ACTA MATHEMATICA VIETNAMICA
Volume 31, Number 3, 2006, pp. 261-267

ON LEVY'S CONVERGENCE THEOREMS OF TWO-PARAMETER MULTIVALUED RANDOM PROCESSES

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Abstract. In this paper we prove the Levy's upward and downward theorems for the convergence of two-parameter multivalued random processes in Hausdorff's sense.

1. Introduction and preliminaries

The Levy's convergence theorems of one-parameter multivalued processes were presented in the works of Z. P. Wang and X. H. Xue [6], D. Wenlong and W. Zhenpeng [7]. In this paper we will extend these results to the two-parameter cases.

Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space and $\mathfrak{X}$ a real separable Banach space with norm $\|.\|$. Let $P_c(\mathfrak{X})$ denote the family of all nonempty bounded closed convex subsets of $\mathfrak{X}$. For $A, B, C \in P_c(\mathfrak{X})$, the Hausdorff distance $h(A, B)$, the radius $|C|$ of the set C are defined as in [4, 6]. The concepts and notations such as $\mathfrak{X}$-valued Bochner integrable random variables space $L^1(\Omega, \mathfrak{X})$, measurable multifunction, Aumann integral, conditional multivalued expectation, etc. are the same as in the above references.

For each $p \geqslant 1$, let $\mathcal{L}_c^p[\Omega, \mathfrak{X}]$ denote the family of measurable multifunction $F : \Omega \to P_c(\mathfrak{X})$ satisfying $\int_\Omega |F(\omega)|^p d\mathbb{P} < \infty$, where two multifunctions F, G are identical if $F(\omega) = G(\omega)$ a.e. Let $F, G \in \mathcal{L}_c^p[\Omega, \mathfrak{X}]$. Since

$$h^p(F(\omega), G(\omega)) \leqslant (|F(\omega)| + |G(\omega)|)^p \leqslant 2^{p-1}(|F(\omega)|^p + |G(\omega)|^p),$$

the function $\omega \mapsto h^p(F(\omega), G(\omega))$ is in $L^p(\mathbb{R})$ and we define

$$\Delta_p(F, G) = \Big(\int_\Omega h^p(F, G) d\mathbb{P} \Big)^{1/p}.$$

For $F, G, H \in \mathcal{L}_c^p[\Omega, \mathfrak{X}]$ we have

$$\begin{aligned}\Delta_p(F, G) &= \|h(F, G)\|_p \leqslant \|h(F, H) + h(H, G)\|_p \\ &\leqslant \|h(F, H)\|_p + \|h(H, G)\|_p = \Delta_p(F, H) + \Delta_p(H, G),\end{aligned}$$

Received February 13, 2006.
1991 *Mathematics Subject Classification.* Primary 60G48.
Key words and phrases. Two-parameter multivalued random processes, commuting filtration, maximal inequality.

then $(\mathcal{L}_c^p[\Omega, \mathfrak{X}], \Delta_p)$ is a metric space. A measurable multivalued function F is called *simple* if there exists a finite measurable partition $\{A_1, \ldots, A_n\}$ of Ω and nonempty subsets $X_1, \ldots, X_n$ of $\mathfrak{X}$ such that $F(\omega) = \sum_{i=1}^n I_{A_i}(\omega) X_i$ for all $\omega \in \Omega$. We denote by $\mathbb{L}_c^p[\Omega, \mathfrak{X}]$ the closure of the set of all simple functions in $\mathcal{L}_c^p[\Omega, \mathfrak{X}]$. It should be mention that $\mathbb{L}_c^p[\Omega, \mathfrak{X}] \subsetneq \mathcal{L}_c^p[\Omega, \mathfrak{X}]$ (see Example 3.4 [4]).

