ACTA MATHEMATICA VIETNAMICA
Volume 31, Number 1, 2006, pp. 61-75

MIXED MULTIPLICITIES OF IDEALS AND OF REES ALGEBRAS ASSOCIATED WITH RATIONAL NORMAL CURVES

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Introduction

Let c, d be positive integers with $c \geqslant d$, and let $X_1, \ldots, X_{c+d}$ be indeterminates over a field k. Let $R := k[X_1, \ldots, X_{c+d}]$ be a polynomial ring over a field k. Consider the matrix

$$\begin{pmatrix} X_1 & X_2 & \ldots & X_c \\ X_{1+d} & X_{2+d} & \ldots & X_{c+d} \end{pmatrix}.$$

Let I be the ideal generated by the 2×2 minors of this matrix. Then I is the defining ideal of a rational normal scroll of dimension d in $\mathbb{P}^{c+d-1}$. Consider the Rees algebra of I:

$$R[It] := \underset{v \geqslant 0}{\oplus} I^v t^v.$$

This Rees algebra has a natural bigraded structure whose Proj is the blow-up of $\mathbb{P}^{c+d-1}$ along the rational normal scroll.

Conca, Herzog and Valla [4] applied the Sagbi basic theory to study the Rees algebra and the fibre ring of I. They showed that the natural generators of these algebras form Sagbi-bases and computed their relations. In particular, they used this information to compute the Hilbert function and the multiplicity of the fibre ring. They did not compute the Hilbert function and the multiplicity of the Rees algebra $R[It]$, perhaps due to the more complicated structure of the Rees algebra. Recall that the fibre ring of I is the quotient ring of $R[It]$ by the maximal graded ideal of R, which has a simpler presentation than $R[It]$. Conca [3] computed the Hilbert function of the powers of I in the case I is the ideal of the rational normal curve (i.e. $d = 1$). Using this result one can compute the Hilbert function of $R[It]$ and the mixed multiplicities of I. The computation of Conca is based on information on the minimal free resolutions of I^n, which is not available in the general case ($d > 1$).

In this paper, we will use the Gröbner technique to compute the mixed multiplicities of the ideal I in the case of a rational normal curve. This technique can be used in the general case. It transfers the computation into a combinatorical problem. We will introduce a filtration of the Rees algebra $R[It]$ and associate a simplicial complex with this filtration. Furthermore, we will also compute the

Received March 30, 2005; in revised form August 28, 2005.
The author is partially supported by the National Basic Research Program.

mixed multiplicities of the naturally bigraded algebra $R[It]$. This allows us to compute the degree of certain embeddings of the blowup of the space $\mathbb{P}^c$ along the ideal sheaf of I.

1. Mixed multiplicities of ideals of rational normal curves

Let $R[It]$ be the Rees algebra associated with a rational normal curve in $\mathbb{P}^c$. Let $M = (\mathfrak{m}, It)$ be the maximal graded ideal of $R[It]$, where $\mathfrak{m} := (X_1, \ldots, X_{c+1})$. The associated graded ring $gr_M R[It] := \bigoplus_{n\geqslant 0} M^n/M^{n+1}$ has a natural bigrading with

$$(gr_M R[It])_{(u,v)} = \mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v.$$

By Bhattacharya [1], the numerical function $\dim_k \mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v$ is given by a polynomial in u and v for all large values of u and v. Let s be the degree of this polynomial and write the terms of the bigzest total degree as

$$\sum_{i+j=s} \frac{a_{ij}}{i!j!} u^i v^j,$$

where a_{ij} are non negative integers. Following [9] we call the number a_{is-i} a mixed multiplicity of the pair $(\mathfrak{m}, I)$ and denoted it by $e_i(\mathfrak{m}|I)$.

Now we will present a method for the computation of $e_i(\mathfrak{m}|I)$.

Let $S = k[X,T] := k[X_1, \ldots, X_{c+1}, T_{12}, \ldots, T_{1c} \ldots, T_{c-1c}]$ be a polynomial ring over k. Mapping X_h to X_h, $1 \leqslant h \leqslant c+1$ and T_{ij} to $X_i X_{j+1} - X_j X_{i+1}$, $1 \leqslant i < j \leqslant c$, we obtain a representation of the Rees algebra

$$R[It] \cong \frac{S}{J},$$

where J is the ideal of S generated by the forms vanishing at 2×2-minors of the matrix

$$\begin{pmatrix} X_1 & X_2 & \ldots & X_c \\ X_2 & X_3 & \ldots & X_{c+1} \end{pmatrix}.$$

We set

$$\operatorname{bideg} X_h = (1,0), \quad 1 \leqslant h \leqslant c+1,$$

$$\operatorname{bideg} T_{ij} = (0,1), \quad 1 \leqslant i < j \leqslant c.$$

Then S is a standard bigraded algebra and the above isomorphism is a bigraded isomorphism.

