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ENDPOINT BOUNDEDNESS FOR SOME MULTILINEAR OPERATORS

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Abstract. In this paper, we prove the endpoint estimates for some multilinear operators related to the Littlewood-Paley operator and Marcinkiewiecz integral operator.

1. Introduction and theorems

In this paper, we will consider a class of multilinear operators related to some integral operators, whose definitions are given below.

Fix any $\delta > 0$. Let $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$. Denote the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} D^\alpha A(x)(x-y)^\alpha.$$

Definition 1.1. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leqslant C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ whenever $2|y| < |x|$;

The multilinear Littlewood-Paley operator is defined by

$$g_S^A(f)(x) = \Big[\iint_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\Big]^{1/2},$$

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where

$$F_t^A(f)(x,y)=\int\limits_{R^n}\frac{R_{m+1}(A;x,z)}{|x-z|^m}f(z)\psi_t(y-z)dz$$

and $\psi_t(x)=t^{-n+\delta}\psi(x/t)$ for $t>0$. Set $F_t(f)(y)=f*\psi_t(y)$. We put

$$g_S(f)(x)=\Big(\iint\limits_{\Gamma(x)}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

which is the Littlewood-Paley S operator (see [13]).

Consider the variant of g_S^A, which is defined by

$$\tilde{g}_S^A(f)(x)=\Big[\iint\limits_{\Gamma(x)}|\tilde{F}_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\Big]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y)=\int\limits_{R^n}\frac{Q_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz.$$

Let H be the Hilbert space defined by

$$H=\Big\{h:\|h\|=\Big(\iint\limits_{R_+^{n+1}}|h(t)|^2dydt/t^{n+1}\Big)^{1/2}<\infty\Big\}.$$

Then for each fixed $x\in R^n$, $F_t^A(f)(x,y)$ can be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$g_S^A(f)(x)=\left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|,$$
$$\tilde{g}_S^A(f)(x)=\left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\right\|,$$
$$g_S(f)(x)=\left\|\chi_{\Gamma(x)}F_t(f)(y)\right\|.$$

Definition 1.2. Let $0<\gamma\leqslant 1$ and Ω be homogeneous of degree zero on R^n such that $\int\limits_{S^{n-1}}\Omega(x')d\sigma(x')=0$. Assume that $\Omega\in Lip_\gamma(S^{n-1})$, that is there exists a constant $M>0$ such that for any $x,y\in S^{n-1}$, $|\Omega(x)-\Omega(y)|\leqslant M|x-y|^\gamma$. The multilinear Marcinkiewicz operator and its variant are defined by

$$\mu_S^A(f)(x)=\Big[\iint\limits_{\Gamma(x)}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+3}}\Big]^{1/2}$$

and

$$\tilde{\mu}_S^A(f)(x)=\Big[\iint\limits_{\Gamma(x)}|\tilde{F}_t^A(f)(x,y)|^2\frac{dydt}{t^{n+3}}\Big]^{1/2},$$

where

$$F_t^A(f)(x,y)=\int\limits_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\frac{R_{m+1}(A;x,z)}{|x-z|^m}f(z)dz$$

and

$$\tilde{F}_t^A(f)(x,y) = \int\limits_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)dz.$$

Set

$$F_t(f)(y) = \int\limits_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

We define

$$\mu_S(f)(x) = \Big(\iint\limits_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\Big)^{1/2},$$

which is the Marcinkiewicz operator (see [14]).

Let H be the Hilbert space defined by

$$H = \Big\{h : \|h\| = \Big(\iint\limits_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+3}\Big)^{1/2} < \infty\Big\}.$$

Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ can be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right\|,$$
$$\tilde{\mu}_S^A(f)(x) = \left\|\chi_{\Gamma(x)} \tilde{F}_t^A(f)(x,y)\right\|,$$
$$\mu_S(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|.$$

Note that when $m = 0$ and $\delta = 0$, g_S^A and μ_S^A are just the commutators of F_t and A (see[10], [11] and [14]). Let T be the Calderon-Zygmund singular integral operator. The classical result of Coifman, Rochberg and Weiss [6] states that the commutator

$$[b,T] = T(bf) - bTf$$

(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$, Chanillo [1] proves a similar result when T is replaced by a fractional integral operator. In [9], the boundedness properties of the commutators for the extreme values of p are obtained. It is well known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3]-[5], [7]). The purpose of this paper is to discuss the endpoint estimates of the multilinear operators g_S^A and μ_S^A.

First, let us introduce some notations (see [8], [12]).

