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MINIMIZING POLYNOMIAL FUNCTIONS

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ABSTRACT. We effectively compute a finite set containing all critical values and the infimum value of a real multivariate polynomial function. Besides, some relations between Newton polytopes and bounded below polynomials are also established.

1. INTRODUCTION

This note is concerned with the following basic problem. Given a multivariate polynomial function $f \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, x_2, \ldots, x_n]$ which is bounded from below on $\mathbb{R}^n$, find the global infimum

$$f^* := \inf\{f(x) \mid x \in \mathbb{R}^n\}.$$

If the polynomial f attains its infimum f^*, then it is well known that this problem can be solved by methods of Computational Algebra. In fact, consider the ideal generated by the partial derivatives of f

$$I := \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \subseteq \mathbb{R}[\mathbf{x}].$$

The zeros of the ideal I in complex n-space $\mathbb{C}^n$ are the *critical points* of f. Their number (counted with multiplicity) is the dimension over $\mathbb{R}$ of the residue ring $\mathbb{R}[\mathbf{x}]/I$:

$$\mu_f := \dim_{\mathbb{R}} \mathbb{R}[\mathbf{x}]/I = \#V_{\mathbb{C}}(I),$$

where $V_{\mathbb{C}}(I)$ is the set of critical points of f.

Consider the subset of *real critical points* $V_{\mathbb{R}}(I) := V_{\mathbb{C}}(I) \cap \mathbb{R}^n$. Then

$$\begin{aligned} f^* &= \min\{f(x) \mid x \in V_{\mathbb{R}}(I)\} \\ &= \min\{t \mid t \text{ is a critical value of } f\}. \end{aligned}$$

There are at least three techniques to compute the value f^* : Gröbner bases and eigenvalues, Resultants and discriminants, and Homotopy methods. Exact methods can be found in Hägglöf et al. (1995), Li (1997), Uteshev and Cherkasov (1998), and Parrilo and Sturmfels (2003). These algorithms work when the given polynomial has a minimum, without considering an approach for finding the infimum.

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Different approaches, based on solving a certain convex relaxation of the problem, can be found in Shor (1998), Lasserre (2001), Sturmfels (2001), Parrilo (2003) and Parrilo and Sturmfels (2003). Such methods seem to have better computational properties. However, in general, they only guarantee finding a lower bound of the infimum.

The aim of this note is to investigate the general case: *the polynomial f may not attain its infimum.* In Section 2, we shall construct a finite set in $\mathbb{R}$ containing all real critical values and the infimum value of f. This set can be computed effectively. Here "effectively" means that we give an algorithm (based on Gröbner basis) which works actually effectively on a computer. In addition, in Section 3, we show some relations between Newton polytopes and bounded below polynomials; and especially, necessary and almost sufficient conditions for a polynomial to be bounded from below are given.

2. Minimum of polynomials and bifurcation values

Let us consider the polynomial f as a map from $\mathbb{C}^n$ to $\mathbb{C}$. Thom proved that f is a C^∞-fibration outside a finite set (see Thom, 1969; Verdier, 1976). The smallest of such sets is called the *bifurcation set* of f. We denote it by $B(f)$. Recall that in general, the set $B(f)$ is bigger than $K_0(f)$-the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcation values at infinity. Briefly speaking, the set $B_\infty(f)$ consists of values at which f is not a locally trivial fibration at infinity (i.e., outside a large ball).

The set $B_\infty(f)$ is closely related to the following set

$$\begin{aligned} K_\infty(f) := \{t \in \mathbb{C} \quad | \quad & \text{there is a sequence } x_k \in \mathbb{R}^n \text{ such that } x_k \to \infty, \\ & f(x_k) \to t \text{ and } \|\mathrm{grad} f(x_k)\| \|x_k\| \to 0\}. \end{aligned}$$

If $t \in K_\infty(f)$ then it is usual to say that f does not satisfy *Malgrange's condition* at t (see Malgrange, 1980; Parusinski, 1995). It was proved (see, for example, Parusinski, 1995) that $B_\infty(f) \subset K_\infty(f)$. Put $K(f) := K_0(f) \cup K_\infty(f)$. Thus we have that in general $B(f) \subset K(f)$. Moreover, it is proved that $B_\infty(f) = K_\infty(f)$ if $n = 2$ (see Hà, 1990 and 2001) or if f has isolated singularities at infinity (see Parusinski, 1995 and 1997).