We denote by $\mathbb{N}$ (resp., $-\mathbb{N}$) the set of nonnegative (resp., nonpositive) integers. The ordering on $J = \mathbb{N} \times \mathbb{N}$ (resp., $K = (-\mathbb{N}) \times (-\mathbb{N})$) is defined as the natural one. Namely, for $s = (s_1, s_2)$ and $t = (t_1, t_2)$, we put $s \leqslant t$ whenever $s_1 \leqslant t_1$ and $s_2 \leqslant t_2$. We also denote $s \wedge t = (s_1 \wedge t_1, s_2 \wedge t_2)$. For each $n \in \mathbb{N}$ or $n \in -\mathbb{N}$, let $\overline{n} = (n, n)$. $\mathcal{F} = \{\mathcal{F}_t\}$ is a *two-parameter filtration of* Σ if for each t, $\mathcal{F}_t$ is a sub-σ-field of Σ and whenever $s \leqslant t$, $\mathcal{F}_s \subset \mathcal{F}_t$. $\mathcal{F}$ is *commuting* if for all s, t, $\mathcal{F}_s$ and $\mathcal{F}_t$ are conditionally independent, given $F_{s \wedge t}$.

Definition 1.1. A two-parameter process $\{M_t\}$ is a *martingale with respect to the filtration* $\mathcal{F}$ if for each t, M_t is $\mathcal{F}_t$-measurable, $\mathbb{E}|M_t| < \infty$ and whenever $s \leqslant t$, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ a.s.

In the sequel, we will frequently use the following well known result (see [1]).

Lemma 1.1. (Cairoli's Maximal inequality) *Let $\mathcal{F} = \{\mathcal{F}_t, t \in J\}$ be a commuting filtration and $M = \{M_{i,j} : i, j \in \mathbb{N}\}$ a two-parameter martingale. If $p > 1$, then for all $m, n \in \mathbb{N}$, we have*

$$\mathbb{E}\big(\max_{(i,j) \leqslant (n,m)} |M_{i,j}|^p\big) \leqslant \Big(\frac{p}{p-1}\Big)^{2p} \mathbb{E}|M_{n,m}|^p.$$

2. Main results

Theorem 2.1. *Suppose that $p > 1$, $F \in \mathbb{L}_c^p[\Omega, \mathfrak{X}]$ and $\mathcal{F} = \{\mathcal{F}_t, t \in J\}$ is a commuting filtration of Σ. Let $F_t = \mathcal{E}[F|\mathcal{F}_t]$ and $F_\infty = \mathcal{E}[F|\mathcal{F}_\infty]$ where $\mathcal{F}_\infty = \sigma\big(\cup_{n \in \mathbb{N}} \mathcal{F}_{\overline{n}}\big)$. Then F_t converges to F_∞ a.s. in the Hausdorff's sense.*

Proof. Without loss of generality, we may assume that F is $\mathcal{F}_\infty$-measurable. For any $\epsilon > 0$, pick a simple function $H \in \mathcal{L}_c^p$ such that H is $\mathcal{F}_\infty$-measurable and $\Delta_p(F, H) < \epsilon^2$. Assume that $H = \sum_{i=1}^K H_i I_{A_i}$, where $\{A_i : i = 1, \ldots, K\}$ is a measurable partition of Ω and $H_k \in P_c(\mathfrak{X})$. Pick $\delta > 0$ such that $\delta < \epsilon^{2p}\big(2^p \max_{1 \leqslant i \leqslant K} |H_i|^p\big)^{-1}$. Choose $n_1 < n_2 < \ldots < n_K$ such that for each i, there exists $B_i \in \mathcal{F}_{\overline{n}_i}$ satisfying $\mathbb{P}(A_i \Delta B_i) < \delta/2K$. Let

$$C_i = B_i \backslash \big(\bigcup_{1 \leqslant j < i} B_j\big),\ 1 \leqslant i < K, \quad C_K = \Omega \backslash \bigcup_{1 \leqslant i < K} C_i,$$

and $G(\omega) = \sum_{i=1}^K H_i I_{C_i}$. Then

$$[\Delta_p(G, H)]^p = \mathbb{E}(h^p(G, H)) = \sum_{j,i=1}^K \mathbb{E}\big(h^p(G, H) I_{C_i \cap A_j}\big)$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{K} \mathbb{E}(h^p(H_i, H_j) I_{C_i \cap A_j}) \\
&\leqslant \sum_{j,i=1, j\neq i}^{K} \mathbb{E}\big(2^{p-1}(|H_i|^p + |H_j|^p) I_{C_i \cap A_j}\big) \\
&\leqslant 2^p \max_{1\leqslant i\leqslant K} |H_i|^p \sum_{j=1}^{K} \mathbb{P}\big(C_j \cap \bigcup_{i\neq j} A_i\big) \\
&= 2^p \max_{1\leqslant i\leqslant K} |H_i|^p \Big[\sum_{j=1}^{K-1} \mathbb{P}(C_j \cap A_j^c) + \mathbb{P}(C_K \cap A_K^c)\Big]. \qquad (2.1)
\end{aligned}
$$

