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GENERALIZED ℓ-ISOMORPHISMS OF ℓ-GROUPS

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Abstract. By using large and small convex ℓ-subgroups of an ℓ-group introduced in [3] and [4], we extend ℓ-isomorphisms of ℓ-groups to quasi-ℓ-isomorphisms and co-quasi-ℓ-isomorphisms. Then we will establish some sufficient and necessary conditions under which two ℓ-groups are quasi-ℓ-isomorphic or co-quasi-ℓ-isomorphic.

1. Background

Throughout, G will denote an ℓ-group. Recall from [3] and [4] that an ℓ-subgroup L of G is *large* in G if $L \cap C \neq 0$ for every $0 \neq C \in C(G)$. A convex ℓ-subgroup L of G is *large* if L is large in G as an ℓ-subgroup. We denote by $\ell(G)$ the set of all large ℓ-subgroups of G and by $L(G)$ the set of all large convex ℓ-subgroups of G. As the dual of large convex ℓ-subgroups, we define small convex ℓ-subgroups similarly, and denote by $S(G)$ the set of all small convex ℓ-subgroups of G. Recall also that, for any two ℓ-groups G and H, an ℓ-homomorphism $f : G \to H$ is called an *ℓ-isomorphism* if $Kerf = 0$ and $Imf = H$.

In the present paper, we will extend an ℓ-isomorphism of ℓ-groups to the following two more general cases:

Case I. If the condition that $Kerf = 0$ is changed by the condition that $Kerf \in S(G)$, then we call f a *quasi-ℓ-isomorphism*.

Case II. If the condition that $Imf = H$ is changed by the condition that $Imf \in \ell(H)$, then we call f a *co-quasi-ℓ-isomorphism*.

The main purpose of this paper is to establish some sufficient and necessary conditions under which two ℓ-groups are quasi-ℓ-isomorphic or co-quasi-ℓ-isomorphic.

2. Preliminaries

In this section, let us simply review some of the basic terms and concepts of ℓ-groups. The reader is referred to [2] for the general theory of ℓ-groups.

A *partially ordered group* G is a group that is also a partially ordered set such that for any $a, b, c, d \in G$, $c + a + d \leqslant c + b + d$ whenever $a \leqslant b$. Here we use $+$ to denote the group operation, but the group need not to be commutative. A

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partially ordered group G is an *ℓ-group* if the underlying order endows G with a lattice structure. An ℓ-group G is *o-group* if for any $x, y \in G$, either $x \leqslant y$ or $y \leqslant x$. An ℓ-group G is *archimedean* if for any $x, y \in G$, the condition $nx \leqslant y$ for all integers n implies $x = 0$. In view of Hölder's Theorem, any archimedean o-group is o-isomorphic to a subgroup of the additive group $\mathbb{R}$ of reals.

A subgroup A of an ℓ-group G is an *ℓ-subgroup* if A is also a sublattice of G. An ℓ-subgroup C of G is *convex* if for $g \in G$ and $c \in C$, $0 \leqslant g \leqslant c$ implies $g \in C$. For any $g \in G$, we denote by $G(g)$ the convex ℓ-subgroup of G generated by g; $G(g)$ is called a *principal convex ℓ-subgroup*. The set of all convex ℓ-subgroups of G is denoted by $C(G)$, which is a distributive Brouwerian lattice. $I \in C(G)$ is an *ℓ-ideal* if I is a normal convex ℓ-subgroup of G, i.e., $I \lhd G$. The set of all ℓ-ideals of G is denoted by $I(G)$. Clearly if I is an ℓ-ideal of G, then the factor group G/I is an ℓ-group by the following coset ordering: $I + x \geqslant I + y$ if there exists $z \in I$ such that $x \geqslant z + y$. $G_\lambda \in C(G)$ is *regular* if it is maximal with respect to not containing some $g \in G$. In this case, G_λ is a *value* of g. The set of all regular subgroups of G is denoted by $\Gamma(G)$. An ℓ-group G is *normal-valued* if for any $G_\lambda \in \Gamma(G)$, $G_\lambda \lhd G^\lambda$, where G^λ denotes the cover of G_λ in $C(G)$. In general, if there exists a minimal convex ℓ-subgroup of G properly containing $C \in C(G)$, this minimal convex ℓ-subgroup is unique and is called the cover of C. In view of ([2], Theorem 41.1), an ℓ-group G is normal-valued if and only if for any $A, B \in C(G)$, $A \vee B = A + B = B + A$.