Let us return to our problem on minimizing polynomial functions. It turns out that, as we shall show below, the set $K(f)$ contains all critical values and also the infimum of f. Fortunately, the set $K(f)$ can be computed effectively, as Jelonek and Kurdyka (2003) have shown it very recently.

2.1. **Case $n = 2$.**
We consider the case $n = 2$ separately, because for polynomials in two variables we can give concrete information.

If f does not attain the value f^* then as it was proved in Hà (2001) that $f^* \in B_\infty(f)$. The bifurcation values at infinity of the given polynomial of two variables can be computed explicitly. In fact, let d be the degree of f, then we may always assume that f is of the form

$$f(x_1, x_2) = cx_2^d + \text{ terms of degrees in } x_2 \text{ less than } d,$$

where $c \in \mathbb{R}, c \neq 0$. Let t be a new indeterminate and form the discriminant of the polynomial $f(x_1, x_2) - t$ with respect to x_2 :

$$\Delta(x_1, t) := \operatorname{Res}_{x_2}\left(f(x_1, x_2) - t, \frac{\partial f}{\partial x_2}(x_1, x_2)\right),$$

where Res_{x_2} is the resultant of polynomials in variable the x_2 (see Cox *et al.*, 1997). Then we can write

$$\Delta(x_1, t) = q_0(t)x_1^{\beta} + \text{ terms of degrees in } x_1 \text{ less than } \beta,$$

where q_0 is a polynomial in t of degree less or equal to $d-1$. Hà (1989) proved that if $\mu_f < \infty$ then

$$B_\infty(f) = \{t \in \mathbb{C} \mid q_0(t) = 0\}.$$

With the above notations we formulate our result in the case of two variables as follows.

Proposition 2.1. *Let f be a polynomial of two real variables. Assume that f is bounded from below. If $\mu_f < \infty$ then*

$$f^* \in (K_0(f) \cup B_\infty(f)) \cap \mathbb{R} = (K_0(f) \cup q_0^{-1}(0)) \cap \mathbb{R}.$$

Example 2.1. Let us compute the set $K_0(f) \cup B_\infty(f)$ for the following polynomial (see also Hà, 2001)

$$f(x_1, x_2) := 2x_2^4(x_2 + x_1)^4 + x_2^2(x_2 + x_1)^2 + 2x_2(x_2 + x_1) + x_2^2.$$

Using MAPLE we obtained that $\Delta(x_1, t) = q_0(t)x_1^{14} + q_1(t)x_1^{13} + \cdots + q_{14}(t)$, where

$$\begin{aligned} q_0(t) &= -16777216t^3 - 4194304t^2 - 19136512t - 14417920 \\ &= -8192(8t+5)(16t - 3 - 7\sqrt{-7})(16t - 3 + 7\sqrt{-7}). \end{aligned}$$

Therefore $B_\infty(f) \cap \mathbb{R} = \{-\frac{5}{8}\}$. On the other hand, it is easy to check that the set of critical points of f is $\{(0,0)\}$, and so $K_0(f) = \{f(0,0) = 0\}$. It remains to show that $-\frac{5}{8}$ is really the infimum value of f. Indeed, taking $x_1(t) = t + \frac{1}{2t}, x_2(t) = -t$, we see that

$$\lim_{t\to 0} f\Big[x_1(t), x_2(t)\Big] = \lim_{t\to 0}\Big[-\frac{5}{8} + t^2\Big] = -\frac{5}{8}.$$

Thus $f^* = -\frac{5}{8}$ and the polynomial f does not attain its infimum value.

Remark 2.1. A program in MAPLE for computing $B(f)$ has been written by Bailly-Maitre (see Bailly-Maitre, 2000) based on our discriminant formula (see Hà, 1989) that we have mentioned above.

Remark 2.2. There is another algorithm to compute the set $B(f)$ in Lê and Weber (1994) (via resolution of singularities).

2.2. **Case $n \geqslant 2$.** According to Kurdyka *et al.* (2000), for a polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ and any $N \in \mathbb{N} - \{0\}$ we define

$$\begin{aligned} K_\infty^N(f) := \{t \in \mathbb{R} \quad | \quad & \text{there is a sequence } x_k \in \mathbb{R}^n \text{ such that } x_k \to \infty, \\ & f(x_k) \to t \text{ and } \|\mathrm{grad} f(x_k)\| \|x_k\|^{1+\frac{1}{N}} \to 0\}. \end{aligned}$$

Clearly, $K_\infty^N(f) \subset K_\infty(f)$.

