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ON THE ASYMPTOTIC EQUIVALENCE OF LINEAR DELAY EQUATIONS IN BANACH SPACE

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Abstract. In this paper we present some sufficient conditions for the asymptotic equivalence of linear evolution equations with time-delay in Banach spaces. Besides, we discuss the asymptotic relationship between C_0 groups and strongly continuous evolutionary processes. The obtained results extend the Levinsons theorem on the asymptotic equivalence of linear differential equations.

1. Introduction

Let us consider the following linear differential equations:

$$\frac{dx(t)}{dt} = Ax(t), \quad t \geqslant 0, \tag{1}$$

and

$$\frac{dy(t)}{dt} = C(t)y(t), \quad t \geqslant 0, \tag{2}$$

where $x(t), y(t) \in X$, X is a complex Banach space, A and $C(t)$ are linear operators acting on X for each $t \in R_+$. Under suitable conditions, equations (1) and (2) are well posed (see [1], [7], [9]). We recall that equations (1) and (2) are said to be asymptotically equivalent if there exists a bijection between the set of solutions $\{x(t)\}$ of (1) and the one of $\{y(t)\}$ of (2) such that

$$\lim_{t\to\infty} \|x(t) - y(t)\| = 0. \tag{3}$$

An interesting proplem in the qualitative theory of solutions to differential equations is to find conditions such that (1) and (2) are asymptotically equivalent. The first results on this problem were given by N. Levinson in 1946 (see [5]). Then, the results have been developed by many authors (see [2], [3], [4], [5], [6], [10]). In recent years, much attention has been devoted to the qualitative theory of solutions to differential equations with time delay (see [1], [6], [7], [12], [13], [14], [15]). In this direction, several authors have focused on extending the classical results of the asymptotic behavior of solutions to differential equations. In the models related to mechanics and biology (see [12], [13], [14], [15]), one usually

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studies retarded differential equations in the form

$$\frac{dy(t)}{dt} = Ay(t) + \mu \sum_{k=1}^{q} B_k(t)y(t+\tau_k). \tag{4}$$

In this paper, we are interested in finding conditions such that the solutions of (4) in the case $\mu = 0$ are asymptotically equivalent to the solutions of (4) in the case $\mu \neq 0$. We will consider the case $q = 1$. The case $q > 1$ can be treated in a similar way.

2. Main results

For a Banach space X we denote by $\mathcal{L}(X)$ the Banach space of all bounded linear operators on X. Together with (1), we consider the differential equation

$$\frac{dy(t)}{dt} = Ay(t) + B(t)y(t+\theta), \quad t \geq 0, \tag{5}$$

where $x(t) \in X,\ y(t) \in X,\ -h \leqslant \theta \leqslant 0,\ A \in \mathcal{L}(X)$ and $B(.) : [0, +\infty) \to \mathcal{L}(X)$ satisfies the condition

$$\int_0^{+\infty} \|B(t)\| dx < +\infty. \tag{6}$$

Denote by $T(t)$ and $N(t, t_0)$ the solution operators of (1) and (5), respectively. Then $(T(t))_{t\geqslant 0}$ is a C_0-semigroup in X, $(N(t, t_0))_{t\geqslant s}$ is a strongly continuous evolutionary process (see [16], [17]).

Definition 2.1. The equations (1) and (5) are said to be asymptotically equivalent if for every solution $x(t)$ of (1), there is a solution $y(t)$ of (5) such that

$$\lim_{t\to+\infty} \|y(t) - x(t)\| = 0, \tag{7}$$

and conversely for each solution $y(t)$ of (5) there is a solution $x(t)$ of (1) such that (7) holds.

Suppose that $\phi(.) \in C([-h, 0], X)$, by Grownwall–Belmann's lemma and the methods in [12] we can get the following result.

Theorem 2.1.

(a) *For a given* $\phi(.) \in C([-h, 0], X)$*, there exists a unique solution of* (4) *on* $[-h, +\infty]$ *satisfying* $y(t) = \phi(t)$, $(t \in [-h, 0])$.

(b) *If* $\|T(t)\| \leqslant M$ *for all* $t \geqslant 0$ *then* $N(t, t_0)$ *is a bounded operator, i.e., there exists* $K > 0$ *such that*

$$\|N(t, t_0)\| \leqslant K, \quad \textit{for all}\ \ t \geqslant t_0 > 0.$$

In order to prove the asymptotic equivalence of (1) and (4) we consider the following lemma.