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let

$$f_Q = |Q|^{-1} \int\limits_Q f(x)dx$$

and
$$f^{\#}(x)=\sup_{x\in Q}|Q|^{-1}\int\limits_{Q}|f(y)-f_Q|dy.$$
Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#}\in L^{\infty}(R^n)$. Let $\|f\|_{BMO}=\|f^{\#}\|_{L^{\infty}}$.

Now we consider the concepts of the atom and H^1 space. A function a is called a $H^1(R^n)$ atom if there exists a cube Q such that a is supported on Q, $\|a\|_{L^{\infty}}\leqslant |Q|^{-1}$ and $\int a(x)dx=0$. It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see[8], [12]).

We shall prove the following theorems in Section 2.

Theorem 1.1. *Let $0<\delta<n$ and $D^{\alpha}A\in BMO(R^n)$ for $|\alpha|=m$.*

(i) *If for any H^1-atom a supported on certain cube Q and $u\in 3Q\setminus 2Q$, there is a constant C such that*
$$\int\limits_{(4Q)^c}\Big\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^{\alpha}}{|x-u|^m}\psi_t(y-u)\int\limits_{Q}D^{\alpha}A(z)a(z)dz\Big\|^{n/(n-\delta)}dx\leqslant C,$$
then g_S^A is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$;

(ii) *If for any cube Q and $u\in 3Q\setminus 2Q$, there is a constant C such that*
$$\frac{1}{|Q|}\int\limits_{Q}\Big\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^{\alpha}A(x)-(D^{\alpha}A)_Q)\int\limits_{(4Q)^c}\frac{(u-z)^{\alpha}}{|u-z|^m}\psi_t(u-z)f(z)dz\Big\|dx$$
$$\leqslant C\|f\|_{L^{n/\delta}},$$
then $\tilde{g}_S^A$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.

Theorem 1.2. *Let $0<\delta<n$ and $D^{\alpha}A\in BMO(R^n)$ for $|\alpha|=m$.*

(i) *If for any H^1-atom a supported on certain cube Q and $u\in 3Q\setminus 2Q$, there is a constant C such that*
$$\int\limits_{(4Q)^c}\Big\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^{\alpha}}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\int\limits_{Q}D^{\alpha}A(z)a(z)dz\quad\Big\|^{n/(n-\delta)}dx$$
$$\leqslant C,$$
then μ_S^A is bounded from $H^1(R^n)$ to $L^{n/\delta}(R^n)$;

(ii) *If for any cube Q and $u\in 3Q\setminus 2Q$, there is a constant C such that*
$$\frac{1}{|Q|}\int\limits_{Q}\Big\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^{\alpha}A(x)-(D^{\alpha}A)_Q)$$
$$\times\int\limits_{(4Q)^c}\frac{(u-z)^{\alpha}}{|u-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}}f(z)dz\Big\|dx\leqslant C\|f\|_{L^{n/\delta}},$$
then $\tilde{\mu}_S^A$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.

2. Proofs of Theorems 1.1 and 1.2

We begin with two lemmas.

Lemma 2.1. (see [5]) *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leqslant C|x-y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \Big)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2. *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then g_S^A and μ_S^A are all bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof. For g_S^A, by Minkowski inequality and the condition on ψ, we get

$$g_S^A(f)(x) \leqslant \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \Big(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \Big)^{1/2} dz$$

$$\leqslant C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \Big(\int_0^\infty \int_{|x-y|\leqslant t} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \frac{dydt}{t^{1+n}} \Big)^{1/2} dz$$

$$\leqslant C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \Big(\int_0^\infty \int_{|x-y|\leqslant t} \frac{2^{2n+2} t^{1-n}}{(2t+|y-z|)^{2n+2-2\delta}} dydt \Big)^{1/2} dz.$$

Noting that $2t + |y-z| \geqslant 2t + |x-z| - |x-y| \geqslant t + |x-z|$ when $|x-y| \leqslant t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$g_S^A(f)(x) \leqslant C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \Big(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} \Big)^{1/2} dz$$
$$= C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^{m+n-\delta}} dz;$$