Let G be an ℓ-group. Two positive elements $x, y \in G$ are *a-equivalent* if there exist positive integers m and n such that $x < ny$ and $y < mx$. G is an *a-extension* of an ℓ-subgroup A if for each $g \in G$, there exists $a \in A$ such that a is a-equivalent to g. In particular, if G is an a-extension of A, then the map $\tau : C(G) \to C(A) : C \to C \cap A$ is a lattice isomorphism.

Let G and H be both ℓ-groups. A function $f : G \to H$ is an *ℓ-homomorphism* if f is both a group and a lattice homomorphism. If, in addition, f is both surjective and injective, we say that f is an *ℓ-isomorphism*. For general ℓ-groups, we also have three basic isomorphism theorems ([2], Theorem 8.6).

Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a set of ℓ-groups for all λ. On $\times G_\lambda$, place the componentwise lattice and group operations. The resulting ℓ-group, denoted by $\prod\limits_{\lambda\in\Lambda} G_\lambda$, is called the cardinal product of the set $\{G_\lambda\}_{\lambda\in\Lambda}$ and each G_λ is called a *cardinal summand* of $\prod\limits_{\lambda\in\Lambda} G_\lambda$. Let $G = \prod\limits_{\lambda\in\Lambda} G_\lambda$, and let $\sum\limits_{\lambda\in\Lambda} G_\lambda = \{g \in G : g_\lambda = 0$ for all but a finite number of indices $\lambda\}$. A direct computation shows that $\sum\limits_{\lambda\in\Lambda} G_\lambda$ is an ℓ-ideal of $\prod\limits_{\lambda\in\Lambda} G_\lambda$, and is called the *cardinal sum* of the ℓ-groups $\{G_\lambda\}_{\lambda\in\Lambda}$.

An ℓ-group G is a *subdirect product* of ℓ-groups $\{G_\lambda\}_{\lambda\in\Lambda}$ if there exists an injective ℓ-homomorphism $\sigma : G \to \prod\limits_{\lambda\in\Lambda} G_\lambda$ such that for each projection $\rho_\mu : \prod\limits_{\lambda\in\Lambda} G_\lambda \to G_\mu$, $\rho_\mu \cdot \sigma$ is surjective.

3. Small and large convex ℓ-subgroups

In this section, we investigate some properties of small convex ℓ-subgroups and large convex ℓ-subgroups of an ℓ-group, which we will need in the later sections. Let us first recall from [4]:

Definition 3.1. Let G be an ℓ-group. $S \in C(G)$ is called small in G if $S \vee W = G$ for some $W \in C(G)$, then $W = G$.

For an ℓ-group G, we denote by $S(G)$ the set of all small convex ℓ-subgroups of G. A direct computation shows that $S(G)$ forms a lattice, and is a sublattice of $C(G)$, i.e., for any $S, T \in S(G)$, $S \vee T, S \cap T \in S(G)$, where $\vee$ and $\cap$ are in $C(G)$.

Example 1. Let $\mathbb{R}$ denote the additive group of reals with usual order, and set $\mathbb{R}_i \cong \mathbb{R}$ for any $i \geqslant 1$. Consider the following three ℓ-groups:

$$G_1 = \overleftarrow{\bigoplus_{i=1}^{\infty}} \mathbb{R}_i, \quad G_2 = \bigoplus_{i=1}^{\infty} \mathbb{R}_i, \quad \text{and} \quad G_3 = (\mathbb{R}_1 \oplus \mathbb{R}_2) \overleftarrow{\oplus} \mathbb{R}_3.$$

By a direct computation, we see that

(1) for any positive integer n, $\overleftarrow{\bigoplus_{i=1}^{n}} \mathbb{R}_i \in S(G)$. It follows that every convex ℓ-subgroup of G_1 is small in G except G.

(2) every convex ℓ-subgroup of G_2 is not small in G except 0.