Lemma 2.1. *Suppose that f is bounded from below. If f does not attain its infimum value f^*, then there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$, $f^* \in K_\infty^N(f)$. In particular, f does not satisfy Malgrange's condition at f^*; that is $f^* \in K_\infty(f)$.*

Proof. Put

$$A := \{x \in \mathbb{R}^n \mid f(x) = \min\{f(y) \mid \|y\| = \|x\|,\ y \in \mathbb{R}^n\}\}.$$

Then we can prove that

(a1) A is an unbounded semi-algebraic set (this follows from Tarski's theorem);
(a2) For all $x \in A$ there is $\lambda(x) \in \mathbb{R}$ such that $\mathrm{grad} f(x) = \lambda(x)x$; and
(a3) For every sequence $x_k \in A, x_k \to \infty$, we have $f(x_k) \to f^*$ (since f does not attain its infimum).

Then, by using a version at infinity of the Curve Selection Lemma (see Milnor, 1968; Némethi and Zaharia, 1992), there is a curve $\varphi\colon (0, \epsilon] \longrightarrow \mathbb{R}^n,\ t \mapsto \varphi(t)$, such that

(b1) $\varphi(t) \in A$ for $t \in (0, \epsilon]$;
(b2) $\|\varphi(t)\| \to +\infty$ as $t \to +0$; and
(b3) φ is a real meromorphic mapping.

We can write

$$\begin{aligned} f[\varphi(t)] - f^* &= c_1 t^\nu + \text{ higher order terms in } t, \\ \|\varphi(t)\| &= c_2 t^\rho + \text{ higher order terms in } t. \end{aligned}$$

where c_1, c_2 are non-zero real numbers. It is clear from (a3), (b1), and (b2) that

$$\nu > 0, \qquad \rho < 0. \tag{2.1}$$

We have

$$\begin{aligned} \frac{df[\varphi(t)]}{dt} &= \left\langle \mathrm{grad} f[\varphi(t)], \frac{d\varphi(t)}{dt} \right\rangle \\ &= \lambda[\varphi(t)] \left\langle \varphi(t), \frac{d\varphi(t)}{dt} \right\rangle. \end{aligned}$$

(The second equality follows from (a2).) This gives

$$2\frac{df[\varphi(t)]}{dt} = \lambda[\varphi(t)] \frac{d\|\varphi(t)\|^2}{dt}.$$

Thus,

$$2\left|\frac{df[\varphi(t)]}{dt}\right| = \frac{\|\mathrm{grad} f[\varphi(t)]\|}{\|\varphi(t)\|} \frac{d\|\varphi(t)\|^2}{dt}.$$

Therefore,

$$|2c_1\nu t^{\nu-1} + \cdots| = \|\mathrm{grad} f[\varphi(t)]\||2\rho||c_2 t^{\rho-1} + \cdots|.$$

This is equivalent to

$$|2c_1\nu t^{\nu} + \cdots| = \|\mathrm{grad} f[\varphi(t)]\||2\rho||c_2 t^{\rho} + \cdots|.$$

As a consequence,

$$\|\mathrm{grad} f[\varphi(t)]\|\|\varphi(t)\| \simeq |t|^{\nu} \simeq \|\varphi(t)\|^{\frac{\nu}{\rho}} \quad \text{as} \quad t \to 0.$$

Let N_0 be the smallest integer $> -\frac{\rho}{\nu}$. Then, by a direct computation, for all $N \geqslant N_0$ we have

$$\|\mathrm{grad} f[\varphi(t)]\|\|\varphi(t)\|^{1+\frac{1}{N}} \to 0 \quad \text{as} \quad t \to 0,$$

which shows that $f^* \in K_\infty^N(f)$. Consequently, $f^* \in K_\infty(f)$ because $K_\infty^N(f) \subset K_\infty(f)$. The claim is proved. □

Remark 2.3. (i) After the preparation of this paper we have learnt that Lemma 2.1 was also proved by Schweighofer [21] using an actually different argument. In fact, our proof, based on the ideas of Kuo and Łojasiewicz, uses only the Curve Selection Lemma as a tool.