For μ_S^A, since $|x-z| \leqslant 2t$, $|y-z| \geqslant |x-z|-t \geqslant |x-z|-3t$ whenever $|x-y| \leqslant t$, $|y-z| \leqslant t$, we get

$$
\begin{aligned}
\mu_S^A(f)(x) &\leqslant \int_{R^n} \Big[\iint_{|x-y|\leqslant t} \Big(\frac{|\Omega(y-z)\|R_{m+1}(A;x,z)\|f(z)|}{|y-z|^{n-1-\delta}|x-z|^m} \Big)^2 \chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}} \Big]^{1/2} dz \\
&\leqslant C \int_{R^n} \frac{|R_{m+1}(A;x,z)\|f(z)|}{|x-z|^m} \Big[\iint_{|x-y|\leqslant t} \frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}}{(|x-z|-3t)^{2n-2-2\delta}} dydt \Big]^{1/2} dz \\
&\leqslant C \int_{R_n} \frac{|R_{m+1}(A;x,z)\|f(z)|}{|x-z|^{m+3/2}} \Big[\int_{|x-z|/2}^{\infty} \frac{dt}{(|x-z|-3t)^{2n-2-2\delta}} \Big]^{1/2} dz \\
&\leqslant C \int_{R_n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} |f(z)| dz.
\end{aligned}
$$

Thus, the lemma follows from [7]. □

Proof of Theorem 1.1.

(i) It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a with supp $a \subset Q = Q(x_0, d)$,

$$\|g_S^A(a)\|_{L^{n/(n-\delta)}} \leqslant C.$$

Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and

$$D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_Q$$

for all α with $|\alpha| = m$. We write, by the vanishing moment of a and for $u \in 3Q \setminus 2Q$,

$$
\begin{aligned}
&F_t^A(a)(x,y) \\
=&\chi_{4Q}(x)F_t^A(a)(x,y) \\
&+ \chi_{(4Q)^c}(x) \int_{R^n} \Big[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-y|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m} \Big] a(z)dz \\
&- \chi_{(4Q)^c}(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-u)(x-u)^\alpha}{|x-u|^m} \Big] D^\alpha \tilde{A}(z)a(z)dz \\
&- \chi_{(4Q)^c}(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-u)^\alpha}{|x-u|^m} \psi_t(y-u) D^\alpha \tilde{A}(z)a(z)dz,
\end{aligned}
$$

then

$$\begin{aligned}
g_S^A(a)(x) &= \Big\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\Big\| \\
&\leqslant \chi_{4Q}(x)\Big\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\Big\| \\
&\quad + \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}(y,t)\int_{R^n}\Big[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m} \\
&\qquad\qquad - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\Big]a(z)dz\Big\| \\
&\quad + \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} \\
&\qquad\qquad - \frac{\psi_t(y-u)(x-u)^\alpha}{|x-u|^m}\Big]D^\alpha\tilde{A}(z)a(z)dz\Big\| \\
&\quad + \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-u)D^\alpha\tilde{A}(z)a(z)dz\Big\| \\
&= I_1(x) + I_2(x,u) + I_3(x,u) + I_4(x,u).
\end{aligned}$$

By the (L^p, L^q)-boundedness of g_S^A for $n/(n-\delta) < q$ and $1/q = 1/p - \delta/n$ (see Lemma 2.1), we get

$$\|I_1(\cdot)\|_{L^{n/(n-\delta)}} \leqslant \|g_S^A(a)\|_{L^q}|4Q|^{(n-\delta)/n-1/q} \leqslant C\|a\|_{L^p}|Q|^{1-1/p} \leqslant C.$$

For $I_2(x,u)$, we write

$$\begin{aligned}
&\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m} \\
&= \left[\frac{1}{|x-z|^m} - \frac{1}{|x-u|^m}\right] R_m(\tilde{A};x,z)\psi_t(y-z) \\
&\quad + (\psi_t(y-z)-\psi_t(y-u))\frac{R_m(\tilde{A};x,z)}{|x-u|^m} + \frac{\psi_t(y-u)}{|x-u|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x,u)]
\end{aligned}$$

Note that $|x-z| \sim |x-u| \sim |x-x_0|$ for $z \in Q$ and $x \in R^n \setminus 4Q$. By Lemma 2.1 and the following inequality (see [12])

$$|b_{Q_1} - b_{Q_2}| \leqslant C\log(|Q_2|/|Q_1|)\|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $z \in Q$ and $x \in 2^{k+1}Q \setminus 2^kQ$,

$$\begin{aligned}
|R_m(\tilde{A};x,z)| &\leqslant C|x-z|^m\sum_{|\alpha|=m}(\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{Q(x,z)} - (D^\alpha A)_Q|) \\
&\leqslant Ck|x-z|^m\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO};
\end{aligned}$$

By the formula

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x,u)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};z,u)(x-z)^\beta$$