Set $s := \dfrac{c(c-1)}{2}$. For every $h = (\alpha_1, \ldots, \alpha_{c+1}, \beta_{12}, \ldots, \beta_{1c}, \ldots, \beta_{c-1c}) \in \mathbb{N}^{c+1+s}$, put

$$S_h := k{X_1}^{\alpha_1} \ldots {X_{c+1}}^{\alpha_{c+1}} {T_{12}}^{\beta_{12}} \ldots {T_{1c}}^{\beta_{1c}} \ldots {T_{c-1c}}^{\beta_{c-1c}}.$$

Then $S = \bigoplus_{h \in \mathbb{N}^{c+1+s}} S_h$ is an $\mathbb{N}^{c+1+s}$-graded algebra. This $\mathbb{N}^{c+1+s}$-grading is finer than the above bigrading because

$$S_{(u,v)} = \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1}=u \\ \beta_{12}+\ldots+\beta_{c-1c}=v}} S_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{1c},\ldots,\beta_{c-1c})},$$

for all $(u,v) \in \mathbb{N}^2$.

Let δ be a monomial order on S. For any $f \in S$ we define the initial term of f, written f_*, to be the greatest term of f with respect to the order δ. Let J_* denote the ideal of S generated by the elements f_*, $f \in J$. A set of elements $f_1, \ldots, f_k \in J$ is called a Gröbner basic for J if $J_* = ((f_1)_*, \ldots, (f_k)_*)$. Conca, Herzog and Valla [4] introduced a monomial order on S which is the product of term orders on $k[X]$ and on $k[T]$ and computed a Gröbner basic for J with respect to this order.

Lemma 1.1. [4, Lemma 3.4] *There exists a term order δ for the monomials in S such that the folowing polynomials form a Gröbner basic of J:*

- $X_hT_{ij} - X_iT_{hj} + X_jT_{hi},\ \ 1 \leqslant h < i < j \leqslant c,$
- $X_hT_{ij} - X_{j+1}T_{i,h-1} + X_{i+1}T_{h-1,j},\ \ 1 \leqslant i < h-1 < j \leqslant c,$
- $T_{ij}T_{hk} - T_{ik}T_{hj} + T_{ih}T_{kj},\ \ 1 \leqslant i < h < k < j \leqslant c,$
- $T_{ij}T_{hk} - T_{i,h-1}T_{j+1,k} - T_{i,j+1}T_{h,k-1} - T_{i+1,j}T_{h-1,k} + T_{i+1,j+1}T_{h-1,k-1} + T_{ih}T_{j+1,k-1} + T_{i+1,h-1}T_{jk} - T_{i+1,h}T_{j,k-1},\ 1 \leqslant i < j < h < k \leqslant c, h-j > 1.$

In particular, J_* is generated by the monomials

- $X_hT_{ij},\ \ 1 \leqslant h < i < j \leqslant c,$
- $X_hT_{ij},\ \ 1 \leqslant i < h-1 < j \leqslant c,$
- $T_{ij}T_{hk},\ \ 1 \leqslant i < h < k < j \leqslant c,$
- $T_{ij}T_{hk},\ \ 1 \leqslant i < j < h < k \leqslant c, h-j > 1.$

We use δ to define a term order τ for the monomials of S in the following way: Let u_1f_1 and u_2f_2 be monomials of S, where u_1, u_2 are monomials in the X_h and f_1, f_2 are monomials in the T_{ij}. Then we set $u_1f_1 \underset{\tau}{<} u_2f_2$ if $\deg u_1f_1 < \deg u_2f_2$, or $\deg u_1f_1 = \deg u_2f_2$ and $\deg u_1 < \deg u_2$, or $\deg u_1f_1 = \deg u_2f_2$, $\deg u_1 = \deg u_2$ and $u_1f_1 \underset{\delta}{>} u_2f_2$. The term order τ induces an order $<$ on $\mathbb{N}^{c+1+s}$ as follows. Let

$$h = (\alpha_1, \ldots, \alpha_{c+1}, \beta_{12}, \ldots, \beta_{1c}, \ldots, \beta_{c-1c})$$

and

$$h' = (\alpha'_1, \ldots, \alpha'_{c+1}, \beta'_{12}, \ldots, \beta'_{1c}, \ldots, \beta'_{c-1c}).$$

Then $h < h'$ if $(XT)^h \underset{\tau}{<} (XT)^{h'}$. Set

$$F_hS := \bigoplus_{h' \geqslant h} S_{h'}.$$

It is clear that $F = \{F_hS\}_{h \in \mathbb{N}^{c+1+s}}$ is a filtration of S. The filtration F imposes a filtration on S/J which we also denote by F.

For every polynomial $f \in S$, let f^* denote the initial term of f, i.e $f^* := f_h$ if $f = \sum_{h' \in \mathbb{N}^{c+1+s}} f_{h'}$ and $h = \min\{h' \mid f_{h'} \neq 0\}$. Let J^* denote the ideal of S generated by the elements f^*, $f \in J$. Then

$$gr_F(S/J) \cong S/J^*.$$

Since J^* is an $\mathbb{N}^{c+1+s}$-graded ideal of S, J^* is also a bigraded ideal of S. Hence S/J^* is a bigraded algebra over k with respect to the bigrading induced from S.

We shall see that the Bhattacharya function of $(\mathfrak{m}, I)$ coincides with the Hilbert function of S/J^*.