(ii) For a polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ and any $N \in \mathbb{N}$, we let (see [21])

$$S(\mathrm{grad} f, N) := \{x \in \mathbb{R}^n \mid \|\mathrm{grad} f(x)\|^{2N}(1 + \|x\|^2)^{N+1} \leqslant 1\}.$$

Suppose that f^* is not attained by f, then the above proof shows also that $\varphi(t) \in S(\mathrm{grad} f, N)$ with $0 < t \ll 1$ and for all $N \geqslant N_0$. Hence

$$f^* = \inf\{f(x) \mid x \in S(\mathrm{grad} f, N)\}.$$

On the other hand, it is clear that $f^* \in K_0(f)$ provided the infimum value f^* is attained by f on $\mathbb{R}^n$. In this case, for all $N \in \mathbb{N}$, we also have

$$f^* = \inf\{f(x) \mid x \in S(\mathrm{grad} f, N)\}.$$

Therefore, by what has already been proved, we obtain the following immediate corollary.

Corollary 2.1. (see [21, Theorem 44]) *Let $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ be bounded from below. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$,*

$$f^* = \inf\{f(x) \mid x \in S(\mathit{grad} f, N)\}.$$

Let $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. For each $r > 0$, consider the polynomial

$$f_r(x) := f(x) + r\|x\|^2.$$

Lemma 2.2. *Suppose $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is bounded from below. Then for each $r > 0$, f_r is proper and bounded from below.*

Proof. By the hypothesis,

$$f^* := \inf\{f(x) \mid x \in \mathbb{R}^n\} > -\infty.$$

Then for each $r > 0$, we have

$$f_r(x) \geqslant f^* + r\|x\|^2 \geqslant f^* \quad \text{for all } x \in \mathbb{R}^n,$$

which proves f_r is bounded from below.

Moreover, $\{x \in \mathbb{R}^n \mid f_r(x) \leqslant c\}$ is a compact set because it is contained in the ball $\{\|x\|^2 \leqslant (c - f^*)/r\}$. This implies that f is proper. $\square$

By the above lemma, for every $r > 0$, the polynomial f_r attains a minimum on $\mathbb{R}^n$:

$$f_r^* := \min\{f_r(x) \mid x \in \mathbb{R}^n\} > -\infty.$$

Proposition 2.2. *Let $f \in \mathbb{R}[x_1, x_2, \dots, x_n]$ be bounded from below. If f does not attain its infimum f^* then*

$$\lim_{r \to +0} f_r^* = f^*.$$

Proof. The proof of Lemma 2.1 shows also that

(c1) $\|\varphi(t)\|^2$ is increasing to $+\infty$ as t is decreasing to $+0$;
(c2) $f[\varphi(t)]$ is decreasing to f^* as t is decreasing to $+0$;
(c3) grad $f[\varphi(t)] = \lambda[\varphi(t)]\varphi(t)$ for $t \in (0, \epsilon]$;
(c4) $2\frac{df[\varphi(t)]}{dt} = \lambda[\varphi(t)]\frac{d\|\varphi(t)\|^2}{dt}$ for $t \in (0, \epsilon]$; and
(c5) $\|\text{grad} f[\varphi(t)]\|\|\varphi(t)\| \to 0$ as $t \to +0$.

It follows from (c1), (c2) and (c4) that

$$r(t) := -\lambda[\varphi(t)] > 0 \quad \text{for} \quad 0 < t \ll 1.$$

Moreover, by (c3) and (c5), we have

$$r(t)\|\varphi(t)\|^2 = \|\text{grad} f[\varphi(t)]\|\|\varphi(t)\| \to 0 \quad \text{as} \quad t \to +0.$$

Hence

$$f_{r(t)}[\varphi(t)] = f[\varphi(t)] + r(t)\|\varphi(t)\|^2 \to f^* \quad \text{as} \quad t \to +0.$$

On the other hand,

$$f_{r(t)}[\varphi(t)] \geqslant f_{r(t)}^* \geqslant f^*.$$

These imply that

$$f_{r(t)}^* \to f^* \quad \text{as} \quad t \to +0,$$

which proves the proposition. $\square$

By Lemma 2.1, the value f^* belongs to the set $K(f) = K_0(f) \cup K_\infty(f)$ in any case: if f attains f^* then $f^* \in K_0(f)$, if not, $f^* \in K_\infty(f)$. Now to achieve our aim we only need to apply the result of Jelonek and Kurdyka (2003): the set $K(f)$ can be computed effectively.

