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TIKHONOV REGULARIZATION ALGORITHM FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. We establish a convergence theorem for the Tikhonov regularization algorithm applied to finite-dimensional pseudomonotone variational inequalities. Thus the open question stated by Facchinei and Pang in [1, p. 1129] is answered in affirmative. Several examples are given to analyze the obtained results.

1. Introduction

The simplest variational inequality (VI for brevity) is defined as follows.

Definition 1.1. *Given a nonempty closed convex subset K of the n-dimensional Euclidean space $\mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, the variational inequality defined by K and F, which is denoted by $\mathrm{VI}(K, F)$, is the problem of finding a vector $x \in K$ such that*

$$\langle F(x), y - x\rangle \geqslant 0 \ \ \forall y \in K.$$

The solution set of this problem is denoted by $SOL(K, F)$.

One often considers VIs with some additional properties imposed on the mapping F such as continuity, strong monotonicity, monotonicity, pseudomonotonicity and quasimonotonicity of F. Let us recall some well-known definitions.

Definition 1.2. *A mapping $F : K \to \mathbb{R}^n$ is said to be*

(a) *strongly monotone on K if*

$$\langle F(x) - F(y), x - y\rangle \geqslant c\|x - y\|^2$$

for all $x, y \in K$, where $c > 0$ is a constant;

(b) *monotone on K if*

$$\langle F(x) - F(y), x - y\rangle \geqslant 0$$

for all $x, y \in K$;

(c) *pseudomonotone on K if, for all $x, y \in K$, the inequality $\langle F(y), x - y\rangle \geqslant 0$ implies $\langle F(x), x - y\rangle \geqslant 0$;*

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(d) *quasimonotone on* K *if, for all* $x, y \in K$, *the inequality* $\langle F(y), x-y\rangle > 0$ *implies* $\langle F(x), x-y\rangle \geqslant 0$.

It is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). The reverse implications are not true in general. The following proposition describes the properties of the solution set of $\mathrm{VI}(K,F)$.

Theorem 1.1. *If* F *is continuous and pseudomonotone on* K, *then the solution set* $\mathrm{SOL}(K,F)$ *is closed convex. Furthermore, if* F *is strongly monotone on* K, *then* $\mathrm{SOL}(K,F)$ *is a singleton.*

The proof of this proposition can be found in many books (see for instance [1], [2]).

Consider the problem $\mathrm{VI}(K,F)$. For each $\varepsilon > 0$, we put $F_\varepsilon = F + \varepsilon I$, where I is the identity mapping in $\mathbb{R}^n$. If the problem $\mathrm{VI}(K,F_\varepsilon)$ has a unique solution, denoted by $x(\varepsilon)$, then the set $\{x(\varepsilon) : \varepsilon > 0\}$ is called *the Tikhonov trajectory* of $\mathrm{VI}(K,F)$. Under some suitable conditions, the limit $\lim\limits_{\varepsilon\to 0^+} x(\varepsilon)$ exists and it is a solution to $\mathrm{VI}(K,F)$. The procedure of solving the problem $\mathrm{VI}(K,F)$ via calculating a sequence of points $\{x(\varepsilon_k)\}$ ($k \in \mathbb{N}, \varepsilon_k \to 0^+$ as $k \to \infty$) in the Tikhonov trajectory and taking the limit $\lim\limits_{k\to\infty} x(\varepsilon_k)$ is called *the Tikhonov regularization method.*

The following theorem discusses the convergence of the Tikhonov regularization method in the case where F is continuous and monotone on K. Since $F_\varepsilon = F+\varepsilon I$ is a strongly monotone operator, by Theorem 1.1 (see also [1, Theorem 2.3.3]), the problem $\mathrm{VI}(K,F_\varepsilon)$ has a unique solution for every $\varepsilon > 0$.

Theorem 1.2 (see [1, Theorem 12.2.3]) *Let* $K \subset \mathbb{R}^n$ *be nonempty closed convex,* $F : K \longrightarrow \mathbb{R}^n$ *be continuous and monotone on* K. *Let* $\{x(\epsilon) : \epsilon > 0\}$ *be the Tikhonov trajectory. The following three statements are equivalent:*

(a) $\lim\limits_{\varepsilon\to 0^+} x(\varepsilon)$ *exists;*

(b) $\limsup\limits_{\varepsilon\to 0^+} \| x(\varepsilon) \| < \infty$;

(c) $\mathrm{SOL}(K,F)$ *is nonempty.*

Moreover, if any one of these statements holds, the limit in (a) *is equal to the least-norm solution of the problem* $\mathrm{VI}(K,F)$.