Lemma 1.2. *For all $(u, v) \in \mathbb{N}^2$ we have*

$$\dim_k(\mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v) = \dim_k(S/J^*)_{(u,v)}.$$

Proof. We know that

$$\mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v = (gr_M R[It])_{(u,v)}.$$

Let $\mathfrak{M} = (X_1, \ldots, X_{c+1}, T_{12}, \ldots, T_{c-1c})$ be the maximal graded ideal of S. Then

$$gr_M R[It] \cong gr_{\mathfrak{M}}(S/J).$$

The bigrading on $gr_M R[It]$ imposes a bigrading on $gr_{\mathfrak{M}}(S/J)$ with

$$\begin{aligned} gr_{\mathfrak{M}}(S/J)_{(u,v)} = \Big(& \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1} \geqslant u \\ \beta_{12}+\ldots+\beta_{c-1c} \geqslant v}} S_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})} + J \Big) \Big/ \\ & \Big(\bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1} \geqslant u \\ \beta_{12}+\ldots+\beta_{c-1c} \geqslant v+1}} S_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})} \\ & + \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1} \geqslant u+1 \\ \beta_{12}+\ldots+\beta_{c-1c} \geqslant v}} S_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})} + J \Big) \\ \cong & \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1} = u \\ \beta_{12}+\ldots+\beta c-1c = v}} S_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})} + J/J. \end{aligned}$$

Using the filtration F on S/J we can decompose the latter module into a series of graded pieces of the associated ring $gr_F(S/J) \cong S/J^*$ and we obtain

$$\begin{aligned}\dim_k gr_{\mathfrak{M}}(S/J)_{(u,v)} &= \sum_{\substack{\alpha_1+\ldots+\alpha_{c+1}=u\\ \beta_{12}+\ldots+\beta_{c-1c}=v}} \dim_k (S/J^*)_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})}\\ &= \dim_k \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1}=u\\ \beta_{12}+\ldots+\beta_{c-1c}=v}} (S/J^*)_{(\alpha_1,\ldots,\alpha_{c+1},\beta_{12},\ldots,\beta_{c-1c})}\\ &= \dim_k (S/J^*)_{(u,v)}.\end{aligned}$$

□

According to Lemma 1.2 we can use the Hilbert function of S/J^* to compute the mixed multiplicities $e_i(\mathfrak{m}|I)$.

Now we will compute the ideal J^*. Let $\prec$ be the term order on $\mathbb{N}^{c+1+s}$ induced from the term order δ. The order $\prec$ is also artinian, that is, every non-ascending sequence of elements of $\mathbb{N}^{c+1+s}$ with respect to $\prec$ becomes stationary. Clearly, if f is a generator of J as in Lemma 1.1, then $f^* = f_*$. We can pass from J_* to J^* by the following result.

Lemma 1.3. [6, Lemma 1.3] *Let B be an $\mathbb{N}^r$-graded algebra over k and $\mathfrak{J}$ a homogeneous ideal of B. Let $\mathfrak{J}^*$ and $\mathfrak{J}_*$ be the ideals generated by the initial forms f^* and f_* of the elements $f \in \mathfrak{J}$ with respect to the term orders $<$ and $\prec$ of $\mathbb{N}^r$ respectively. Suppose that every bounded non-ascending sequence of elements of $\mathbb{N}^r$ with respect to $\prec$ becomes stationary, and that there exists a set Z of generators of $\mathfrak{J}$ such that*

(i) *$f_* = f^*$ for all $f \in Z$,*

(ii) *$\mathfrak{J}_*$ is generated by the elements f_*, $f \in Z$.*

Then $\mathfrak{J}^* = \mathfrak{J}_*$.

The ideal J^* is generated by square-free monomials. Therefore we can associate with J^* a simplicial complex Δ as follows. Setting

$$T := \{T_{ij} |\ 1 \leqslant i \leqslant j \leqslant c\} \cup \{T_{(0,0)}\}.$$

For convenience, we identify X_j with T_{j-1j-1} for $1 \leqslant j \leqslant c+1$. Then J^* is generated by the following monomials:

(1) $T_{hh}T_{ij},\ 1 \leqslant h+1 < i < j \leqslant c,$

(2) $T_{hh}T_{ij},\ 1 \leqslant i < h < j \leqslant c,$

(3) $T_{ij}T_{hk},\ 1 \leqslant i < h < k < j \leqslant c,$

(4) $T_{ij}T_{hk},\ 1 \leqslant i < j < h < k \leqslant c, h-j > 1.$

Let

$$A := \{(i,j) \in \mathbb{N}^2 |\ 1 \leqslant\ i \leqslant j \leqslant c\} \cup \{(0,0)\}.$$

For any subset H of A we define

$$T^H := \prod_{(i,j)\in H} T_{ij}.$$

Then

$$\Delta = \{H \subseteq A |\ T^H \notin J^*\}.$$

The ring $k[T]/J^* = k[\Delta]$ is called the Stanley-Reisner ring associated with Δ. We equip A with the partial order: $(i,j) \leqslant (h,k)$ if $i \leqslant h$ and $j \leqslant k$. Using this partial order we can describe the facets F of Δ with

$$\dim F = \dim k[\Delta] - 1 = \dim R[It] - 1 = c + 1.$$

Remark. Since monomials $T_{hh}T_{ij}$ and $T_{ij}T_{hk}$ in (2) and (3) correspond to all pairs of incomparable elements of A, F is a chain of Δ.