Lemma 2.1. *Suppose that there is a projector $P : X \to X$ such that:*
(a) $\|PT(t)\| \leqslant Me^{-\omega t}$, *for all* $t \in R^+$,
(b) $\|(I-P)T(t)\| \leqslant m$, *for all* $t \in R$,
where M, m, ω are positive constants. Then the operator $F : X \to X$ defined by

$$Fx := \int_{t_0}^{+\infty} (I-P)T(t_0-s)B(s)N(s+\theta,t_0)xds$$

is bounded. Moreover, there exists a positive constant Δ such that

$$\|F\| < 1, \quad \forall t_0 \geqslant \Delta > 0.$$

Proof. Putting

$$U(t) = PT(t), \quad V(t) = (I-P)T(t),$$

we get

$$T(t) = U(t) + V(t).$$

By (6), for any $\alpha < 1$ we can find a number $\Delta > 0$ such that

$$\int_{t_0}^{+\infty} \|B(s)\|ds \leqslant \frac{\alpha}{m.K}, \quad \forall\ t_0 > \Delta > 0,$$

where the constant K is defined as in Theorem 2.1. By the above inequality and Theorem 2.1, we have

$$\|F\| \leqslant \int_{t_0}^{\infty} \|V(t_0-s)\|.\|B(s)\|.\|N(s+\theta,t_0)\|ds$$
$$\leqslant m.K \int_{t_0}^{\infty} \|B(s)\|ds \leqslant \alpha < 1, \quad \forall t_0 \geqslant \Delta > 0.$$

□

From Theorem 2.1 we obtain the following result.

Theorem 2.2. *Suppose that $(T(t))_{t\geqslant 0}$ satisfies all the conditions of Lemma 2.1. Moveover, the operator P commutes with $T(t)$ for all $t \geqslant 0$. Then (1) and (5) are asymptotically equivalent.*

Proof. By Theorem 2.1, for each $\phi(.) \in C\big([-h,0],X\big)$ equation (5) has an unique solution satisfying

(8) $$y(t) = T(t)\phi(0) + \int_0^t T(t-s)B(s)y(s+\theta)ds, \quad t \geqslant 0,$$

$$y(t) = \phi(t), \quad -h \leqslant t \leqslant 0.$$

Let $y(t_0) = y_0$, $t_0 > 0$. Then the solution $y(t)$ of (5) can be written in the form

$$y(t) = T(t-t_0)y_0 + \int_{t_0}^t T(t-s)B(s)y(s+\theta)ds,$$

Moreover, by Lemma 2.1, we have

$$T(t) = U(t) + V(t),$$
$$V(t-s) = T(t-t_0)V(t_0-s).$$

Let $Qx = (I+F)x$, $x \in X$. Then the operator $Q : X \to X$ is invertible. Assume that $y(t)$ is a solution of (5). For each sufficiently large $t_0 \in R^+$ and $y(t_0) \in X$, taking $x(t_0) = Qy(t_0)$ we have

$$x(t_0) = Qy(t_0) = y(t_0) + \int_{t_0}^{+\infty} V(t_0 - s)B(s)y(s+\theta)ds.$$

Hence,

$$\begin{aligned} x(t) &= T(t-t_0)x(t_0) \\ &= T(t-t_0)y(t_0) + \int_{t_0}^{+\infty} V(t-s)B(s)y(s+\theta)ds. \end{aligned}$$

Consequently,

$$\|y(t) - x(t)\| = \Big\| \int_{t_0}^{t} U(t-s)B(s)y(s+\theta)ds - \int_{t}^{\infty} V(t-s)B(s)y(s+\theta)ds \Big\|.$$

Therefore,

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant M.K\|y_0\| \int_{t_0}^{t} e^{-\omega(t-s)} \Big\|B(s)\Big\| ds + m.K\|y_0\| \int_{t}^{\infty} \Big\|B(s)\Big\| ds \\ &\leqslant M_1 \int_{t_0}^{t} e^{-\omega(t-s)} \Big\|B(s)\Big\| ds + M_2 \int_{t}^{\infty} \Big\|B(s)\Big\| ds, \quad \forall t \geqslant s, \end{aligned}$$

where $M_1 = M.K\|y_0\|$, $M_2 = mK\|y_0\|$.