For $1 \leqslant j < K-1$,

$$
\mathbb{P}(C_j \cap A_j^c) \leqslant \mathbb{P}(B_j \cap A_j^c) \leqslant \mathbb{P}(B_j \Delta A_j) \leqslant \frac{\delta}{2K}. \qquad (2.2)
$$

Since

$$
C_K = \Omega \backslash \underset{1\leqslant i<K}{\cup} C_i = \Omega \backslash \underset{1\leqslant i<K}{\cup} B_i
$$

we have

$$
(C_K \cap A_K^c)^c = C_K^c \cup A_K = (\underset{1\leqslant i<K}{\cup} B_i) \cup A_K
$$

and

$$
\begin{aligned}
\mathbb{P}\big((\underset{1\leqslant i<K}{\cup} B_i) \cup A_K\big) &\geqslant \mathbb{P}\big((\underset{1\leqslant i<K}{\cup} B_i A_i) \cup A_K\big) = \sum_{1\leqslant i<K} \mathbb{P}(A_i B_i) + \mathbb{P}(A_K) \\
&\geqslant \mathbb{P}(A_K) + \sum_{1\leqslant i<K} \Big(\mathbb{P}(A_i) - \mathbb{P}(A_i \Delta B_i)\Big) \\
&= 1 - \sum_{1\leqslant i<K} \mathbb{P}(A_i \Delta B_i) \\
&\geqslant 1 - (K-1)\frac{\delta}{2K}.
\end{aligned}
$$

Therefore

$$
\mathbb{P}(C_K \cap A_K^c) \leqslant \frac{(K-1)\delta}{2K}. \qquad (2.3)
$$

According to (2.1), (2.2) and (2.3), we have

$$
[\Delta_p(G,H)]^p \leqslant 2^p \max_{1\leqslant i\leqslant K} |H_i|^p \; \frac{2(K-1)\delta}{2K} \leqslant \epsilon^{2p}.
$$

Then, $\Delta_p(G,H) \leqslant \epsilon^2$ and

$$
\Delta_p(G,F) \leqslant \Delta_p(G,H) + \Delta_p(H,F) \leqslant 2\epsilon^2. \qquad (2.4)
$$

For any $t \geqslant \overline{n}_k$, by Lemma 2.6 of [3], we have

$$
h(F_t, G) = h\big(\mathcal{E}[F|\mathcal{F}_t], \mathcal{E}[G|\mathcal{F}_t]\big) \leqslant \mathbb{E}(h(F,G)|\mathcal{F}_t) = h_t.
$$

For any $m > n_k$, using Markov's inequality, Lemma 1.2, Jensen's inequality and (2.4) we have

$$\begin{aligned}\mathbb{P}\big(\sup_{\overline{n}_k \leqslant t \leqslant \overline{m}} h_t > \epsilon\big) &\leqslant \frac{1}{\epsilon^p}\mathbb{E}\big(\sup_{\overline{n}_k \leqslant t \leqslant \overline{m}} h_t^p\big) \leqslant \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\mathbb{E}(h_{\overline{m}}^p)\\ &= \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\mathbb{E}\Big(\mathbb{E}^p(h(F,G)|\mathcal{F}_{\overline{m}})\Big)\\ &\leqslant \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\mathbb{E}\Big(\mathbb{E}(h^p(F,G)|\mathcal{F}_{\overline{m}})\Big)\\ &= \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\mathbb{E}h^p(F,G) \leqslant (2\epsilon)^p\Big(\frac{p}{p-1}\Big)^{2p}.\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\mathbb{P}\big(\sup_{t \geqslant \overline{n}_k} h_t > \epsilon\big) \leqslant (2\epsilon)^p\Big(\frac{p}{p-1}\Big)^{2p}.$$