Let us recall the definition of Gröbner basis. On the set of monomials x^α in $\mathbb{R}[x_1, x_2, \dots, x_n]$ let us consider the order induced by the lexicographic order in $\mathbb{N}^n$; i. e., we say $x^\alpha > x^\beta$ if in the difference $\alpha - \beta \in \mathbb{Z}^n$, the left-most nonzero entry is positive. (Lexicographic order is analogous to the ordering of words used in dictionaries.)

By $\mathrm{in} f := a_d x^d$ we will denote the initial form of a polynomial

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \in \mathbb{R}[x_1, x_2, \ldots, x_n],$$

where $d := \max\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \mid a_\alpha \neq 0\}$ (the maximum is taken with respect to the lexicographic order). By definition, a finite subset $\mathcal{A} \subset J \subset \mathbb{R}[x_1, x_2, \ldots, x_n]$ of an ideal J is called a *Gröbner basis* of this ideal if the set $\{\mathrm{in} f, f \in \mathcal{A}\}$ generates the ideal generated by all initial forms of the ideal J (see, for example Cox *et al.*, 1997).

Following Jelonek and Kurdyka (2003) consider the ideal

$$J := \left\langle f(x) - t, y_1 - \frac{\partial f}{\partial x_1}, y_2 - \frac{\partial f}{\partial x_2}, \cdots, y_n - \frac{\partial f}{\partial x_n}, y_{ij} - x_i \frac{\partial f}{\partial x_j}, i, j = 1, 2, \ldots, n \right\rangle$$

in $\mathbb{R}[x_1, x_2, \ldots, x_n, t, y_1, y_2, \ldots, y_n, y_{11}, y_{12}, \ldots, y_{nn}]$. Let $\mathcal{A}$ be a Gröbner basis of J. Put

$$\mathcal{B} := \mathcal{A} \cap \mathbb{R}[t, y_1, y_2, \ldots, y_n, y_{11}, y_{12}, \ldots, y_{nn}].$$

Then Jelonek and Kurdyka (2003) showed that

$$K(f) = \{t \in \mathbb{C} \mid h(t, 0, 0, \ldots, 0) = 0 \text{ for every } h \in \mathcal{B}\}.$$

With the notation as above we obtain

Proposition 2.3. *Assume that f is bounded from below. Then*

$$f^* \in K(f) \cap \mathbb{R} = \{t \in \mathbb{R} \mid h(t, 0, 0, \ldots, 0) = 0 \textit{ for every } h \in \mathcal{B}\}.$$

Example 2.2. Consider an example with three variables:

$$f^* = \inf\{f(x_1, x_2, x_3) := (x_1 x_2 - 1)^2 + x_2^2 + x_3^2 \mid (x_1, x_2, x_3) \in \mathbb{R}^3\}.$$

Using MAPLE we obtained that the basis $\mathcal{B}$ after substituting $y_1 = y_2 = \cdots = y_{12} = 0$ reduces to one polynomial in the variable t, namely to $4t^2 - 4t$. Hence $K(f) = \{0, 1\}$. It is easy to check that the only critical point of f is $(0, 0, 0)$, and so $K_0(f) = \{f(0, 0, 0) = 1\}$. It remains to show that 0 is really the infimum value of f. Indeed, taking $x_1(t) = \frac{1}{t}, x_2(t) = t, x_3(t) = 0$, we see that $\lim_{t \to 0} f[x_1(t), x_2(t), x_3(t)] = \lim_{t \to 0} t^2 = 0$. Thus $f^* = 0$ and the polynomial f does not attain its infimum value.

Remark 2.4. Let f be a polynomial of degree $d > 0$. Let $a := \#K_\infty(f), b := \#K(f)$. Then Jelonek and Kurdyka (2003) proved that

$$da + b \leqslant d^n - 1.$$

Remark 2.5. We can use SINGULAR (see Greuel et. al., 2002) to compute the set $K(f)$. Some examples are given in Jelonek and Kurdyka (2003).

3. Newton polytopes and bounded below polynomials

In this section we give some relations between Newton polytopes and bounded below polynomials. First we recall some notations about Newton polytopes. Let f be a polynomial in $\mathbb{R}[\mathbf{x}]$ and write

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

Put

$$\operatorname{supp}(f) := \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n \mid a_\alpha \neq 0\}.$$

The *Newton polytope (at infinity)* $\Gamma_-(f)$ of f is the convex hull of $\{0\} \cup \operatorname{supp}(f)$. Clearly, $\Gamma_-(f)$ is a compact, convex polytope of dimension at most n.