Lemma 1.4. *Let F be a facet of Δ. Let*

$$(i_0, j_0) := \max\{(i,j) \in F |\ i < j\}.$$

Then $\dim F = c + 1$ *if and only if* $i_0 = 1$ *and*

$$F = \{(0,0), (1,1), (1,2), \dots , (1, j_0), (j_0, j_0), (j_0 + 1, j_0 + 1), \dots , (c, c)\},$$

or $i_0 \geqslant 2$ *and*

$$F = m \cup \{(j_0, j_0), (j_0 + 1, j_0 + 1), \dots , (c, c)\},$$

where m is a maximal chain from $(1, i_0 - 1)$ to (i_0, j_0).

Proof. By (1), (2), (3), and (4) we have $(h,h) \notin F$ for $h < i_0 - 1$ or $i_0 < h < j_0$, $(h,k) \notin F$ for $i_0 < h < k < j_0$ or $h < i_0$ and $k > j_0$ or $j_0 + 1 < h < k \leqslant c$ or $k < i_0 - 1$.

Let $\dim F = c + 1$. We consider the following cases.

Case 1: $i_0 = 1$. It is easy to check that

$$F = \{(0,0), (1,1), (1,2), \dots , (1, j_0), (j_0, j_0), (j_0 + 1, j_0 + 1), \dots , (c, c)\}.$$

Case 2: $i_0 \geqslant 2$. We set

$$A(i_0, j_0) := \{(i,j) \in A |\ 1 \leqslant i \leqslant i_0,\ i_0 - 1 \leqslant j \leqslant j_0\}.$$

From the above remark we have $(i,j) \notin F$ if

$$(i,j) \notin A(i_0, j_0) \cup \{(j_0, j_0), (j_0 + 1, j_0 + 1), \dots , (c, c)\}.$$

On the other hand, if F' is a chain of $A(i_0, j_0)$ then $|F'| \leqslant j_0 + 1$, hence

$$\begin{aligned} c + 2 = |F| &= |m| + \#\ \{(j_0, j_0), \dots , (c, c)\} \\ &\leqslant j_0 + 1 + c - j_0 + 1 = c + 2. \end{aligned}$$

So we get $|m| = j_0 + 1$ and m is a maximal chain from $(1, i_0 - 1)$ to (i_0, j_0) in $A(i_0, j_0)$.

Conversely, let F be of the above form. Since T^F is non-divisible by any of the monomials in (1), (2), (3), and (4) we have $T^F \in k[\Delta]$. Hence F is a chain of Δ. If $i_0 = 1$, it is easy to see that $\dim F = c + 1$. If $i_0 \geqslant 2$, m being a maximal chain from $(1, i_0 - 1)$ to (i_0, j_0) implies $|m| = j_0 + 1$. Then we get

$$\dim F = |m| + c - j_0 + 1 - 1 = c + 1.$$

□

Let F be a facet of Δ with $\dim F = c+1$. Put

$$P_F := (T_{ij}|\ (i,j) \notin F).$$

By [2, Theorem 5.14], P_F is a minimal prime ideal of J^*. By [7, Theorem 3.4] we have

$$e_i(S/J^*) = \sum_{F\in\Delta, \dim F = c+1} e_i(S/P_F).$$

Since $S/P_F \cong k[F]$, this implies $e_i(S/P_F) = e_i(k[F])$. Let

$$j := \#\{(h,h) \in A|\ T_{hh} \in P_F\}.$$

Then

$$H_{k[F]}(u,v) = \binom{u}{j-1}\binom{v}{c-j+1} + \text{terms of degree} < \text{c}.$$

Hence we have

$$e_i(k[F]) = \begin{cases} 0 & \text{if } i \neq j-1, \\ 1 & \text{if } i = j-1. \end{cases}$$

Therefore, to compute the $e_i(S/J^*)$ we will compute the number of the facets of Δ which contain $i+1$ vertices of the form (h,h). We set

$$M_i := \{F \in \Delta|\ \dim F = c+1 \text{ and } \#\{(h,h) \in F\} = i+1\}.$$

Then

$$e_i(m|I) = e_i(S/J^*) = |M_i|.$$

To compute $|M_i|$ we shall need the following lemma.

Lemma 1.5. [4, Lemma 4.4] *Let a, b, k be positive integers with $k \leqslant \min\{a,b\}$. Consider the sublattice*

$$L(a,b,k) = \{(t,j) \in \mathbb{N}^2|\ 1 \leqslant t \leqslant a, 1 \leqslant j \leqslant b, j \geqslant t-a+k\}.$$

Denote by $e(a,b,k)$ the number of maximal chains of $L(a,b,k)$. Then

$$e(a,b,k) = \binom{a+b-2}{a-1} - \binom{a+b-2}{a+b-k}.$$

Using Lemma 1.4 and Lemma 1.5 we can compute the mixed multiplicities $e_i(\mathfrak{m}|I)$.

Theorem 1.6. *Let I be the defining ideal of a rational normal curve in $\mathbb{P}^c$. Then*

$$e_i(\mathfrak{m}|I) = \begin{cases} 2^c - c^2 + c - 2 & \text{if } i = 0, \\ 2^{c-1} - c & \text{if } i = 1, \\ 2^{c-i} & \text{if } 2 \leqslant i \leqslant c. \end{cases}$$

Proof. Let $F \in M_i$. Put

$$(i_0, j_0) := \max\{(t,j) \in F|\ t < j\}.$$

By Lemma 1.3, we have

$$\{(j_0,j_0), (j_0+1, j_0+1), \ldots, (c,c)\} \subseteq F,$$

and

$$F \cap \{(h,h) \in A \mid h < j_0\} \subseteq \{(i_0-1, i_0-1), (i_0, i_0)\}.$$

Let B be a subset of $\{(i_0-1, i_0-1), (i_0, i_0)\}$ and $q := \#B$. Then $0 \leqslant q \leqslant 2$ and

$$i + 1 = c - j_0 + 1 + q,$$

where $2 \leqslant j_0 \leqslant c$.