Finally, we have

$$\begin{aligned}\mathbb{P}\big(\sup_{t\geqslant \overline{n}_k} h(F_t,F) > 2\epsilon\big) &\leqslant \mathbb{P}\big(\sup_{t\geqslant \overline{n}_k} h(F_t,G) > \epsilon\big) + \mathbb{P}\big(h(F,G) > \epsilon\big)\\ &\leqslant \mathbb{P}\big(\sup_{t\geqslant \overline{n}_k} h_t > \epsilon\big) + \frac{\mathbb{E}h^p(F,G)}{\epsilon^p}\\ &\leqslant (2\epsilon)^p\Big(1 + \big(\frac{p}{p-1}\big)^{2p}\Big),\end{aligned}$$

and by Lemma 2 of [5], we obtain that $h - \lim F_t = F$ a.s. The theorem is proved. ☐

We have proved the upward case of Levy's convergence theorem. For the downward case, we need first the following technical lemma.

Lemma 2.1. *Let M be a square integrable random variable, $\mathcal{F} = \{\mathcal{F}_t : t \in K\}$ a commuting filtration. Then $\mathbb{E}(M|\mathcal{F}_t) \to \mathbb{E}(M|\mathcal{F}_{-\infty})$ a.s. where $\mathcal{F}_{-\infty} = \bigcap_{n=-1}^{-\infty} \mathcal{F}_{\overline{n}}$.*

Proof. To prove this convergence, we consider the set

$$\mathbb{G} = \{X \in \mathrm{Ł}^2(\mathbb{R}) : \mathbb{E}(X|\mathcal{F}_t) = \mathbb{E}(X|\mathcal{F}_{-\infty}) \text{ for some } t \in K\}.$$

Then, we claim that the closed linear spand of $\mathbb{G}$ is all of $L^2(\mathbb{R})$. Suppose that there exists a random variable $Y \in \mathrm{Ł}^2(\mathbb{R})$ such that $Y \perp \mathbb{G}$, it means that $\mathbb{E}XY = 0$ for all $X \in \mathbb{G}$. For each $t \in K$, since $X = Y - \mathbb{E}(Y|\mathcal{F}_t) \in \mathbb{G}$ then we have

$$\mathbb{E}(Y(Y - \mathbb{E}(Y|\mathcal{F}_t)) = 0 \Leftrightarrow \mathbb{E}(Y - \mathbb{E}(Y|\mathcal{F}_t))^2 = 0.$$

It implies that Y is $\mathcal{F}_t$-measurable. Since this is true for all $t \in K$, Y is $\mathcal{F}_{-\infty}$-measurable. On the other hand, $\mathbb{G}$ contains all $\mathcal{F}_{-\infty}$-measurable random variables, so $Y \perp \mathbb{G}$ implies $Y = 0$.

Next, we show that the set

$$\mathbb{H} = \{X \in \mathrm{Ł}^2(\mathbb{R}) : \mathbb{E}(X|\mathcal{F}_t) \to \mathbb{E}(X|\mathcal{F}_{-\infty}) \text{ a.s.}\}$$

is a closed linear space of $\mathrm{Ł}^2(\mathbb{R})$. Indeed, for any random variable X in the closed hull of $\mathbb{H}$ and $\epsilon > 0$, there exists $Y \in \mathbb{H}$ such that $\|X - Y\|_2^2 < \epsilon^3/4$. For each $t \in K$, put $X_t = \mathbb{E}(X|\mathcal{F}_t)$, $Y_t = \mathbb{E}(Y|\mathcal{F}_t)$. Then for any $n \in -\mathbb{N}$, we have

$$\begin{aligned}\mathbb{P}\big(\sup_{\overline{n}\leqslant t\leqslant \overline{-1}} |X_t - Y_t| \geqslant \epsilon\big) &\leqslant \frac{1}{\epsilon^2}\mathbb{E}\big(\sup_{\overline{n}\leqslant t\leqslant \overline{-1}} |X_t - Y_t|^2\big)\\ &\leqslant \frac{4}{\epsilon^2}\mathbb{E}|X_{\overline{-1}} - Y_{\overline{-1}}|^2 \leqslant \frac{4}{\epsilon^2}\|X - Y\|_2^2 \leqslant \epsilon.\end{aligned}$$