(3) G_3 has three nontrivial small convex ℓ-subgroups $\mathbb{R}_1$, $\mathbb{R}_2$ and $\mathbb{R}_1 \oplus \mathbb{R}_2$, i.e., $S(G_3) = \{0, \mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_1 \oplus \mathbb{R}_2\}$.

The following will show that small convex ℓ-subgroups of an ℓ-group are closely related to its maximal convex ℓ-subgroups. For convenience, we denote by $Max(G)$ the set of all maximal convex ℓ-subgroups of an ℓ-group G.

Remark 1. If an ℓ-group G has no maximal convex ℓ-subgroups, then we always define $\bigcap Max(G) = G$.

Lemma 3.1. Let G be an ℓ-group.

(1) For any $g \in G$, $G(g) \notin S(G)$ if and only if there exists some $M \in Max(G)$ such that $g \notin M$.

(2) $\bigvee S(G) = \bigcap Max(G)$.

Proof. (1) First, for $g \in G$, suppose there exists some $M \in Max(G)$ such that $g \notin M$. Then, by the maximality of M, $G(g) \vee M = G$. But then $M \neq G$. Thus $G(g) \notin S(G)$.

Conversely, suppose $G(g) \notin S(G)$; then, by definition, there exists some $A \in C(G)$ with $A \neq G$ such that $G(g) \vee A = G$. Now, consider the set $\Omega = \{A \in C(G) : A \neq G \text{ and } G(g) \vee A = G\}$. An easy application of Zorn's Lemma shows that Ω has a maximal element, denoted by M. Suppose there exists $W \in C(G)$

such that $M \subset W \subseteq G$. By the maximality of M in Ω, we have $W \notin \Omega$, so that $W = G$. Hence M is a maximal convex ℓ-subgroup of G.

(2) If G contains no maximal convex ℓ-subgroups, then, by the above remark, $\bigcap Max(G) = G$. In this case, we easily obtain by (1) that every principal convex ℓ-subgroup of G is small in G. Hence $\bigvee S(G) = \bigvee_{g \in G} G(g) = G$. So, $\bigvee S(G) = \bigcap Max(G)$. If G contains maximal convex ℓ-subgroups, then (1) shows that G must contain small convex ℓ-subgroups. Now, let S be any small convex ℓ-subgroup and M any maximal convex ℓ-subgroup. Then, by definition, $S \vee M \neq G$. So, by the maximality of M, we get $S \vee M = M$, so that $S \subseteq M$. Hence $\bigvee S(G) \subseteq \bigcap Max(G)$. For the reverse inclusion, let $g \in \bigcap Max(G)$. Suppose that $G(g) \notin S(G)$; then, by (1), there must exist some $M \in Max(G)$ such that $g \notin M$, which is a contradiction. Therefore $\bigvee S(G) = \bigcap Max(G)$. $\square$

Theorem 3.1. The following conditions are equivalent for an ℓ-group G:

(1) G contains no small convex ℓ-subgroups except 0.

(2) For any $0 \neq g \in G$, there exists some maximal convex ℓ-subgroup M of G such that $g \notin M$.

(3) $\bigcap Max(G) = 0$.

If G is normal-valued, then any one of the above conditions is equivalent to the following condition:

(4) G is a subdirect product of archimedean o-groups.

Proof. (1)⇒(2)⇒(3) is clear by Lemma 3.1. For (3)⇒(1), suppose that there exists $0 \neq C \in C(G)$ such that $C \in S(G)$. Then we may pick $0 \neq g \in C$. By (3), there exists some $M \in Max(G)$ such that $g \notin M$. So, by the maximality of M, we see that $G(g) \notin S(G)$. On the other hand, let $G(g) \vee W = G$ for some $W \in C(G)$. Then clearly $C \vee W = G$. Since $C \in S(G)$, we then have $W = G$, so that $G(g) \in S(G)$, which is a contradiction. Hence G contains no small convex ℓ-subgroups except 0.

Now, if G is normal-valued, then for any $M \in Max(G)$, $M \lhd M^* = G$, and so by Hölder's Theorem, G/M is an archimedean o-group. Therefore G is a subdirect product of archimedean o-groups $\{G/M : M \in Max(G)\}$ if and only if $\bigcap Max(G) = 0$. This completes the proof. $\square$

Theorem 3.2. Let G be an a-extension of an ℓ-subgroup A. Then the map $\sigma : S(G) \to S(A) : S \mapsto S \cap A$ is a lattice isomorphism.