(see [5]) and Lemma 2.1, we have

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,u)|\leqslant C\sum_{|\beta|<m}\sum_{|\alpha|=m}|z-u|^{m-|\beta|}|x-z|^{|\beta|}\|D^\alpha A\|_{BMO}.$$

Thus, arguing similarly as in the proof of Lemma 2.2, we get

$$\begin{aligned}
\|I_2(\cdot,u)\|_{L^{\frac{n}{(n-\delta)}}} &\leqslant C\sum_{k=2}^{\infty}\Bigg[\int_{2^{k+1}Q\setminus 2^kQ}\Big(\int_Q\Big(\frac{|z-u|}{|x-z|^{m+n+1-\delta}}+\frac{|z-u|^\varepsilon}{|x-z|^{m+n+\varepsilon-\delta}}\Big)\\
&\qquad\times|R_m(\tilde{A};x,z)\||a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Bigg]^{\frac{(n-\delta)}{n}}\\
&\quad+\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\Big[\int_{2^{k+1}Q\setminus 2^kQ}\Big(\int_Q\frac{|z-u|}{|x-z|^{n+1-\delta}}|a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\Big[\int_{2^{k+1}Q\setminus 2^kQ}\Big(\int_Q k\Big(\frac{|z-u|}{|x-z|^{n+1-\delta}}\\
&\qquad+\frac{|z-u|^\varepsilon}{|x-z|^{n+\varepsilon-\delta}}\Big)|a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\Big[\int_{2^{k+1}Q\setminus 2^kQ}k\Big(\frac{d}{(2^kd)^{n+1-\delta}}\\
&\qquad+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon-\delta}}\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\|a\|_{L^\infty}|Q|\\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-\varepsilon k})\leqslant C.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|I_3(\cdot,u)\|_{L^{\frac{n}{(n-\delta)}}} &\leqslant C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\Big[\int_{2^{k+1}Q\setminus 2^kQ}\Big(\int_Q\Big(\frac{|z-u|}{|x-z|^{n+1-\delta}}\\
&\qquad+\frac{|z-u|^\varepsilon}{|x-z|^{n+\varepsilon-\delta}}\Big)|D^\alpha\tilde{A}(z)\||a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}
\end{aligned}$$

$$\leqslant C \sum_{|\alpha|=m} \sum_{k=2}^{\infty} \Big(\frac{d}{(2^k d)^{n+1-\delta}} + \frac{d^{\varepsilon}}{(2^k d)^{n+\varepsilon-\delta}} \Big) \Big(\frac{1}{|Q|} \int_Q |D^\alpha \tilde{A}(y)| dy \Big) \|a\|_{L^\infty} |Q| |2^k Q|^{\frac{(n-\delta)}{n}}$$

$$\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=2}^{\infty} (2^{-k} + 2^{-\varepsilon k}) \leqslant C.$$

Thus, by using the condition on $I_4(x,u)$, we obtain

$$\|g_S^A(a)\|_{L^{\frac{n}{(n-\delta)}}} \leqslant C.$$

(ii) It is suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |\tilde{g}_S^A(f)(x) - C_Q| dx \leqslant C \|f\|_{L^{n/\delta}}$$

holds for any cube Q. By the equality

$$Q_{m+1}(A;x,z) = R_{m+1}(A;x,z) + \sum_{|\alpha|=m} \frac{1}{\alpha!} (x-z)^\alpha (D^\alpha A(x) - D^\alpha A(z))$$

and by some arguments similar to those in the proof of Lemma 2.2, we have

$$\tilde{g}_S^A(a)(x) \leqslant g_S^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x-z|^{n-\delta}} |a(z)| dz,$$

Thus, $\tilde{g}_S^A$ is (L^p, L^q)-bounded for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$ by Lemma 2.2 and [1]. For any cube $Q = Q(x_0, d)$, let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned} \tilde{F}_t^A(f)(x,y) =& \tilde{F}_t^A(f_1)(x,y) + \int_{R^n} \frac{R_m(\tilde{A};x,z)}{|x-z|^m} \psi_t(y-z) f_2(z) dz \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \Big[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} \\ & \qquad - \frac{\psi_t(u-z)(u-z)^\alpha}{|u-z|^m} \Big] f_2(z) dz \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(u-z) f_2(z) dz. \end{aligned}$$