Letting n tend to $-\infty$, we have

$$\mathbb{P}\big(\sup_t |X_t - Y_t| \geqslant \epsilon\big) \leqslant \epsilon.$$

On the other hand,

$$\begin{aligned}\sup_t |\mathbb{E}(Y_t|\mathcal{F}_{-\infty}) - \mathbb{E}(X|\mathcal{F}_{-\infty})| &= |\mathbb{E}(Y|\mathcal{F}_{-\infty}) - \mathbb{E}(X|\mathcal{F}_{-\infty})|\\ &= |\mathbb{E}(Y - X|\mathcal{F}_{-\infty})| \leqslant \mathbb{E}(|Y - X||\mathcal{F}_{-\infty}).\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{P}\big(\sup_t |\mathbb{E}(Y_t|\mathcal{F}_{-\infty}) - \mathbb{E}(X|\mathcal{F}_{-\infty})| \geqslant \epsilon\big) &\leqslant \mathbb{P}\big(\mathbb{E}(|Y - X||\mathcal{F}_{-\infty}) \geqslant \epsilon\big)\\ &\leqslant \frac{1}{\epsilon^2}\|Y - X\|_2^2 \ \leqslant \ \epsilon.\end{aligned}$$

Since $Y_t \to \mathbb{E}(Y|\mathcal{F}_{-\infty}) = \mathbb{E}(Y_t|\mathcal{F}_{-\infty})$ a.s., there exists $t_0 \in K$ such that

$$\mathbb{P}\big(\sup_{t\leqslant t_0} |Y_t - \mathbb{E}(Y_t|\mathcal{F}_{-\infty})| \geqslant \epsilon\big) \leqslant \epsilon.$$

Hence

$$\mathbb{P}\big(\sup_{t\leqslant t_0} |X_t - \mathbb{E}(X|\mathcal{F}_{-\infty})| \geqslant 3\epsilon\big) \leqslant 3\epsilon,$$

which implies that $X_t \to \mathbb{E}(X|\mathcal{F}_{-\infty})$ a.s., so $X \in \mathbb{H}$ and $\mathbb{H}$ is closed.

Moreover, $\mathbb{G} \subset \mathbb{H}$ then $\mathrm{Ł}^2(\mathbb{R}) = \overline{\mathbb{G}} \subset \mathbb{H} \subset \mathrm{Ł}^2(\mathbb{R})$, it implies that $\mathbb{H} = \mathrm{Ł}^2(\mathbb{R})$. The proof is complete. □

Theorem 2.2. *Suppose that $\mathcal{F}$ is a commuting filtration, $F \in \mathbb{L}_c^p$, $F_t = \mathcal{E}[F|\mathcal{F}_t]$, $t \in K$. Then $F_t \xrightarrow{h} F_{-\infty}$ a.s., where $F_{-\infty} = \mathcal{E}[F|\mathcal{F}_{-\infty}]$.*

Proof. Without loss of generality we may assume that F is $\mathcal{F}_{(-1,-1)}$-measureable. First, we suppose that F is a simple function in $\mathcal{L}_c^p$, i.e. $F = \sum_{i=1}^k H_i I_{A_i}$, where (A_i) is a measurable partition of Ω and $H_k \in P_c(\mathfrak{X})$. For any $F_1, F_2, G_1, G_2 \in P_c(\mathfrak{X})$ it is known that

$$h(F_1 + F_2, G_1 + G_2) \leqslant h(F_1, G_1) + h(F_2, G_2).$$

Thus, by Lemma 2.2 we have

$$
\begin{aligned}
h(F_t, F_{-\infty}) &= h\Big(\sum_{i=1}^{k} H_i\mathbb{E}(I_{A_i}|\mathcal{F}_t), \sum_{i=1}^{k} H_i\mathbb{E}(I_{A_i}|\mathcal{F}_{-\infty})\Big) \\
&\leqslant \sum_{i=1}^{k} h(H_i\mathbb{E}(I_{A_i}|\mathcal{F}_t), H_i\mathbb{E}(I_{A_i}|\mathcal{F}_{-\infty})) \\
&\leqslant \big(\max_{1\leqslant i\leqslant K}|H_i|\big)\sum_{i=1}^{k}|\mathbb{E}(I_{A_i}|\mathcal{F}_t) - \mathbb{E}(I_{A_i}|\mathcal{F}_{-\infty})| \longrightarrow 0 \text{ a.s.}
\end{aligned}
$$

