(t))^2})M_2 > \frac{\alpha}{2}$,

(3.4)
- $\frac{a_2(t)}{M_1+k_2(t)}-\frac{a_1(t)}{m_1+k_1(t)} > \frac{\beta}{2}$,

for all $t \geqslant T$.

Consider a Lyapunov function defined by

$$V(t) = |\ln x_1(t) - \ln x_2(t)| + |\ln y_1(t) - \ln y_2(t)|, \quad t \geqslant t_0.$$

A direct calculation of the derivative $D^+V(t)$ of $V(t)$ along the solution of (2.1) leads to

$$\begin{aligned}&D^+\left|\ln x_1(t)-\ln x_2(t)\right|\\=&\Big[-b_1(t)x_1(t)-\frac{a_1(t)y_1(t)}{x_1(t)+k_1(t)}+b_1(t)x_2(t)+\frac{a_1(t)y_2(t)}{x_2(t)+k_1(t)}\Big]\operatorname{sgn}(x_1(t)-x_2(t)).\end{aligned}$$

Applying the finite increment formula of Lagrange to the function $f(x,y)=\frac{y}{x+k_1}$ we obtain

$$\begin{aligned}&\frac{y_1(t)}{x_1(t)+k_1(t)}-\frac{y_2(t)}{x_2(t)+k_2(t)}\\=&\frac{-\eta_1(t)}{(\xi_1(t)+k_1(t))^2}(x_1(t)-x_2(t))+\frac{1}{\xi_1(t)+k_1(t))}(y_1(t)-y_2(t)),\end{aligned}$$

where $\xi_1(t)$ is a certain point on the interval $(x_1(t),x_2(t))$ and $\eta_1(t)\in(y_1(t),y_2(t))$. Therefore, for $t\geqslant T$, we have

$$\begin{aligned}&D^+\left|\ln x_1(t)-\ln x_2(t)\right|\\=&-b_1(t)\left|x_1(t)-x_2(t)\right|-a_1(t)\big[\frac{-\eta_1(t)}{(\xi_1(t)+k_1(t))^2}(x_1(t)-x_2(t))\\&+\frac{1}{\xi_1(t)+k_1(t))}(y_1(t)-y_2(t))\big]\operatorname{sgn}(x_1(t)-x_2(t))\\\leqslant&-b_1(t)\left|x_1(t)-x_2(t)\right|+\frac{a_1(t)}{(m_1+k_1(t))^2}M_2|x_1(t)-x_2(t)|\\&+\frac{a_1(t)}{m_1+k_1(t)}|y_1(t)-y_2(t)|.\end{aligned}$$

Similarly,

$$\begin{aligned}&D^+\left|\ln y_1(t)-\ln y_2(t)\right|\\=&\big[-\frac{a_2(t)}{x_1(t)+k_2(t)}y_1(t)+\frac{a_2(t)}{x_2(t)+k_2(t)}y_2(t)\big]\operatorname{sgn}(y_1(t)-y_2(t))\\=&\Big[\frac{a_2(t)y_2(t)}{x_2(t)+k_2(t)}-\frac{a_2(t)y_1(t)}{x_1(t)+k_2(t)}\Big]\operatorname{sgn}(y_1(t)-y_2(t))\\=&\big[-\frac{a_2(t)\eta_2(t)}{(\xi_2(t)+k_2(t))^2}(x_2(t)-x_1(t))\\&+\frac{a_2(t)}{\xi_2(t)+k_2(t)}(y_2(t)-y_1(t))\big]\operatorname{sgn}(y_1(t)-y_2(t))\\\leqslant&\frac{a_2(t)M_2}{(m_1+k_2(t))^2}|x_1(t)-x_2(t)|-\frac{a_2(t)}{M_1+k_2(t)}|y_1(t)-y_2(t)|\end{aligned}$$

with $\xi_2(t)\in(x_1(t),x_2(t))$ and $\eta_2(t))\in(y_1(t),y_2(t))$.