A *supporting hyperplane* of $\Gamma_-(f)$ is a hyperplane minimizing the value of some linear function on $\Gamma_-(f)$. The *faces* of the boundary of the Newton polytope $\Gamma_-(f)$ are intersections of $\Gamma_-(f)$ with a supporting hyperplane. They are compact, convex polytopes of dimension at most $n-1$. *Vertices* are faces of dimension 0 (i.e., points). Denote by $\Gamma(f)$ the union of closed faces which do not contain 0, and we call $\Gamma(f)$ the *Newton diagram (at infinity)* of f.

For a face $\gamma \in \Gamma(f)$, the restriction

$$f_\gamma(x) := \sum_{\alpha \in \gamma} a_\alpha x^\alpha$$

is called the *quasi-homogeneous component* of f.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is called *convenient*, if for any $i = 1, 2, \dots, n$, the monomial $x_i^{\alpha_i}, \alpha_i \geqslant 1$, appears in f with non-zero coefficient.

The purpose of this section is to verify the following theorem.

Theorem 3.1. *Let f be a real multivariate polynomial.*

(i) *If f is bounded from below then all quasi-homogeneous components of f are non-negative.*

(ii) *Suppose that f is convenient. If all quasi-homogeneous components of f are strictly positive outside the coordinate planes, then f is bounded from below. Furthermore, there are constants c_1, c_2 ($c_2 > 0$) such that*

$$f(x) \geqslant c_1 + c_2 \sum_{\alpha \in V(f)} x^\alpha,$$

where $V(f)$ is the set of vertices of $\Gamma_-(f)$.

Proof. (i) Suppose, on the contrary, that there exists a face $\gamma \in \Gamma(f)$ such that

$$f_\gamma(x^0) < 0 \tag{3.1}$$

for some $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}^n$.

Let

$$H := \{\alpha \in \mathbb{R}^n \mid \langle m, \alpha \rangle := m_1\alpha_1 + m_2\alpha_2 + \dots + m_n\alpha_n = \nu\}$$

be some supporting hyperplane of $\Gamma_-(f)$ which contains the face γ. Since the vertices of $\Gamma_-(f)$ have integer coordinates, without loss of generality, we can suppose that $m_1, m_2, \ldots, m_n$ and ν are integer numbers. By definition, the supporting hyperplane H has the property that $\Gamma_-(f)$ lies in the half-space $\{\alpha \in \mathbb{R}^n \mid \langle m, \alpha \rangle \geqslant \nu\}$. This implies that

$$\langle m, \alpha \rangle > \nu$$

for all $\alpha \in \Gamma_-(f) \setminus \gamma$. In particular, for $\alpha = 0$ we get

$$0 > \nu. \tag{3.2}$$

Take $0 < \epsilon \ll 1$ and we define an algebraic curve $\varphi \colon (0, \epsilon] \longrightarrow \mathbb{R}^n, t \mapsto \varphi(t)$, as follows

$$\varphi(t) := \begin{cases} x_1(t) = x_1^0 t^{m_1}, \\ x_2(t) = x_2^0 t^{m_2}, \\ \qquad \vdots \\ x_n(t) = x_n^0 t^{m_n}. \end{cases}$$

Then it is easy to check that

$$\begin{aligned} f[\varphi(t)] &= f_\gamma[\varphi(t)] + \sum_{\alpha \notin \gamma} a_\alpha [\varphi(t)]^\alpha \\ &= t^\nu f_\gamma(x^0) + \text{ higher order terms in } t \\ &\simeq t^\nu f_\gamma(x^0), \text{ for } 0 < t \ll 1, \end{aligned}$$

where $A \simeq B$ means that the ratio of the two sides is between two positive constants. So from (3.1) and (3.2) we get

$$\lim_{t \to +0} f[\varphi(t)] = -\infty$$

which contradicts the hypothesis.

Thus all quasi-homogeneous components of f are non-negative. In particular, all vertices of the Newton polytope $\Gamma_-(f)$ have even coordinates.