Now, we suppose that $F \in \mathbb{L}_c^p$. For any $\epsilon > 0$, there exists a simple function $H \in \mathcal{L}_c^p$ such that H is $\mathcal{F}_{(-1,-1)}$-measurable and $\Delta_p(F, H) \leqslant \big(\frac{p-1}{p}\big)^2\epsilon^{(p+1)/p}$. Suppose that $H = \sum_{i=1}^{k} H_i I_{A_i}$, where (A_i) is a $\mathcal{F}_{(-1,-1)}$-measurable partition of Ω and $H_i \in P_c(\mathfrak{X})$. For each $t \in K$, denote $H_t = \mathcal{E}[H|\mathcal{F}_t]$ and $H_{-\infty} = \mathcal{E}[H|\mathcal{F}_{-\infty}]$. Since $h(H_t, H_{-\infty}) \to 0$ a.s, there exists $t_0 \in K$ such that

$$
\mathbb{P}\Big(\sup_{t\leqslant t_0}(h(H_t, H_{-\infty})) \geqslant \epsilon\Big) < \epsilon.
$$

For any $t \in K$, we have

$$
\begin{aligned}
h(F_t, H_t) &= h(\mathcal{E}[F|\mathcal{F}_t], \mathcal{E}[H|\mathcal{F}_t]) \leqslant \mathbb{E}(h(F, H)|\mathcal{F}_t) = h_t, \\
h(F_{-\infty}, H_{-\infty}) &= h(\mathcal{E}[F|\mathcal{F}_{-\infty}], \mathcal{E}[H|\mathcal{F}_{-\infty}]) \leqslant \mathbb{E}(h(F, H)|\mathcal{F}_{-\infty}) = h_{-\infty}.
\end{aligned}
$$

Since $\{h_t, \mathcal{F}_t, \overline{n} \leqslant t \leqslant \overline{-1}\}$ is a real martingale for any $n \in -\mathbb{N}$, by Lemma 1.2 we have

$$
\begin{aligned}
\mathbb{P}\Big(\max_{\overline{n}\leqslant t\leqslant\overline{-1}} h_t \geqslant \epsilon\Big) &\leqslant \frac{1}{\epsilon^p}\mathbb{E}\big(\max_{\overline{n}\leqslant t\leqslant\overline{-1}} h_t^p\big) \leqslant \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\mathbb{E}(h_{\overline{-1}}^p) \\
&\leqslant \frac{1}{\epsilon^p}\Big(\frac{p}{p-1}\Big)^{2p}\Delta_p^p(F, H) \leqslant \epsilon.
\end{aligned}
$$

Letting $n \to -\infty$, we obtain $\mathbb{P}\big(\sup_t h_t \geqslant \epsilon\big) \leqslant \epsilon$. Moreover,

$$
\mathbb{P}\big(h(F_{-\infty}, H_{-\infty}) \geqslant \epsilon\big) \leqslant \frac{1}{\epsilon^p}\mathbb{E}(h_{-\infty}^p) \leqslant \frac{1}{\epsilon^p}\Delta_p^p(F, H) < \epsilon.
$$

Then, for any $\epsilon > 0$, there exists a $t_1 \in K$ such that

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{t\leqslant t_1} h(F_t, F_{-\infty}) \geqslant 3\epsilon\Big) \leqslant& \mathbb{P}\Big(\sup_{t\leqslant t_1} h(F_t, H_t) \geqslant \epsilon\Big) + \mathbb{P}\Big(\sup_{t\leqslant t_1} h(H_t, H_{-\infty}) \geqslant \epsilon\Big) \\
&+ \mathbb{P}\big(h(H_{-\infty}, F_{-\infty}) \geqslant \epsilon\big) < 3\epsilon,
\end{aligned}
$$

which give $F_t \to F_{-\infty}$ a.s. The theorem is proved. □

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