Remark 12.2.4 in the book [1] says: "*It is not clear if Theorem* 12.2.3 [Theorem 1.2 above] *will remain valid if* F *is pseudomonotone on* K. *In this case, although the* $\mathrm{VI}(K,F)$ *still possesses a unique least-norm solution, due to the convexity of* $\mathrm{SOL}(K,F)$ [provided that this set is nonempty], *the existence and uniqueness of the Tikhonov trajectory is in jeopardy. We leave this as an unresolved question*".

This question raised by Facchinei and Pang will be answered in the affirmative in the next section. Some useful examples will be given in the last section.

2. Convergence theorem

The main result of this paper can be formulated as follows.

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, $F : K \longrightarrow \mathbb{R}^n$ be a continuous and pseudomonotone operator on K. Then the following three statements are equivalent:*

(i) $\mathrm{SOL}(K, F_\varepsilon)$ *is nonempty for each* $\varepsilon > 0$ *and* $\lim_{\varepsilon \to 0^+} x(\varepsilon)$ *exists, where* $x(\varepsilon)$ *is arbitrarily chosen in* $\mathrm{SOL}(K, F_\varepsilon)$.

(ii) $\mathrm{SOL}(K, F_\varepsilon)$ *is nonempty for each* $\varepsilon > 0$ *and* $\limsup_{\varepsilon \to 0^+} \|x(\varepsilon)\| < \infty$*, where* $x(\varepsilon)$ *is arbitrarily chosen in* $\mathrm{SOL}(K, F_\varepsilon)$.

(iii) $\mathrm{SOL}(K, F)$ *is nonempty.*

If any one of these statements is valid, then the limit described in (i) is the least-norm element in $\mathrm{SOL}(K, F)$.

To prove this theorem we have to rely on the following solution existence theorems for VIs and pseudomonotone VIs which were established by Facchinei and Pang.

Lemma 2.1. (see [1, Prop. 2.2.3]) *Let $K \subset \mathbb{R}^n$ be nonempty closed convex and $F : K \longrightarrow \mathbb{R}^n$ be continuous on K. Consider the following statements:*

(a) There exists a reference vector $x^{\mathrm{ref}} \in K$ such that the set

$$L_< := \{x \in K : \langle F(x), x - x^{\mathrm{ref}} \rangle < 0\}$$

is bounded (possibly empty).

(b) There exist a bounded open set Ω and a vector $x^{\mathrm{ref}} \in K \cap \Omega$ such that

$$\langle F(x), x - x^{\mathrm{ref}} \rangle \geqslant 0 \quad \forall x \in K \cap \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω.

(c) The problem $\mathrm{VI}(K, F)$ *has a solution.*

It holds that $(a) \Rightarrow (b) \Rightarrow (c)$*. Moreover, if the set*

$$L_\leqslant := \{x \in K : \langle F(x), x - x^{\mathrm{ref}} \rangle \leqslant 0\}$$

which is nonempty and larger than $L_<$, is bounded, then $\mathrm{SOL}(K, F)$ *is nonempty and compact.*

Lemma 2.2. (see [1, Theorem 2.3.4]) *Let $K \subset \mathbb{R}^n$ be nonempty closed convex and $F : K \longrightarrow \mathbb{R}^n$ be continuous and pseudomonotone on K. Then the statements (a),(b), (c) in Lemma 2.1 are equivalent.*

Proof of Theorem 2.1. The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Since $\limsup_{\varepsilon \to 0^+} \|x(\varepsilon)\| < \infty$, the sequence $\{x(\varepsilon)\}_{\varepsilon>0}$ must have at least one accumulation point. Suppose that $x(\varepsilon_k) \to \overline{x}$, where $\overline{x} \in K$ and $\varepsilon_k \to 0^+$ as $k \to \infty$. Since

$$\langle F(x(\varepsilon_k)) + \varepsilon_k x(\varepsilon_k), y - x(\varepsilon_k)\rangle \geqslant 0 \quad \forall y \in K,$$

taking $k \to \infty$ and employing the continuity of F we get

$$\langle F(\overline{x}), y - \overline{x}\rangle \geqslant 0 \quad \forall y \in K.$$

This means $\overline{x} \in \mathrm{SOL}(K, F)$.