If $i_0 = 1$, there is only one $F \in M_i$ which contains two points $(i_0-1, i_0-1), (i_0, i_0)$ (by Lemma 1.3).

If $i_0 = 2$, we set

$$\begin{aligned} A(i_0, j_0) &:= \{(t,j) \in A \mid 1 \leqslant t \leqslant i_0,\ i_0 - 1 \leqslant j \leqslant j_0\}, \\ B(i_0, j_0) &:= \{(t,j) \in A(i_0, j_0) \mid t < j\}. \end{aligned}$$

By Lemma 1.4, the number of maximal chains from $(1, i_0-1)$ to (i_0, j_0) of $A(i_0, j_0)$ and of $B(i_0, j_0)$ is

$$\begin{aligned} m_{i_0 j_0} &:= \binom{j_0}{i_0-1} - \binom{j_0}{j_0} = \binom{j_0}{i_0-1} - 1, \\ n_{i_0 j_0} &:= \binom{j_0}{i_0-1} - \binom{j_0}{j_0-1} = \binom{j_0}{i_0-1} - j_0. \end{aligned}$$

On the other hand, the number of maximal chains from $(1, i_0-1)$ to (i_0, j_0) of $A(i_0, j_0)$ which contain two points (i_0-1, i_0-1), (i_0, i_0) is 1. Hence, the number of maximal chains from $(1, i_0-1)$ to (i_0, j_0) of $A(i_0, j_0)$ which contain only a point (i_0-1, i_0-1) or (i_0, i_0) is

$$m_{i_0 j_0} - n_{i_0 j_0} - 1 = j_0 - 2.$$

If $i = 0$ then $q = 0$, $j = c$ and $3 \leqslant i_0 \leqslant c-1$. So we get

$$\begin{aligned} e_0(\mathfrak{m}|I) &= \sum_{i_0=3}^{c-1} n_{i_0 c} = \sum_{i_0=3}^{c-1} \Big[\binom{c}{i_0-1} - c \Big] \\ &= 2^c - c^2 + c - 2. \end{aligned}$$

If $i = 1$ then $0 \leqslant q \leqslant 1$, $c-1 \leqslant j_0 \leqslant c$. Hence $e_1(m|I)$ can be expressed as the sum of the numbers of maximal chains with $q = 0$ and $q = 1$:

$$e_1(\mathfrak{m}|I) = \sum_{i_0=3}^{c-2} \Big[\binom{c-1}{i_0-1} - (c-1) \Big] + \sum_{i_0=2}^{c-1} (c-2) =$$

$$
\begin{aligned}
&= \sum_{i_0=3}^{c-2} \binom{c-1}{i_0-1} - \sum_{i_0=3}^{c-2} (c-1) + \sum_{i_0=2}^{c-1} (c-2) \\
&= \sum_{i_0=3}^{c-2} \binom{c-1}{i_0-1} - \sum_{i_0=3}^{c-2} 1 + 2(c-2) \\
&= 2^{c-1} - 2 - 2(c-1) - (c-4) + 2(c-2) \\
&= 2^{c-1} - c.
\end{aligned}
$$

If $2 \leqslant i \leqslant c$ then $0 \leqslant q \leqslant 2$, $j_0 = c - i + q$. Hence $e_i(m|I)$ can be expressed as the sum of the numbers of maximal chains with $q = 0$, $q = 1$ and $q = 3$

$$
\begin{aligned}
e_i(\mathfrak{m}|I) &= \sum_{i_0=3}^{c-i-1} \left[\binom{c-i}{i_0-1} - (c-i) \right] + \sum_{i_0=2}^{c-i} (c-i-1) + \sum_{i_0=1}^{c-i+1} 1 \\
&= \sum_{i_0=3}^{c-i-1} \binom{c-i}{i_0-1} - \sum_{i_0=3}^{c-i-1} (c-i) + \sum_{i_0=2}^{c-i} (c-i-1) + c - i + 1 \\
&= 2^{c-i} - 2 - 2(c-i) - \sum_{i_0=3}^{c-i-1} 1 + 2(c-i-1) + c - i + 1 \\
&= 2^{c-i} - 2 - 2(c-i) - (c-i-3) + 2(c-i-1) + c - i + 1 \\
&= 2^{c-i}.
\end{aligned}
$$

□

Corollary 1.7. $e(R[It]) = 2^{c+1} - c^2 - 3.$

Proof. By [10, Theorem 3.1] and Theorem 1.4 we have

$$
\begin{aligned}
e(R[It]) &= \sum_{i=0}^{c} e_i(m|I) \\
&= 2^c - c^2 + c - 2 + 2^{c-1} - c + \sum_{i=2}^{c} 2^{c-i} \\
&= 2^{c+1} - c^2 - 3.
\end{aligned}
$$