Then

$$
\begin{aligned}
&\left|\tilde g_S^A(f)(x)-g_S\left(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|\\
=&\left|\left\|\chi_{\Gamma(x)}\tilde F_t^A(f)(x,y)\right\|-\left\|\chi_{\Gamma(x_0)}F_t\left(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\|\right|\\
\leqslant&\left\|\chi_{\Gamma(x)}(y,t)\tilde F_t^A(f)(x,y)-\chi_{\Gamma(x_0)}(y,t)F_t\left(\frac{R_m(\tilde A;x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\|\\
\leqslant&\left\|\chi_{\Gamma(x)}(y,t)\tilde F_t^A(f_1)(x,y)\right\|+\left\|\Big[\chi_{\Gamma(x)}(y,t)\int_{R^n}\frac{R_m(\tilde A;x,z)}{|x-z|^m}\psi_t(y-z)\right.\\
&\left.-\chi_{\Gamma(x_0)}(y,t)\int_{R^n}\frac{R_m(\tilde A;x_0,z)}{|x_0-z|^m}\psi_t(y-z)\Big]f_2(z)dz\right\|+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)\right.\\
&\left.-(D^\alpha A)_Q)\int_{R^n}\left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}-\frac{\psi_t(u-z)(u-z)^\alpha}{|u-z|^m}\right]f_2(z)dz\right\|\\
&+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{R^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(u-z)f_2(z)dz\right\|\\
=&J_1(x)+J_2(x)+J_3(x,u)+J_4(x,u).
\end{aligned}
$$

By the (L^p,L^q)-boundedness of $\tilde g_S^A$ for $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we get

$$
\frac{1}{|Q|}\int_Q J_1(x)dx\leqslant\ |Q|^{-1/q}\|\tilde g_S^A(f_1)\|_{L^q}\leqslant\ C|Q|^{-1/q}\|f_1\|_{L^p}\leqslant\ C\|f\|_{L^{n/\delta}}.
$$

For $J_2(x)$, we write

$$
\begin{aligned}
&\Big[\chi_{\Gamma(x)}(y,t)\int_{R^n}\frac{R_m(\tilde A;x,z)}{|x-z|^m}\psi_t(y-z)\\
&-\chi_{\Gamma(x_0)}(y,t)\int_{R^n}\frac{R_m(\tilde A;x_0,z)}{|x_0-z|^m}\psi_t(y-z)\Big]f_2(z)dz\\
=&\int_{R^n}\chi_{\Gamma(x)}(y,t)\left[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\right]\psi_t(y-z)R_m(\tilde A;x,z)f_2(z)dz+\\
&+\int_{R^n}\chi_{\Gamma(x)}(y,t)\frac{\psi_t(y-z)}{|x_0-z|^m}[R_m(\tilde A;x,z)-R_m(\tilde A;x_0,z)]f_2(z)dz\\
&+\int_{R^n}(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{R_m(\tilde A;x_0,z)\psi_t(y-z)}{|x_0-z|^m}f_2(z)dz.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big\|\int_{R^n}(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{R_m(\tilde{A};x_0,z)\psi_t(y-z)}{|x_0-z|^m}f_2(z)dz\Big\| \\
&\leqslant C\int_{R^n}\bigg\{\frac{|f_2(z)\|R_m(\tilde{A}(x_0,z)|}{|x_0-z|^m} \\
&\quad\times\Big(\iint_{R^{n+1}_+}|\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t)|^2|\psi_t(y-z)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\bigg\}dz \\
&\leqslant C\int_{R^n}\bigg\{\frac{|f_2(z)\|R_m(\tilde{A}(x_0,z)|}{|x_0-z|^m} \\
&\quad\times\bigg|\iint_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}-\iint_{\Gamma(x_0)}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\bigg|^{1/2}\bigg\}dz \\
&\leqslant C\int_{R^n}\bigg\{\frac{|f_2(z)\|R_m(\tilde{A}(x_0,z)|}{|x_0-z|^m} \\
&\quad\times\Big(\iint_{|y|\leqslant t}\Big|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\Big|\frac{dydt}{t^{n-1}}\Big)^{1/2}\bigg\}dz \\
&\leqslant C\int_{R^n}\frac{|f_2(z)\|R_m(\tilde{A}(x_0,z)|)}{|x_0-z|^m}\Big(\iint_{|y|\leqslant t}\frac{|x-x_0|t^{1-n}dydt}{(t+|x+y-z|)^{2n+3-2\delta}}\Big)^{1/2}dz \\
&\leqslant C\int_{R^n}\frac{|f_2(z)\|R_m(\tilde{A}(x_0,z)|)}{|x_0-z|^m}\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz,
\end{aligned}
$$