(iii) $\Rightarrow$ (i). Let us show that if F is pseudomonotone then $\mathrm{SOL}(K, F_\varepsilon)$ is nonempty for every $\varepsilon > 0$. To prove this, according to Lemma 2.1, it is sufficient to find a vector $x^{\mathrm{ref}} \in K$ such that the set

$$L^\varepsilon_< := \{x \in K : \langle F(x) + \varepsilon x, x - x^{\mathrm{ref}}\rangle < 0\}$$

is bounded. By the hypothesis (iii) and Lemma 2.2, there exist $x^{\mathrm{ref}} \in K$ and $M > 0$ such that $\|x\| \leqslant M$ for every x belonging to the set $L_< := \{x \in K : \langle F(x), x - x^{\mathrm{ref}}\rangle < 0\}$. For each $x \in L^\varepsilon_<$, we have $\langle F(x) + \varepsilon x, x - x^{\mathrm{ref}}\rangle < 0$ or, equivalently,

$$\langle F(x), x - x^{\mathrm{ref}}\rangle < \langle \varepsilon x, x^{\mathrm{ref}} - x\rangle.$$

If $\langle \varepsilon x, x^{\mathrm{ref}} - x\rangle \leqslant 0$, then $\langle F(x), x - x^{\mathrm{ref}}\rangle < 0$; that is $x \in L_<$, hence $\|x\| \leqslant M$. Otherwise, $\langle \varepsilon x, x^{\mathrm{ref}} - x\rangle > 0$. This implies $\langle x, x^{\mathrm{ref}}\rangle > \langle x, x\rangle$ and therefore we have $\|x\| \leqslant \|x^{\mathrm{ref}}\|$. Thus

$$\|x\| \leqslant \max\{M, \|x^{\mathrm{ref}}\|\} \quad \forall x \in L^\varepsilon_<,$$

and we can assert that $L^\varepsilon_<$ is bounded.

Now for an arbitrary sequence $\varepsilon_k \to 0^+$, we have $\mathrm{SOL}(K, F_{\varepsilon_k}) \neq \emptyset$ for each $k \in \mathbb{N}$. Take any $x_k = x(\varepsilon_k) \in \mathrm{SOL}(K, F_\varepsilon)$. Since $\mathrm{VI}(K, F)$ has a solution and F is continuous and pseudomonotone, $\mathrm{SOL}(K, F)$ must be nonempty closed and convex by Theorem 1.1. Therefore, the projection of $0 \in \mathbb{R}^n$ onto the set $\mathrm{SOL}(K, F)$, the least-norm element in $\mathrm{SOL}(K, F)$, is uniquely defined. We denote this element by $\overline{x}$.

Since $x_k \in \mathrm{SOL}(K, F_{\varepsilon_k})$ and $\overline{x} \in \mathrm{SOL}(K, F)$, we have

$$\langle F(x_k) + \varepsilon_k x_k, y - x_k\rangle \geqslant 0 \quad \forall y \in K$$

and

$$\langle F(\overline{x}), y - \overline{x}\rangle \geqslant 0 \quad \forall y \in K.$$

Substituting $y = \overline{x}$ into the first inequality and $y = x_k$ into the second one, we deduce that

$$\langle F(x_k) + \varepsilon_k x_k, \overline{x} - x_k\rangle \geqslant 0 \tag{2.1}$$

and

$$\langle F(\overline{x}), x_k - \overline{x}\rangle \geqslant 0. \tag{2.2}$$

By the pseudomonotonicity of F, from (2.2) it follows that

$$\langle F(x_k), x_k - \overline{x}\rangle \geqslant 0. \tag{2.3}$$

Adding the inequalities (2.1) and (2.3) we get

$$\langle \varepsilon_k x_k, \overline{x} - x_k \rangle \geqslant 0.$$

This implies $\|\overline{x}\| \geqslant \|x_k\|$ for all k. Hence $\{x_k\}$ is a bounded sequence, so it has a convergent subsequence $\{x_{k_j}\}$. Suppose that $x_{k_j} \to \widehat{x}$, $\widehat{x} \in K$. It is easy to see that $\|\widehat{x}\| \leqslant \|\overline{x}\|$. Moreover,

$$\langle F(x_{k_j}) + \varepsilon_{k_j} x_{k_j}, y - x_{k_j} \rangle \geqslant 0 \quad \forall y \in K.$$