□

2. Mixed multiplicities of Rees algebras associated with rational normal curves

Let I be the defining ideal of a rational normal curve in $\mathbb{P}^c$. The Rees algebra $R[It]$ has a natural bigrading by setting

$$R[It]_{(u,v)} := (I^v)_u t^v,$$

for all $(u,v) \in \mathbb{N}^2$. The Hilbert function of $R[It]$ with respect to this bigrading is the function

$$H_{R[It]}(u,v) := \dim_k R[It]_{(u,v)}.$$

By [7, Theorem 1.1], there exist integers u_0, v_0 such that for $u \geqslant 2v + u_0$ and $v \geqslant v_0$, the Hilbert function $H_{R[It]}(u,v)$ is equal to a polynomial $P_{R[It]}(u,v)$ with total degree c. Moreover, if $P_{R[It]}(u,v)$ is written in the form

$$P_{R[It]}(u,v) := \sum_{i=0}^{c} \frac{e_i(R[It])}{i!(c-i)!} u^i v^{c-i} + \text{lower-degree terms},$$

then $e_i(R[It])$ is an integer for $i = 0, \ldots, c$. Follwing [9] we call $e_i(R[It])$ the mixed multiplicities of the bigraded algebra $R[It]$.

Let $S := R[T_{ij}, 1 \leqslant i < j \leqslant c]$. We set

$$\operatorname{bideg} X_i = (1,0), \quad 1 \leqslant i \leqslant c+1,$$
$$\operatorname{bideg} T_{ij} = (2,1), \quad 1 \leqslant i < j \leqslant c.$$

Then S is a bigraded algebra and the isomorphism $R[It] \cong S/J$ is a bigraded isomorphism. Hence

$$H_{R[It]}(u,v) = H_{S/J}(u,v).$$

To compute $H_{S/J}(u,v)$ we introduce a multigraded structure which is finer than the above bigraded structure.

Set $s := \dfrac{c(c-1)}{2}$. Then $S = \bigoplus_{h \in \mathbb{N}^{c+1+s}} S_h$ is an $\mathbb{N}^{c+1+s}$- graded algebra. This $\mathbb{N}^{c+1+s}$-grading is finer than the above bigrading because

$$S_{(u,v)} = \bigoplus_{\substack{\alpha_1+\ldots+\alpha_{c+1}+2v=u \\ \beta_{12}+\ldots+\beta_{c-1c}=v}} S(\alpha_1, \ldots, \alpha_{c+1}, \beta_{12}, \ldots, \beta_{1c}, \ldots, \beta_{c-1c}),$$

for all $(u,v) \in \mathbb{N}^2$.

Let δ and τ be the terms order for the monomials in S as in Section 1. Let J_* be the ideal generated by the initial forms f_* of the elements $f \in J$ with respect to given term order δ. The term order τ induces an order $<$ on $\mathbb{N}^{c+1+s}$ as follows:

Let

$$h = (\alpha_1, \ldots, \alpha_{c+1}, \beta_{12}, \ldots, \beta_{1c}, \ldots, \beta_{c-1c})$$

and

$$h' = (\alpha'_1, \ldots, \alpha'_{c+1}, \beta'_{12}, \ldots, \beta'_{1c}, \ldots, \beta'_{c-1c}).$$

Then $h < h'$ if

$$\sum_{i=1}^{c+1} \alpha_i + 2 \sum_{1 \leqslant i<j \leqslant c} \beta_{ij} < \sum_{i=1}^{c+1} \alpha'_i + 2 \sum_{1 \leqslant i<j \leqslant c} \beta'_{ij},$$

or

$$\sum_{i=1}^{c+1} \alpha_i + 2 \sum_{1 \leqslant i<j \leqslant c} \beta_{ij} = \sum_{i=1}^{c+1} \alpha'_i + 2 \sum_{1 \leqslant i<j \leqslant c} \beta'_{ij}$$

and

$$\sum_{1\leqslant i<j\leqslant c}\beta_{ij} < \sum_{1\leqslant i<j\leqslant c}\beta_{ij}^{'},$$

or

$$\sum_{i=1}^{c+1}\alpha_i + 2\sum_{1\leqslant i<j\leqslant c}\beta_{ij} = \sum_{i=1}^{c+1}\alpha_i^{'} + 2\sum_{1\leqslant i<j\leqslant c}\beta_{ij}^{'},$$

$$\sum_{1\leqslant i<j\leqslant c}\beta_{ij} = \sum_{1\leqslant i<j\leqslant c}\beta_{ij}^{'}$$

and $(XT)^h \underset{\tau}{<} (XT)^{h^{'}}$. The order $<$ is a term order on $\mathbb{N}^{c+1+s}$, i.e. $h < h^{'}$ implies $h + g < h^{'} + g$ for any $g \in \mathbb{N}^{c+1+s}$. Note that this term order is different from that in Section 1.

For every polynomial $f \in S$, let f^* denote the initial term of f, i.e $f^* := f_h$ if $f = \sum\limits_{h^{'}\in\mathbb{N}^{c+1+s}} f_{h^{'}}$ and $h = \min\{h^{'} |\ f_{h^{'}} \neq 0\}$. Let J^* denote the ideal of S generated by the elements f^*, $f \in J$. Then S/J^* is a bigraded algebra. This algebra has a simpler structure than that of S/J. We can use S/J^* to compute the Hilbert function $H_{R[It]}(u, v)$ by the following proposition.