Proof. Since G is an a-extension of an ℓ-subgroup A, the map $\tau : C(G) \to C(A) : C \to C \cap A$ is a lattice isomorphism. So it suffices to show that $S \in S(G)$ if and only if $S \cap A \in S(A)$, then we have $\tau|_{S(G)} = \sigma$. It follows that σ is a lattice isomorphism. For convenience, we denote by $\vee_G$ and $\vee_A$ the operations in $C(G)$ and in $C(A)$, respectively.

Now, suppose $S \in S(G)$ and write $T = S \cap A$. Now, let $W \in C(A)$ be such that $T \vee_A W = A$. Then there must exist some $U \in C(G)$ such that $W = U \cap A$. Notice that τ is a lattice isomorphism, we have

$$A \cap (S \vee_G U) = \tau(S \vee_G U) = \tau(S) \vee_A \tau(U) = (A \cap S) \vee_A (A \cap U) = T \vee_A W = A.$$

It follows that $S \vee_G U = G$ since τ is a lattice isomorphism and $\tau(S \vee_G U) = \tau(G)$. By assumption, $S \in S(G)$, so that $U = G$. Hence $W = A$. So $S \cap A \in S(A)$. For the converse, we can similarly obtain the desired result. $\square$

In [1], Byrd proved that if G is an a-extension of an ℓ-subgroup A, then G is normal-valued if and only if A is also normal-valued. Further, by Theorem 3.1 and Theorem 3.2, we have

Corollary 3.1. Let G be an a-extension of an ℓ-subgroup A.

(1) G contains no small convex ℓ-subgroups if and only if A contains no small convex ℓ-subgroups.

(2) G is a subdirect product of archimedean o-groups if and only if A is a subdirect product of archimedean o-groups.

We now give the corresponding results for the case of large convex ℓ-subgroups. Since their proofs are completely dual, we will omit them.

Lemma 3.2. Let G be an ℓ-group.

(1) $\bigcap L(G) = \bigvee Min(G)$, where $Min(G)$ denotes the set of all minimal convex ℓ-subgroups of G.

(2) G contains no large convex ℓ-subgroups except G if and only if G is a cardinal sum of archimedean o-groups.

Theorem 3.3. Let G be an a-extension of an ℓ-subgroup A.

(1) The map $\sigma : L(G) \to L(A) : L \mapsto L \cap A$ is a lattice isomorphism.

(2) G contains no large convex ℓ-subgroups except G if and only if A contains no large convex ℓ-subgroups except A. Moreover, G is a cardinal sum of archimedean o-groups if and only if A is a cardinal sum of archimedean o-groups.

As a corollary of Theorem 3.1 and Theorem 3.3, we have

Corollary 3.2. Let G be an ℓ-group. If G contains no large convex ℓ-subgroups except G, then it also contains no small convex ℓ-subgroups except 0.

4. Quasi-ℓ-isomorphisms of ℓ-groups

In this section, we extend ℓ-isomorphisms of ℓ-groups to quasi-ℓ-isomorphisms by using small convex ℓ-subgroups. Let us first state the main definition of this section.

Definition 4.1. Let G and H be two ℓ-groups. G and H are called quasi-ℓ-isomorphic if there exists a surjective ℓ-homomorphism $f : G \to H$ such that $Kerf \in S(G)$.

From Definition 4.1, we see that if G and H are ℓ-isomorphic, then they are clearly quasi-ℓ-isomorphic. But the converse does not hold in general. Consider the following example.

Example 2. Let $\mathbb{Z}$ denote the additive group of integers with usual order. Let $G = \mathbb{Z}\overleftarrow{\oplus}\mathbb{Z}$ and let $H = 0\overleftarrow{\oplus}\mathbb{Z}$. Consider the following map

$$f : G \to H \text{ defined by the rule: } f(x, y) = (0, y) \text{ for any } (x, y) \in G.$$

We easily obtain that G and H are quasi-ℓ-isomorphic since $Kerf = \mathbb{Z}\overleftarrow{\oplus}0 \in S(G)$. But clearly G and H are not ℓ-isomorphic.