(ii) We now suppose that all quasi-homogeneous components of f are strictly positive outside the coordinate planes. We will prove that f is bounded from below. In fact, if it is not so, then the semi-algebraic set

$$A := \{x \in \mathbb{R}^n \mid f(x) \leqslant 0\}$$

is unbounded. By a version at infinity of the Curve Selection Lemma (see Milnor, 1969; Némethi and Zaharia, 1992), there exists a real meromorphic mapping

$$\varphi \colon (0, \epsilon] \longrightarrow \mathbb{R}^n, \ t \mapsto \varphi(t),$$

such that

(d1) $\varphi(t) \in A$ for $t \in (0, \epsilon]$;
(d2) $\|\varphi(t)\| \to +\infty$ as $t \to +0$; and
(d3) $f[\varphi(t)] \to -\infty$ as $t \to +0$.

Without loss of generality, we can suppose that for $\epsilon > 0$ sufficiently close to 0, the set $\varphi((0,\epsilon])$ is contained in the coordinate planes $\{x_k = 0\}, k = l+1, l+2, \ldots n$, but not in another one for some $1 \leqslant l \leqslant n$. Then we can write

$$\varphi(t) := \begin{cases} x_1(t) = x_1^0 t^{m_1} + \text{ higher order terms in } t, \\ x_2(t) = x_2^0 t^{m_2} + \text{ higher order terms in } t, \\ \qquad \vdots \\ x_l(t) = x_l^0 t^{m_l} + \text{ higher order terms in } t, \\ x_{l+1}(t) = x_{l+2}(t) = \cdots = x_n(t) = 0, \end{cases}$$

where $x_k^0, k = 1, 2, \ldots, l$ are non-zero real numbers. We note from (d2) that

$$\min_{k=1,2,\ldots,l} m_k < 0. \tag{3.3}$$

Let

$$\Gamma' := \Gamma_-(f) \cap \{\alpha \in \mathbb{N}^n \mid \alpha_{l+1} = \alpha_{l+2} = \cdots = \alpha_n = 0\}.$$

Then Γ' is a compact, convex polytope of dimension at most l. Let γ (resp., ν) be the set of minimal solutions (resp., the minimal value) of the linear programming problem

$$\begin{array}{ll} \text{minimize} & \langle m, \alpha \rangle \\ \text{subject to} & \alpha \in \Gamma', \end{array}$$

where $m := (m_1, m_2, \ldots, m_l, 0, 0, \ldots, 0)$. Then γ is some face of Γ' and hence by (3.3) we get $\gamma \in \Gamma(f)$ because f is convenient.

It follows from (d3) that $f[\bar{\varphi}(t)] < 0$ for $t > 0$ sufficiently small, where the map $\bar{\varphi}\colon (0,\epsilon] \longrightarrow \mathbb{R}^n, t \mapsto \bar{\varphi}(t)$, is defined by

$$\bar{\varphi} := \begin{cases} \bar{x}_1(t) = x_1^0 t^{m_1}, \\ \bar{x}_2(t) = x_2^0 t^{m_2}, \\ \qquad \vdots \\ \bar{x}_l(t) = x_l^0 t^{m_l}, \\ \bar{x}_{l+1}(t) = \bar{x}_{l+2}(t) = \cdots = \bar{x}_n(t) = 0. \end{cases}$$

On the other hand, for $0 < t \ll 1$ we have

$$\begin{aligned} f[\bar{\varphi}(t)] &\simeq f_\gamma[\bar{\varphi}(t)] \\ &= t^\nu f_\gamma(x_1^0, x_2^0, \ldots, x_l^0, 0, 0, \ldots, 0) \\ &= t^\nu f_\gamma(x_1^0, x_2^0, \ldots, x_l^0, 1, 1, \ldots, 1), \end{aligned}$$

where the last equality follows from the independence of the quasi-homogeneous component f_γ in the variables $x_{l+1}, x_{l+2}, \ldots, x_n$.

From the above discussion,

$$f_\gamma(x_1^0, x_2^0, \ldots, x_l^0, 1, 1, \ldots, 1) < 0,$$

which is a contradiction. Therefore, f is bounded from below; and hence there is a constant c_1 such that $f(x) > c_1$ for all $x \in \mathbb{R}^n$. So, from the assumption and a result of Gindikin (1974), there is a positive constant c_2 such that

$$f(x) - c_1 \geqslant c_2 \sum_{\alpha \in V(f)} x^\alpha$$

for all $x \in \mathbb{R}^n$; or equivalently

$$f(x) \geqslant c_1 + c_2 \sum_{\alpha \in V(f)} x^\alpha.$$

The proof is complete. □

Remark 3.1. Theorem 3.1 can be considered as a global analog of a result of Vassiliev (see Vassiliev, 1977 Theorem 1.5).

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