similarly as in the proof of Lemma 2.2 and $I_2(x,u)$, we get

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q J_2(x)dx &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\frac{1}{|Q|}\int_Q\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}k\bigg(\frac{|x-x_0|}{|x_0-z|^{n+1-\delta}} \\
&\qquad+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-z|^{n+\varepsilon-\delta}}\bigg)|f(z)|dzdx \\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}\sum_{k=1}^\infty k(2^{-k}+2^{-\varepsilon k}) \\
&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}
\end{aligned}
$$

and

$$\frac{1}{|Q|}\int\limits_Q J_3(x,u)dx \leqslant \sum_{|\alpha|=m}\frac{C}{|Q|}\int\limits_Q |D^\alpha A(x)-(D^\alpha A)_Q)| \times \sum_{k=2}^{\infty}\int\limits_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|x-u|}{|x-z|^{n+1-\delta}}+\frac{|x-u|^\varepsilon}{|x-z|^{n+\varepsilon-\delta}}\right)|f(z)|dzdx$$
$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}(2^{-k}+2^{-\varepsilon k})\|f\|_{L^{n/\delta}} \leqslant C\|f\|_{L^{n/\delta}}.$$

Thus, by using the condition on $J_4(x,u)$, we obtain

$$\frac{1}{|Q|}\int\limits_Q\left|\tilde{g}_S^A(f)(x)-g_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|dx \leqslant C\|f\|_{L^{n/\delta}}.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2.

(i) It suffices to show that there exists a constant $C>0$ such that for every H^1-atom a with supp$a\subset Q=Q(x_0,d)$, it holds

$$\|\mu_S^A(a)\|_{L^{\frac{n}{(n-\delta)}}} \leqslant C.$$

Let $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$. We write, by the vanishing moment of a and for $u\in 3Q\setminus 2Q$,

$$\begin{aligned}
&F_t^A(a)(x,y)\\
=&\chi_{4Q}(x)F_t^A(a)(x,y)+\chi_{(4Q)^c}(x)\int\limits_{R^n}\Big[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\chi_{\Gamma(y)}(z,t)\\
&-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\chi_{\Gamma(y)}(u,t)\Big]a(z)dz\\
&-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\limits_{R^n}\Big[\frac{(x-z)^\alpha}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\chi_{\Gamma(y)}(z,t)\\
&-\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\chi_{\Gamma(y)}(u,t)\Big]D^\alpha\tilde{A}(z)a(z)dz\\
&-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\limits_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\chi_{\Gamma(y)}(u,t)D^\alpha\tilde{A}(z)a(z)dz.
\end{aligned}$$

Then

$$\begin{aligned}
\mu_S^A(a)(x) =& \Big\|\chi_{\Gamma(x)} F_t^A(a)(x,y)\Big\| \\
\leqslant& \chi_{4Q}(x)\Big\|\chi_{\Gamma(x)} F_t^A(a)(x,y)\Big\| \\
&+ \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}\int\limits_{R^n}\Big[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1-\delta}} \\
&- \frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1-\delta}}\Big]a(z)dz\Big\| \\
&+ \chi_{(4Q)^c}(x)\Big\|\sum_{|\alpha|=m}\frac{\chi_{\Gamma(x)}}{\alpha!}\int\limits_{R^n}\Big[\frac{(x-z)^\alpha}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1-\delta}} \\
&- \frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1-\delta}}\Big]D^\alpha\tilde{A}(z)a(z)dz\Big\| \\
&+ \chi_{(4Q)^c}(x)\Big\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\limits_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\chi_{\Gamma(y)}(u,t) \\
&\times D^\alpha\tilde{A}(z)a(z)dy\Big\| \\
=& K_1(x)+K_2(x,u)+K_3(x,u)+K_4(x,u).
\end{aligned}$$

By the (L^p, L^q)-boundedness of μ_S^A for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, we get

$$\|K_1(\cdot)\|_{L^{\frac{n}{(n-\delta)}}} \leqslant \|\mu_S^A(a)\|_{L^q}|4Q|^{\frac{(n-\delta)}{n}-1/q} \leqslant C\|a\|_{L^p}|Q|^{1-1/p} \leqslant C.$$

For $K_2(x,u)$, we write

$$\begin{aligned}
&\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}\chi_{\Gamma(y)}(z,t) - \frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\chi_{\Gamma(y)}(u,t) \\
=&(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1-\delta}|x-z|^m} \\
&+\left[\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}-\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\right]\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\chi_{\Gamma(y)}(u,t) \\
&+\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1-\delta}}\left(\frac{R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{R_m(\tilde{A};x,u)}{|x-u|^m}\right).
\end{aligned}$$