The following will show the relation between ℓ-isomorphisms and quasi-ℓ-isomorphisms. Since its proof is straightforward, we will omit it.

Theorem 4.1. Let G and H be two ℓ-groups. Then the following conditions are equivalent:

(1) $f : G \to H$ is an ℓ-isomorphism.

(2) f is a quasi-ℓ-isomorphism, and for any $S \in S(G)$, if $f(S) = 0$, then $S = 0$.

By a direct computation, we also have

Lemma 4.1. Let G be an ℓ-group and let $C \in C(G)$. The following conditions are equivalent:

(1) $C \in S(G)$.

(2) If $K \in I(G)$ and $K \subseteq C$, then $K \in S(G)$ and $C/K \in S(G/K)$.

We are now in a position to prove the main result of this section.

Theorem 4.2. Let G and H be two ℓ-groups and let $f : G \to H$ be a surjective ℓ-homomorphism. Then the following conditions are equivalent:

(1) $f : G \to H$ is a quasi-ℓ-isomorphism.

(2) For any $C \in S(H)$, $f^{-1}(C) \in S(G)$.

(3) For an ℓ-ideal I of G, if there exists a surjective ℓ-homomorphism $g : G/I \to H$ such that $f = g\pi$, where $\pi : G \to G/I$ is the natural map, then $I \in S(G)$.

(4) For any proper convex ℓ-subgroup C of G, $f(C) \neq H$.

Proof. (1)$\Rightarrow$(2) Given any $C \in S(H)$, a direct computation shows that $f^{-1}(C) \in C(G)$. Notice that $Kerf \in S(G)$ and $Kerf = f^{-1}(0) \subseteq f^{-1}(C)$. So, in order to show $f^{-1}(C) \in S(G)$, it suffices to show that $f^{-1}(C)/Kerf \in S(G/Kerf)$ by Lemma 4.1. Now, let $W \in C(G)$ with $W \supseteq Kerf$ such that

$$(f^{-1}(C)/Kerf) \vee (W/Kerf) = G/Kerf.$$

So $(f^{-1}(C) \vee W)/Kerf = G/Kerf$, which implies $f^{-1}(C) \vee W = G$. In view of Theorem 7.4 in [2] and the surjectivity of f, we have $C \vee f(W) = f(G) = H$, so that $f(W) = H = f(G)$. Now, for any $g \in G$, there exists some $w \in W$ such that $f(g) = f(w)$, which implies $g - w \in Kerf$. Notice that $Kerf \subseteq W$, so $g \in W$, so that $W = G$. Therefore $f^{-1}(C)/Kerf \in S(G/Kerf)$.

(2)$\Rightarrow$(3) Since $0 \in S(H)$, by (2), we have $Kerg\pi = (g\pi)^{-1}(0) = f^{-1}(0) \in S(G)$. Notice that $I \subseteq Kerg\pi$, so that $I \in S(G)$.

(3)$\Rightarrow$(4) Suppose on the contrary that there exists a proper convex ℓ-subgroup C of G such that $f(C) = H$. As in the proof of (1)$\Rightarrow$(2), we easily obtain that $Kerf + C = G$. Then $G/Kerf \cong f(G) = H$. Naturally, we have the following surjective ℓ-homomorphism

$$g : G/Kerf \to H \text{ defined by the rule } g(x + Kerf) = f(x) \text{ for any } x + Kerf \in G/Kerf.$$

Clearly $f = g\pi$. So, by (3), we have $Kerf \in S(G)$. Since $Kerf \lhd G$, $Kerf + C = Kerf \vee C$, so that $Kerf \vee C = G$. Notice that $Kerf \in S(G)$, so we further get $C = G$, which contradicts the hypothesis.

(4)$\Rightarrow$(1) By definition, it suffices to show that $Kerf \in S(G)$. Let $Kerf \vee W = G$ for some $W \in C(G)$. Then $f(Kerf) \vee f(W) = f(W) = f(G) = H$. By (4), W is a trivial convex ℓ-subgroup of G. Clearly $W \neq 0$, so that $W = G$. Thus $Kerf$ is small in G. So $f : G \to H$ is a quasi-ℓ-isomorphism. □

As a corollary of Theorem 4.2, we can obtain a very nice characterization of quasi-ℓ-isomorphisms of ℓ-groups, as follows:

Corollary 4.1. Let G and H be two ℓ-groups . Then the following conditions are equivalent:

(1) $f : G \to H$ is a quasi-ℓ-isomorphism.