Passing to the limit as $j \to \infty$, we get $\langle F(\widehat{x}), y - \widehat{x} \rangle \geqslant 0$ for all $y \in K$, which shows that $\widehat{x} \in \mathrm{SOL}(K, F)$. Since $\overline{x}$ is the unique least-norm element in $\mathrm{SOL}(K, F)$, from the inequality $\|\widehat{x}\| \leqslant \|\overline{x}\|$ we deduce that $\widehat{x} = \overline{x}$.

Since any subsequence $\{x_{\varepsilon_k}\}$ ($\varepsilon_k \to 0^+$) of $\{x_\varepsilon\}$ converges to $\overline{x}$, we conclude that $\lim\limits_{\varepsilon \to 0^+} x(\varepsilon) = \overline{x}$.

The proof is complete. □

The next statement provides some useful information about the solution set $\mathrm{SOL}(K, F_\varepsilon)$, $\varepsilon > 0$.

In this case, F is pseudomonotone on K and $\mathrm{SOL}(K, F) \neq \emptyset$.

Theorem 2.2. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, $F : K \longrightarrow \mathbb{R}^n$ be a continuous and pseudomonotone operator on K. If $SOL(K, F)$ is nonempty, then the following properties hold:*

(a) For any $\varepsilon > 0$, the set $SOL(K, F_\varepsilon)$ is nonempty and compact.

(b) $\lim\limits_{\varepsilon \to 0^+} \operatorname{diam} SOL(K, F_\varepsilon) = 0$, where $\operatorname{diam} M := \sup\{\|x - y\| : x \in M, y \in M\}$ denotes the diameter of a subset $M \subset \mathbb{R}^n$.

Proof.

(a) Applying the implication (iii) ⇒ (i) in Theorem 2.1, we see that $\mathrm{SOL}(K, F_\varepsilon) \neq \emptyset$ for any $\varepsilon > 0$. Take any $x^{\mathrm{ref}} \in \mathrm{SOL}(K, F)$. According to the last conclusion of Lemma 2.1, if the set

$$L^\varepsilon_{\leqslant} := \{x \in K \,:\, \langle F(x) + \varepsilon x, x - x^{\mathrm{ref}} \rangle \leqslant 0\}$$

is bounded, then $\mathrm{SOL}(K, F_\varepsilon)$ is a compact set. For any $x \in L^\varepsilon_{\leqslant}$ we have

$$\langle F(x), x - x^{\mathrm{ref}} \rangle \leqslant \varepsilon \langle x, x^{\mathrm{ref}} - x \rangle. \tag{2.4}$$

Since $x^{\mathrm{ref}} \in \mathrm{SOL}(K, F)$, it holds $\langle F(x^{\mathrm{ref}}), x - x^{\mathrm{ref}} \rangle \geqslant 0$. By the pseudomonotonicity of F, this implies $\langle F(x), x - x^{\mathrm{ref}} \rangle \geqslant 0$. From the last inequality and (2.4) it follows that

$$\langle x, x^{\mathrm{ref}} - x \rangle \geqslant 0.$$

So

$$\|x\| \|x^{\mathrm{ref}}\| \geqslant \langle x, x^{\mathrm{ref}} \rangle \geqslant \|x\|^2,$$

and we deduce that $\|x\| \leqslant \|x^{\mathrm{ref}}\|$. Thus $L^\varepsilon_{\leqslant}$ is bounded.