By the inequality

$$\Big|\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}-\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\Big| \leqslant C\Big(\frac{|z-u|}{|y-z|^{n-\delta}}+\frac{|z-u|^\gamma}{|y-z|^{n-1+\gamma-\delta}}\Big)$$

(see [14]), noting that

$$\Big\|\chi_{\Gamma(x)}\int\limits_{R^n}(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1-\delta}|x-z|^m}a(z)dz\Big\|$$

$$\leqslant C\int\limits_{R^n}\Big\{\frac{|a(z)\|R_m(\tilde{A};x,z)|}{|x-z|^m}$$

$$\times\Big(\iint\limits_{R^{n+1}_+}\frac{\chi_{\Gamma(x)}(y,t)|\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(u,t)|^2}{|y-z|^{2n-2-2\delta}}\frac{dydt}{t^{n+3}}\Big)^{1/2}\Big\}dz$$

$$\leqslant C\int\limits_{R^n}\Big\{\frac{|a(z)\|R_m(\tilde{A};x,z)|}{|x-z|^m}$$

$$\times\Big|\iint\limits_{\Gamma(x),\ \Gamma(z)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2-2\delta}}-\iint\limits_{\Gamma(x),\ \Gamma(u)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2-2\delta}}\Big|^{1/2}\Big\}dz$$

$$\leqslant C\int\limits_{R^n}\Big\{\frac{|a(z)\|R_m(\tilde{A};x,z)|}{|x-z|^m}$$

$$\times\Big(\iint\limits_{|y|\leqslant t,|x+y-z|\leqslant t}\Big|\frac{1}{|x+y-z|^{2n-2-2\delta}}-\frac{1}{|x+y-u|^{2n-2-2\delta}}\Big|\frac{dydt}{t^{n+3}}\Big)^{1/2}\Big\}dz$$

$$\leqslant C\int\limits_{R^n}\frac{|a(z)\|R_m(\tilde{A};x,z)|}{|x-z|^m}\Big(\iint\limits_{|y|\leqslant t,|x+y-z|\leqslant t}\frac{|u-z|t^{-n-3}dydt}{|x+y-z|^{2n-1-2\delta}}\Big)^{1/2}dz$$

$$\leqslant C\int\limits_{R^n}\frac{|a(z)\|R_m(\tilde{A};x,z)|}{|x-z|^m}\frac{|u-z|^{1/2}}{|x-z|^{n+1/2-\delta}}dz$$

and arguing similarly as in the proof of Theorem 1.1, we obtain

$$\begin{aligned}\|K_2(\cdot,u)\|_{L^{\frac{n}{(n-\delta)}}} &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\Big[\int\limits_{2^{k+1}Q\setminus 2^kQ}\Big(\int\limits_Q k\Big(\frac{|z-u|}{|x-z|^{n+1-\delta}}\\ &\quad+\frac{|z-u|^{1/2}}{|x-z|^{n+1/2-\delta}}+\frac{|z-u|^\gamma}{|x-z|^{n+\gamma-\delta}}\Big)|a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\\ &\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\Big[\int\limits_{2^{k+1}Q\setminus 2^kQ}k\Big(\frac{d}{(2^kd)^{n+1-\delta}}\\ &\quad+\frac{d^{1/2}}{(2^kd)^{n+1/2-\delta}}+\frac{d^\gamma}{(2^kd)^{n+\gamma-\delta}}\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\end{aligned}$$

$$\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\leqslant\ C$$

and

$$\begin{aligned}\|K_3(\cdot,u)\|_{L^{\frac{n}{(n-\delta)}}}\leqslant C&\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\Big[\int\limits_{2^{k+1}Q\setminus 2^kQ}\Big(\int\limits_{Q}\Big(\frac{|z-u|}{|x-z|^{n+1-\delta}}+\frac{|z-u|^{1/2}}{|x-z|^{n+1/2-\delta}}\\&+\frac{|z-u|^\gamma}{|x-z|^{n+\gamma-\delta}}\Big)|D^\alpha\tilde{A}(z)\||a(z)|dz\Big)^{\frac{n}{(n-\delta)}}dx\Big]^{\frac{(n-\delta)}{n}}\\&\leqslant C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\Big(\frac{d}{(2^kd)^{n+1-\delta}}+\frac{d^{1/2}}{(2^kd)^{n+1/2-\delta}}+\frac{d^\gamma}{(2^kd)^{n+\gamma-\delta}}\Big)\\&\times\Big(\frac{1}{|Q|}\int\limits_Q|D^\alpha\tilde{A}(z)|dz\Big)\|a\|_{L^\infty}|Q\|2^{k+1}Q|^{\frac{(n-\delta)}{n}}\\&\leqslant C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}(2^{-k}+2^{-k/2}+2^{-\gamma k})\leqslant\ C.\end{aligned}$$