(2) $f : G \to H$ is a surjective ℓ-homomorphism, and for any $C \in C(H)$, $C \in S(H)$ if and only if $f^{-1}(C) \in S(G)$.

Recall that an ℓ-group G is called *Hamiltonian* if for any $C \in C(G)$, $C \lhd G$.

Theorem 4.3. Let G and H be two Hamiltonian ℓ-groups and let $f : G \to H$ be a surjective ℓ-homomorphism. If for any ℓ-group K, the existence of a surjective ℓ-homomorphism $k : G \to H \oplus K$ implies $K = 0$, then f is a quasi-ℓ-isomorphism.

Proof. By definition, it suffices to show that if $Kerf \vee W = G$ for some $W \in C(G)$, then $W = G$. According to the Second Isomorphism Theorem of ℓ-groups, and noticing that G is Hamiltonian, we have

$$\begin{aligned} &G/(W \cap Kerf) = (Kerf/(W \cap Kerf)) \vee (W/(W \cap Kerf)) \\ &= Kerf/(W \cap Kerf) \oplus W/(W \cap Kerf) \cong (Kerf + W)/W \oplus (Kerf + W)/Kerf \\ &= (Kerf \vee W)/W \oplus (Kerf \vee W)/Kerf = G/W \oplus G/Kerf \\ &\cong G/W \oplus f(G) = G/W \oplus H. \end{aligned}$$

From which we can obtain the following surjective ℓ-homomorphism

$$G \to G/W \oplus H \text{ defined by the rule } a \mapsto (a + W, f(a)).$$

Clearly this is well defined. So, by assumption, we get $G/W = 0$, i.e., $W = G$. It follows that $Kerf$ is small in G. Therefore f is a quasi-ℓ-isomorphism. □

5. Co-quasi-ℓ-isomorphisms of ℓ-groups

As the dual case of quasi-ℓ-isomorphisms of ℓ-groups, we study in this section co-quasi-ℓ-isomorphisms of ℓ-groups, which is, in fact, another generalization of ℓ-isomorphisms of ℓ-groups. Let us recall

Definition 5.1. Let G and H be two ℓ-groups. G and H are called co-quasi-ℓ-isomorphic if there exists an injective ℓ-homomorphism $f : G \to H$ such that $Imf \in \ell(H)$.

From Definition 5.1, we see that if G and H are ℓ-isomorphic, then they are clearly co-quasi-ℓ-isomorphic. But the converse does not hold in general. The following will show the relation between ℓ-isomorphisms and co-quasi-ℓ-isomorphisms. Since its proof is very direct, we will omit it.

Theorem 5.1. Let G and H be two ℓ-groups. Then the following conditions are equivalent:

(1) $f : G \to H$ is an ℓ-isomorphism.

(2) f is a co-quasi-ℓ-isomorphism, and for any $L \in L(H)$, if $Imf \subseteq L$, then $L = H$.

For co-quasi-ℓ-isomorphisms of ℓ-groups, we have the following corresponding characterizations, which are almost dual to Theorem 4.2. For convenience, here we will give its complete explanation.

Theorem 5.2. Let G and H be two ℓ-groups and let $f : G \to H$ be an injective ℓ-homomorphism. Then the following conditions are equivalent:

(1) $f : G \to H$ is a co-quasi-ℓ-isomorphism.

(2) For any $L \in L(G)$, $f(L) \in \ell(H)$.

(3) For an ℓ-subgroup K of H, if there exists an injective ℓ-homomorphism $g : G \to K$ such that $f = ig$, where $i : K \to H$ is the identically embedding, then $K \in \ell(H)$.

(4) For any $0 \neq C \in C(H)$, $f^{-1}(C) \neq 0$.

Proof. (1)⇒(2) Given any $L \in L(G)$, let $f(L) \cap C = 0$ for some $C \in C(H)$. Then we have $L \cap f^{-1}(C) = f^{-1}(0) = 0$. A direct computation shows that $C \in C(G)$ implies $f^{-1}(C) \in C(G)$. Since $L \in L(G)$, we then have $f^{-1}(C) = 0$, so that $C = 0$ since f is injective. Hence $f(L) \in \ell(H)$.