Proposition 2.1. *$H_{R[It]}(u, v) = H_{S/J^*}(u, v)$ for all $(u, v) \in \mathbb{N}^2$.*

Proof. Fix $(u, v) \in \mathbb{N}^2$. Let

$$D := \Big\{(\alpha_1, \dots, \alpha_{c+1}, \beta_{12}, \dots, \beta_{1c}, \dots, \beta_{c-1c}) \in \mathbb{N}^{c+1+s} \,\Big|\, \sum_{i=1}^{c+1}\alpha_i + 2\sum_{1\leqslant i<j\leqslant c}\beta_{ij} = u,\ \sum_{1\leqslant i<j\leqslant c}\beta_{ij} = v\Big\}.$$

Then $S_{(u,v)} = \bigoplus\limits_{h\in D} S_h$. Hence $S_{(u,v)} = 0$ if $D = \emptyset$. If $D \neq \emptyset$, we set

$$h_m := \min\{h |\ h \in D\},$$
$$h_M := \max\{h |\ h \in D\}.$$

By the definition of the order $<$ we have $D = \{h \in \mathbb{N}^{c+1+s} |\ h_m \leqslant h \leqslant h_M\}$. For every $h \in \mathbb{N}^{c+1+s}$ let

$$F_h := \bigoplus_{h^{'}\geqslant h} S_{h^{'}},$$
$$h^* := \min\{h^{'} \in \mathbb{N}^{c+1+s} |\ h^{'} > h\}.$$

Then $J_h^* \cong J \cap F_h / J \cap F_{h^*}$. Moreover, $F_{h_m} = \sum\limits_{h\in D} S_h \oplus F_{(h_M)^*} = S_{(u,v)} \oplus F_{(h_M)^*}$. This implies $J_{(u,v)} = J \cap F_{h_m} / J \cap F_{(h_M)^*}$. Using the chain

$$J \cap F_{h_m} \supset J \cap F_{(h_m)^*} \supset \ldots \supset J \cap F_{h_M} \supset J \cap F_{(h_M)^*}$$

we get

$$\begin{aligned}\dim_k J_{(u,v)} &= \sum_{h_m \leqslant h \leqslant h_M} \dim_k (J \cap F_h / J \cap F_{h^*}) \\ &= \sum_{h \in D} \dim_k J_h^* = \dim_k \bigoplus_{h \in D} J_h^* \\ &= \dim_k J_{(u,v)}^*.\end{aligned}$$

Hence $\dim_k(S/J)_{(u,v)} = \dim_k(S/J^*)_{(u,v)}$. So we get $H_{R[It]}(u,v) = H_{S/J^*}(u,v)$. □

Let $\prec$ be the term order on $\mathbb{N}^{c+1+s}$ induced from the term order δ. Then $\prec$ is artinian. Let Z denote the set of generators of J. Clearly, $f^* = f_*$ for all $f \in Z$. By Lemma 1.3, this implies $J^* = J_*$.

Now we will compute the mixed multiplicities of S/J^* and therefore the mixed multiplicities $e_i(R[It])$. By [7, Theorem 3.4], to compute the mixed multiplicities of S/J^* we only need to compute the mixed multiplicities of the facets of Δ with the highest dimension.

Lemma 2.2. *Let F be a facet of Δ with the highest dimension. Set*

$$j := \#\ \{F \cap \{(h,h) |\ 0 \leqslant h \leqslant c\}\}.$$

Then

$$e_i(k[F]) = \begin{cases} (-1)^{j-i-1} \binom{c-i}{j-i-1} 2^{j-i-1} & \textit{if } 0 \leqslant i < j, \\ 0 & \textit{if } i \geqslant j. \end{cases}$$

Proof. Let $A = k[X_1, \ldots, X_j, Y_1, \ldots, Y_{c+2-j}]$ be a bigraded polynomial ring with $\operatorname{bideg} X_h = (1,0), h = 1, \ldots, j$ and $\operatorname{bideg} Y_n = (2,1),\ \ n = 1, \ldots, c+2-j$. Then $k[F] \cong A$, hence $e_i(k[F]) = e_i(A)$. By [7, Lemma 1.3] we have

$$e_i(k[F]) = \begin{cases} (-1)^{j-i-1} \sum\limits_{j_1+\ldots+j_{c+2-j}=j-i-1} 2^{j_1} \ldots 2^{j_{c+2-j}} & \text{if } \ 0 \leqslant i < j, \\ 0 & \text{if } \ i \geqslant j. \end{cases}$$

It is easy to check that

$$\sum_{j_1+\ldots+j_{c+2-j}=j-i-1} 2^{j_1} \ldots 2^{j_{c+2-j}} = \binom{c-i}{j-i-1} 2^{j-i-1},$$

So we can conclude that

$$e_i(k[F]) = \begin{cases} (-1)^{j-i-1} \binom{c-i}{j-i-1} 2^{j-i-1} & \text{if } 0 \leqslant i < j, \\ 0 & \text{if } i \geqslant j. \end{cases}$$

□

Using Theorem 1.6, Lemma 2.1 and Lemma 2.2 we obtain the following result for the mixed multiplicities of the natural bigration of $R[It]$.