Thus, for any $t > t_0$, we have

$$D^+V(t) \leqslant -\Big[b_1(t) - \frac{a_1(t)}{(m_1+k_1(t))^2}M_2 - \frac{a_2(t)}{(m_1+k_2)^2}M_2\Big]|x_1(t)-x_2(t)| - \Big[\frac{a_2(t)}{M_1+k_2(t)} - \frac{a_1(t)}{(m_1+k_2)}\Big]|y_1(t)-y_2(t)|.$$

Therefore, by using (3.3) and (3.4), we obtain

$$D^+V(t) \leqslant -\gamma[|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|]$$

with $2\gamma = \min\{\alpha, \beta\} > 0$ and $t \geqslant T$. Hence

$$V(t) - V(T) \leqslant -\gamma \int_T^t (|x_1(s)-x_2(s)| + |y_1(s)-y_2(s)|)\, ds, \tag{3.5}$$

which implies that

$$\int_T^t (|x_1(s)-x_2(s)| + |y_1(s)-y_2(s)|)\, ds < \frac{V(T)}{\gamma} < \infty.$$

This means that $|x_1(s)-x_2(s)| + |y_1(s)-y_2(s)| \in L^1([T,+\infty))$. On the other hand, for $t \in [T,+\infty)$ we have $(x_1(t), y_1(t)) \in \mathcal{A}$ and $(x_2(t), y_2(t)) \in \mathcal{A}$, i.e., $m_1 \leqslant x_i(t) \leqslant M_1$ and $m_2 \leqslant y_i(t) \leqslant M_2$ for $i = 1, 2$. Thus, the derivative of $(x_i(s), y_i(s))$, $i = 1, 2$, is bounded on $[T,+\infty)$. Therefore, $|x_1(s)-x_2(s)| + |y_1(s)-y_2(s)|$ is uniformly continuous on $[T,+\infty)$. By Lemma 2.2, we get

$$\lim_{t\to+\infty} |x_1(s)-x_2(s)| + |y_1(s)-y_2(s)| = 0.$$

The proof is complete. □

4. Existence of periodic solutions

Theorem 4.1 (Existence of periodic solution). *Suppose that* $(A1)-(A3)$ *hold. Further, suppose that parameters in System* (2.1) *are periodic functions in* t *with period* ω. *Then System* (2.1) *has a unique positive* ω*-periodic solution which is globally stable.*

Proof. Denote $z(t, t_0, x, y) = (x(t, t_0, x, y), y(t, t_0, x, y))$. For all $(x, y) \in \mathcal{A}$, put

$$T(x, y) = \inf\{s : z(\tau, t_0, x, y) \in \operatorname{int}\mathcal{A} \ \ \forall\, \tau > s\}.$$

By virtue of Theorem 2.1, $T(x, y) < \infty$ for any $(x, y) \in \mathbb{R}^2_+$ and $z(T(x, y), t_0, x, y) \in \mathcal{A}$. Further, by the continuous dependence of solutions on the initial data, the function $T(x, y)$ is continuous in (x, y). Therefore

$$T_1 = \sup\{T(x, y) : (x, y) \in \mathcal{A}\} < \infty.$$

Let $n \in \mathbb{N}$ such that $t_0 + (n-1)\omega \leqslant T_1 < t_0 + n\omega$. Put $T^* = t_0 + n\omega$. Consider the mapping

$$\begin{aligned} \Phi : \mathcal{A} &\longrightarrow \mathcal{A} \\ (x,y) &\longmapsto \Phi(x,y) = z(T^*, t_0, x, y). \end{aligned}$$

It is easy to see that Φ is a continuous function. Since $\mathcal{A}$ is a compact, convex subset of $\mathbb{R}^2$, by Brouwer fixed point theorem Φ has at least one fixed point in $\mathcal{A}$, namely (x^*, y^*). We show that $z(t, t_0, x^*, y^*)$ is a bounded and periodic solution of (2.1) with period $n\omega$. Indeed, for any $t > t_0$, $z(t + n\omega, t_0, x^*, y^*) = z(t + n\omega, T^*, z(T^*, t_0, x^*, y^*)) = z(t + n\omega, T^*, x^*, y^*))$. Since the parameters in System (2.1) are periodic with period ω, we have

$$z(t + n\omega, T^*, x^*, y^*)) = z(t + n\omega, t_0 + n\omega, x^*, y^*)) = z(t, t_0, x^*, y^*)).$$

This means that $z(t, t_0, x^*, y^*)$ is a periodic function with period $n\omega$.