(2)⇒(3) Since $G \in L(G)$, by (2), we have $ig(G) = f(G) \in \ell(H)$. Notice that $g(G) \subseteq K$, so $ig(G) \subseteq i(K) = K$. So, a direct computation will show that $K \in \ell(H)$.

(3)⇒(1) Since $f : G \to H$ is an injective ℓ-homomorphism, we then have $g : G \to f(G)$ defined by the rule: $g(x) = f(x)$ for any $x \in G$, is also an injective ℓ-homomorphism, and $f = ig$, where $i : f(G) \to H$ is the identically embedding. By (3), $f(G) \in \ell(H)$. Thus f is a co-quasi-ℓ-isomorphism.

(1)⇒(4) Since $f : G \to H$ is a co-quasi-ℓ-isomorphism, by definition, $f(G) \in \ell(H)$. Then for any $0 \neq C \in C(H)$, we have $f(G) \cap C \neq 0$. So, we may pick $0 < x \in C$, then there exists $y \in G$ such that $x = f(y)$. Clearly $y \neq 0$, and

$$y \in f^{-1}(f(G) \cap C) = f^{-1}(f(G)) \cap f^{-1}(C) = G \cap f^{-1}(C) = f^{-1}(C).$$

Hence $f^{-1}(C) \neq 0$.

(4)⇒(1) Suppose on the contrary that $f(G) \notin \ell(H)$; then there must exist some $0 \neq C \in C(H)$ such that $f(G) \cap C = 0$. Thus $f^{-1}(f(G) \cap C) = f^{-1}(0) = 0$. On the other hand, we have

$$f^{-1}(f(G) \cap C) = f^{-1}f(G) \cap f^{-1}(C) = G \cap f^{-1}(C) = f^{-1}(C).$$

It follows that $f^{-1}(C) = 0$, which contradicts the assumption. Thus $f(G) \notin \ell(H)$. So f is a co-quasi-ℓ-isomorphism. □

As a corollary of Theorem 5.2, we can similarly obtain a very direct standard of co-quasi-ℓ-isomorphisms of ℓ-groups, as follows:

Corollary 5.1. Let G and H be two ℓ-groups. Then the following conditions are equivalent:

(1) $f : G \to H$ is a co-quasi-ℓ-isomorphism.

(2) $f : G \to H$ is an injective ℓ-homomorphism, and for any $L \in C(G)$, $L \in L(G)$ if and only if $f(L) \in \ell(H)$.

At the end of this paper, we establish a sufficient condition such that an injective ℓ-homomorphism $f : G \to H$ is a co-quasi-ℓ-isomorphism, which is completely dual to Theorem 4.3.

Theorem 5.3. Let G and H be two ℓ-groups and let $f : G \to H$ be an injective ℓ-homomorphism satisfying condition:

for any ℓ-group K, if there exists an injective ℓ-homomorphism $k : G \oplus K \to H$, then $K = 0$.

Then f is a co-quasi-ℓ-isomorphism.

Proof. Suppose on the contrary that $f(G)$ is not large in H; then there exists $0 \neq C \in C(H)$ such that $C \cap f(G) = 0$. Now, let $\Gamma = \{C \in C(H) : C \neq 0$ and $C \cap f(G) = 0\}$. An easy application of Zorn's Lemma shows that Γ has a maximal element, denoted by K. We claim that $f(G) \oplus K \in \ell(H)$. Otherwise, there exists some $0 \neq A \in C(G)$ such that $(f(G) \oplus K) \cap A = 0$. From which we can obtain that $(K \oplus A) \cap f(G) = 0$, which contradicts the fact that K is maximal in Γ. Since $f(G) \oplus K \in \ell(H)$, this will yield an injective ℓ-homomorphism

$$\tau : G \oplus K \to H \text{ defined by the rule } \tau(g, k) = f(g) + k \text{ for any } (g, k) \in G \oplus K.$$

A direct computation shows that this is well-defined and is indeed an injective ℓ-homomorphism. So, by assumption, we have $K = 0$, which contradicts the choice of K. Thus $f(G)$ is large in H. Therefore f is a co-quasi-ℓ-isomorphism. □

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