(b) For every $\varepsilon > 0$, by the property (a) there exist $x(\varepsilon), y(\varepsilon) \in \mathrm{SOL}(K, F_\varepsilon)$ such that

$$\|x(\varepsilon) - y(\varepsilon)\| = \operatorname{diam} \mathrm{SOL}(K, F_\varepsilon).$$

The last conclusion of Theorem 2.1 states that

$$\lim_{\varepsilon \to 0^+} x(\varepsilon) = \lim_{\varepsilon \to 0^+} y(\varepsilon) = \overline{x},$$

where $\overline{x}$ is the the least-norm element in $\mathrm{SOL}(K, F)$. It follows that

$$\lim_{\varepsilon \to 0^+} \operatorname{diam} \mathrm{SOL}(K, F_\varepsilon) = 0.$$

The proof is complete. □

3. Examples

In the case of monotone VIs, the operator F_ε of the perturbed problem $\mathrm{VI}(K, F_\varepsilon)$ is strongly monotone. So $\mathrm{VI}(K, F_\varepsilon)$ has a unique solution. Concerning pseudomonotone VIs, it is of interest to know *if the operator $F_\varepsilon = F + \varepsilon I$ is still pseudomonotone and if the problem $\mathrm{VI}(K, F_\varepsilon)$ has a unique solution.*

The following example shows that, in general, if F is pseudomonotone then there may exist $\varepsilon > 0$ such that F_ε is not pseudomonotone and $\mathrm{VI}(K, F_\varepsilon)$ has two distinct solutions.

Example 3.1. Let m be an arbitrary positive integer and $F : K = [-2, +\infty) \to \mathbb{R}$ be defined by $F(x) = \frac{1}{m}(x^2 + 1)$. It is easy to see that F is not monotone but pseudomonotone on K. For $\varepsilon = \frac{5}{2m}$, we have $F_\varepsilon(x) = \frac{1}{m}(x^2 + \frac{5}{2}x + 1)$. The operator F_ε is not pseudomonotone. Indeed, choose a value $\overline{x} \in (x_1, x_2)$ where $x_1 = -2,\ x_2 = -\frac{1}{2}$ are the roots of the trinomial $F_\varepsilon(x) = \frac{1}{m}(x^2 + \frac{5}{2}x + 1)$. Then we have

$$F_\varepsilon(\overline{x})(\overline{x} - x_1) < 0,$$

although $F_\varepsilon(x_1)(\overline{x} - x_1) = 0$. Moreover, the equalities $F_\varepsilon(x_1) = F_\varepsilon(x_2) = 0$ show that x_1 and x_2 are two distinct solutions of $\mathrm{VI}(K, F_\varepsilon)$.

The next question we want to clarify is *whether it is true that if F is pseudomonotone and continuous on K, then for any $\varepsilon > 0$ the problem $\mathrm{VI}(K, F_\varepsilon)$ has finitely many solutions*? The answer turns out to be negative.

Example 3.2. Let μ, a, b be positive real numbers. Let F be defined on $K = [a, b]$ by setting $F(x) = -\mu x$. Obviously, F is pseudomonotone on K and $\mathrm{SOL}(K, F) = \{b\}$. We have $F_\mu(x) = 0$ for all $x \in K$. So $\mathrm{SOL}(K, F_\mu) = K$, an infinite set.

In the above examples, if ε is chosen small enough then F_ε is pseudomonotone and the problem $\mathrm{VI}(K, F_\varepsilon)$ has a unique solution.

We conclude this section by the following

Open question. *If $K \subset \mathbb{R}^n$ is a nonempty closed convex set, $F : K \to \mathbb{R}^n$ a continuous pseudomonotone operator, and the problem $\mathrm{VI}(K, F)$ has a solution,* *then there exists* $\bar{\varepsilon} > 0$ *such that, for every* $\varepsilon \in (0, \bar{\varepsilon})$*, the operator* $F_\varepsilon := F + \varepsilon I$ *is pseudomonotone and the problem* $\mathrm{VI}(K, F_\varepsilon)$ *has a unique solution?*

NOTE ADDED IN REVISION: One part of the above question has been solved in negative by N. N. Tam, J.-C. Yao and N. D. Yen. Namely, they showed that there exists a continuously differentiable pseudomonotone operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{SOL}(\mathbb{R}^2, F) \neq \emptyset$ but, for any $\varepsilon > 0$, $F_\varepsilon = F + \varepsilon I$ is **not** a pseudomonotone operator; see N. N. Tam, J.-C. Yao and N. D. Yen (2006) *"On some solution methods for pseudomonotone variational inequalities"* (manuscript).

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