We now use the mapping

$$\begin{aligned} \Phi_1 : \mathcal{A} &\longrightarrow \mathcal{A} \\ (x,y) &\longmapsto \Phi_1(x,y) = z(T^* + \omega, t_0, x, y). \end{aligned}$$

By a similar argument as above, we see that there is a fixed point $(\overline{x}, \overline{y}) \in \mathcal{A}$ of the mapping Φ_1 and the solution $z(t, t_0, \overline{x}, \overline{y})$ is periodic with the period $(n+1)\omega$. For any $t > t_0$, from Theorem 3.1 we have

$$\begin{aligned} 0 &= \lim_{\substack{m\to\infty \\ m\in\mathbb{N}}} (z(t + mn(n+1)\omega, t_0, x^*, y^*) - z(t + mn(n+1)\omega, t_0, \overline{x}, \overline{y})) \\ &= z(t, t_0, x^*, y^*) - z(t, t_0, \overline{x}, \overline{y}). \end{aligned}$$

This implies that $z(t, t_0, x^*, y^*) = z(t, t_0, \overline{x}, \overline{y})$. Thus $z(t, t_0, x^*, y^*)$ is periodic with the period $n\omega$ and with the period $(n+1)\omega$. Hence, it is periodic with the period ω. The uniqueness and global stability is deduced from Theorem 3.1. The theorem is proved. $\square$

It is not difficult to show that the example 1 satisfies the conditions from (A1) to (A3). We give another example.

Example 2. Consider the following system

$$\begin{cases} \dot{x}(t) &= x(t)\big[1 - x(t) - \frac{0{,}2\sin 2\pi t + 0{,}25}{x(t)+1} y(t)\big], \\ \dot{y}(t) &= y(t)\big[0{,}4 - \frac{\cos 2\pi t + 2}{x+1} y(t)\big]. \end{cases} \tag{4.1}$$

We have $M_1^* = 1$; $M_2^* = 0{,}8$ $m_1^* = 0{,}64$; $m_2^* \approx 0{,}2186667$. Thus, it is easy to check the conditions $(A_1) - (A_3)$ to be satisfied. Moreover, all parameters are periodic functions. We illustrate the behavior of numerical solutions of System (2.1) by Fig. 1

(A) shows the behavior of two solutions of the system (4.1) for two initial values: the dash line shows the trajectory of the solution $(x_1(t), y_1(t))$ with $x_1(0) = 0.97, y_1(0) = .42$ and the solid line corresponds to the solution $(x_2(t), y_2(t))$ with $x_2(0) = 0.935, y_2(0) = 0.36$.

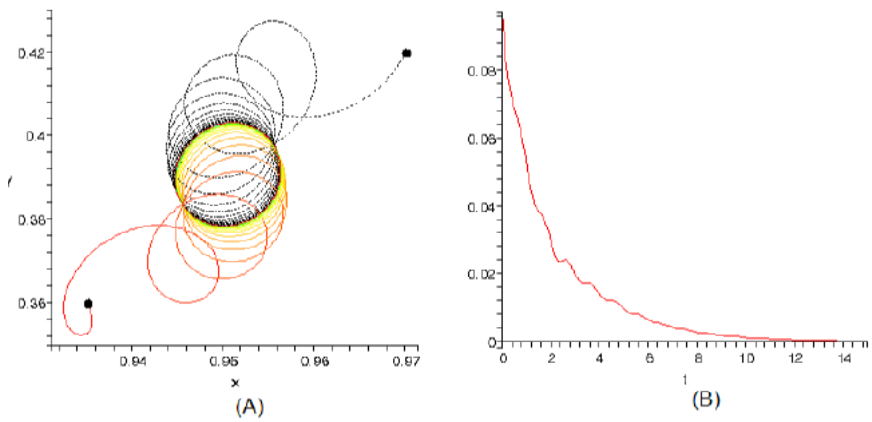

Fig. 1.

(B) shows the graph of $|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|$. We see that $\lim_{t\to\infty}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|=0$. That illustrates the result in Theorem 3.1.

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