Thus using the condition of $K_4(x,u)$ we obtain

$$\|\mu_S^A(a)(x)\|_{L^{\frac{n}{(n-\delta)}}}\leqslant\ C.$$

(ii) For any cube $Q=Q(x_0,d)$, let

$$\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha.$$

We write, for $f=f\chi_{4Q}+f\chi_{(4Q)^c}=f_1+f_2$ and $u\in 3Q\setminus 2Q$,

$$\begin{aligned}\tilde{F}_t^A(f)(x,y)=&\tilde{F}_t^A(f_1)(x,y)+\int\limits_{|y-z\leqslant t}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}f_2(z)dz\\&-\sum_{|\alpha|=m}\frac{1}{\alpha!}\big(D^\alpha A(x)-(D^\alpha A)_Q\big)\\&\times\int\limits_{|y-z|\leqslant t}\Big[\frac{(x-z)^\alpha\Omega(y-z)}{|x-z|^m|y-z|^{n-1-\delta}}-\frac{(u-z)^\alpha\Omega(y-z)}{|u-z|^m|y-z|^{n-1-\delta}}\Big]f_2(z)dz\\&-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int\limits_{|y-z|\leqslant t}\frac{(u-z)^\alpha}{|u-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}f_2(z)dz.\end{aligned}$$

Then

$$
\begin{aligned}
&\Big|\tilde{\mu}_S^A(f)(x)-\mu_S\Big(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(x_0)\Big|\\
=&\Big|\Big\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\Big\|-\Big\|\chi_{\Gamma(x_0)}F_t\Big(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(y)\Big\|\Big|\\
\leqslant&\Big\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)-\chi_{\Gamma(x_0)}F_t\Big(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(y)\Big\|\\
\leqslant&\Big\|\chi_{\Gamma(x)}(y,t)\tilde{F}_t^A(f_1)(x,y)\Big\|+\Big\|\chi_{\Gamma(x)}(y,t)\int\limits_{|y-z\leqslant t}\Big[\frac{R_m(\tilde{A};x,z)\Omega(y-z)}{|x-z|^m|y-z|^{n-1-\delta}}\\
&-\chi_{\Gamma(x_0)}(y,t)\int\limits_{|y-z|\leqslant t}\frac{R_m(\tilde{A};x_0,z)\Omega(y-z)}{|x_0-z|^m|y-z|^{n-1-\delta}}\Big]f_2(z)dz\Big\|\\
&+\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\\
&\times\int\limits_{|y-z|\leqslant t}\Big[\frac{\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1-\delta}|x-z|^m}-\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1-\delta}|u-z|^m}\Big]f_2(z)dz\Big\|\\
&+\Big\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int\limits_{|y-z|\leqslant t}\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1-\delta}|u-z|^m}f_2(z)dz\Big\|\\
=&L_1(x)+L_2(x)+L_3(x,u)+L_4(x,u).
\end{aligned}
$$

Similarly as in the proof of the assertion (i) of Theorem 1.2, we get

$$
\begin{aligned}
\frac{1}{|Q|}\int\limits_Q L_1(x)dx &\leqslant C\|f\|_{L^{n/\delta}};\\
\frac{1}{|Q|}\int\limits_Q L_2(x)dx &\leqslant C\sum_{k=2}^\infty\int\limits_{2^{k+1}Q\setminus 2^kQ}k\Big(\frac{|x-x_0|}{|x-z|^{n+1-\delta}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2-\delta}}\\
&\quad+\frac{|x-x_0|^\gamma}{|x-z|^{n+\gamma-\delta}}\Big)|f(z)|dz\\
&\leqslant C\|f\|_{L^{n/\delta}};\\
\frac{1}{|Q|}\int\limits_Q L_3(x,u)dx &\leqslant C\|f\|_{L^{n/\delta}}.
\end{aligned}
$$

Thus, using the condition on $L_4(x,u)$ we obtain

$$
\frac{1}{|Q|}\int\limits_Q\Big|\tilde{\mu}_S^A(f)(x)-\mu_S\Big(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\Big)(x_0)\Big|w(x)dx\leqslant C\|f\|_{L^{n/\delta}}.
$$

This completes the proof. $\square$

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