Theorem 2.3. *Let $R[It]$ be the Rees algebra associated with a rational curve in $\mathbb{P}^c$. Then*

$$e_i(R[It]) = \begin{cases} c^2+c-2 & \text{if } i=0, \\ c & \text{if } i=1, \\ 0 & \text{if } 2 \leqslant i \leqslant c. \end{cases}$$

Proof. By [7, Theorem 3.4] we have

$$e_i(R[It]) = \sum_{F\in\Delta, \dim F = c+1} e_i(k[F]),$$

$i = 0, 1, \dots, c$. For every $j = 1, \dots, c+1$ we set

$$m_j := \#\ \{F \in \Delta|\ \dim F = c+1 \text{ and } \#\ \{(h,h) \notin F\} = j\}.$$

Then

$$e_i(R[It]) = \sum_{j=i+1}^{c+1} m_j e_i(k[F]),$$

where F is a facet of Δ such that $\#\ \{(h,h) \notin F\} = j$. By Theorem 1.6 we have

$$m_j = \begin{cases} 2^c - c^2 + c - 2 & \text{if } j=1, \\ 2^{c-1} - c & \text{if } j=2, \\ 2^{c-j+1} & \text{if } 3 \leqslant j \leqslant c+1. \end{cases}$$

By Lemma 2.2 we can conclude that

$$\begin{aligned} e_0(R[It]) &= 2^c - c^2 + c - 2 - 2\binom{c-1}{c}(2^{c-1}-c) \\ &\quad + \sum_{j=3}^{c+1} 2^{c+1-j}(-1)^{j-1}\binom{c}{j-1}2^{j-1} \\ &= 2^c - c^2 + c - 2 - c2^c + 2c^2 + 2^c\sum_{j=3}^{c+1}(-1)^{j-1}\binom{c}{j-1} \\ &= 2^c + c^2 + c - c2^c - 2 + 2^c(c-1) \\ &= c^2 + c - 2, \end{aligned}$$

$$\begin{aligned} e_1(R[It]) &= 2^{c-1} - c + \sum_{j=3}^{c+1} 2^{c+1-j}(-1)^{j-2}\binom{c-1}{c-j+1}2^{j-2} \\ &= 2^{c-1} - c + 2^{c-1}\sum_{j=3}^{c+1}(-1)^{j-2}\binom{c-1}{j-2} \\ &= 2^{c-1} + c - 2^{c-1} = c, \end{aligned}$$

and

$$\begin{aligned} e_i(R[It]) &= \sum_{j=i+1}^{c+1} 2^{c+1-j}(-1)^{j-i-1}\binom{c-i}{j-i-1}2^{j-i-1} \\ &= 2^{c-i}\sum_{j=i+1}^{c+1}(-1)^{j-i-1}\binom{c-i}{j-i-1} = 0, \end{aligned}$$

$i = 2, \dots, c$. □

Let V denote the blow-up of $\operatorname{Proj} R$ along the subscheme defined by I. It is known that V can be embedded into a projective space by the linear system $(I^e)_d$ for any pair of positive integers e, d with $d > 2e$ [5, Lemma 1.1]. Let V_{de} denote the embedded variety. By [7, Corollary 4.4] we can compute the degree of V_{de} by means of the mixed multiplicities as follows

Corollary 2.4. *Assume that* $d > 2e$. *Then*

$$\deg V_{de} = (c^2 + c - 2)e^c + cde^{c-1}.$$

Proof. By [7, Corollary 4.4] and Theorem 2.3 we get

$$\begin{aligned} \deg V_{de} &= e_0(R[It])e^c + e_1(R[It])de^{c-1} \\ &= (c^2 + c - 2)e^c + cde^{c-1}. \end{aligned}$$

□

Acknowledgment

The author would like to thank Prof. Ngô Viêt Trung for his guidance.

References

[1] P. B. Bhattacharya, *The Hilbert function of two ideals*, Proc. Cambridge Phil. Soc. **53** (1957), 568-575.

[2] H. Bruns, J. Herzog, *Cohen-Macaulay rings*, Adv. Stud. Math, Cambridge Univ. Press. 1993.

[3] A. Conca, *Hilbert function and resolution of the powers of the ideal of the rationnal normal curve*, J. Pure Appl. Algebra **152** (2000), 65-74.

[4] A. Conca, J. Herzog, and G. Valla, *Sagbi bases with applications to blow-up algebras*, J. Reine Angew. Math. **474** (1996), 113-138.

[5] S. D. Cutkosky and J. Herzog, *Cohen-Macaulay coordinate rings of blowup schemes*, Comment. Math. Helvetici **72** (1997), 605-617.

[6] J. Herzog, N. V. Trung and B. Ulrich, *On the multiplicity of Rees algebras and associated graded rings of d-sequences*, J. Pure Appl. Algebra **80** (1992), 273-297.

[7] N. D. Hoang and N. V. Trung, *Hilbert functions of non-standard bigraded algebras*, Math. Z. **245** (2003), 309-334.

[8] D. Katz, S. Mandal, and J. K. Verma, *Hilbert function of bigraded algebras*, in: A. Simis, N. V. Trung and G. Valla (eds.), Commutative Algebra (ICTP, Trieste, 1992), 291-302, World Scientific, 1994.

[9] B. Teissier, *Cycles évanescent, sections planes, et conditions de Whitney*, Singularités à Cargèse 1972, Astèrisque 7-8 (1973), 285-362.
[10] J. K. Verma, *Rees algebras and mixed multiplicitis*, Proc. Amer. Math. Soc. **104** (1988), 1036-1045.

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