For every positive number $\varepsilon > 0$, there exists a sufficiently large number t, $t > 2t_0$, such that the following inequalities are valid:

$$\int_{t_0}^{\frac{t}{2}} e^{-\omega(t-s)} \Big\|B(s)\Big\| ds \leqslant e^{-\frac{\omega t}{2}} \int_{t_0}^{\infty} \Big\|B(s)\Big\| ds < \frac{\varepsilon}{3M_1},$$

$$\int_{\frac{t}{2}}^{t} \Big\|B(s)\Big\| ds < \frac{\varepsilon}{3M_1},$$

$$\int_{t}^{\infty} \Big\|B(s)\Big\| ds < \frac{\varepsilon}{3M_2}\cdot$$

Hence,

$$\|y(t)-x(t)\| \leqslant M_1\Big(\int_s^{\frac{t}{2}} e^{-\omega(t-s)}\Big\|B(s)\Big\|ds + \int_{\frac{t}{2}}^{t} e^{-\omega(t-s)}\Big\|B(s)\Big\|ds\Big)$$

$$+ M_2\int_t^{\infty}\Big\|B(s)\Big\|ds$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

This means that

$$\lim_{t\to\infty}\|y(t)-x(t)\| = 0.$$

Semilarly, by the existence and uniqueness of solutions of (1) and (5) (see Theorem 2.1) we also get (7) for $x(t) = T(t-t_0)x(t_0)$ and $y(t) = N(t,t_0)y(t_0)$, where $y(t_0) = Q^{-1}x(t_0)$. The theorem is proved. □

Example 2.1. In the space l_2, we consider linear diffrential equations:

$$\frac{dx}{dt} = Ax, \tag{9}$$

$$\frac{dy}{dt} = Ay(t) + B(t)y(t+\theta), \tag{10}$$

where $-h \leqslant \theta \leqslant 0$, $t \geqslant t_0 \geqslant 0$, $x(t),\ y(t) \in l_2$. In an orthonormal basis we define

$$A = \operatorname{diag}(A_1, A_2, \dots, A_n, \dots)$$

with

$$A_n := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \dfrac{-1}{n} \\ 0 & \dfrac{1}{n} & 0 \end{pmatrix}$$

and let $B(.) : R^+ \to \mathcal{L}(\ell_2)$ be such that

$$\int_0^{+\infty} \|B(t)\|dt < +\infty.$$

Then the Cauchy operator $T(t)$ of (9) is in the form

$$T(t) = \operatorname{diag}(T_1, T_2, \dots, T_n, \dots)$$

with

$$T_n = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & \cos\dfrac{t}{n} & -\sin\dfrac{t}{n} \\ 0 & \sin\dfrac{t}{n} & \cos\dfrac{t}{n} \end{pmatrix}.$$

Note that

$$T(t) = \operatorname{diag}(U_1(t), U_2(t), \dots, U_n(t), \dots) + \operatorname{diag}(V_1(t), V_2(t), \dots, V_n(t), \dots),$$

where

$$U_n = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\dfrac{t}{n} & -\sin\dfrac{t}{n} \\ 0 & \sin\dfrac{t}{n} & \cos\dfrac{t}{n} \end{pmatrix}.$$

This shows that $\|U(t)\| \leqslant e^{-t}$ for every $t \in R^+$ and $\|V(t)\| \leqslant m < +\infty$ for every $t \in R$. Hence $T(t)$ satisfies the conditions of Theorem 2.2 . This implies that (9) and (10) are asymptotically equivalent.

The following is a generalization of the Levinson theorem (see [8]) for the asymptotic equivalence of the linear differential equations with time delay.

Theorem 2.3. *The equations* (1) *and* (5) *are asymptotically equivalent if one of following conditions is satisfied:*

(i) $(T(t))_{t\geqslant 0}$ *is a eventually compact, uniformly bounded* C_0*-semigoup.*

ii) $X = R^n$ *and* $(T(t))_{t\geqslant 0}$ *is a uniformly bounded* C_0*-semigoup (Levinson's theorem).*

Proof. (i) We will show that $T(t)$ satisfies all the conditions of Lemma 2.1. Since $(T(t))_{t\geqslant 0}$ is a eventually compact, bounded uniformly C_0-semigroup, we deduce that the spectral set $\sigma\big(T(1)\big)$ is countable and

$$\sigma\big(T(1)\big) \subset \{\lambda \in C : |\lambda| \leqslant 1\}$$

(see [16], [17]). Denoting $\sigma(T(1)) = \sigma_1 \cup \sigma_2$, where

$$\sigma_1 \subset \{\lambda \in C : |\lambda| < 1\}, \quad \sigma_2 \subset \{\lambda \in C : |\lambda| = 1\}.$$

Let P be a projection defined as follows

$$P = \frac{1}{2\pi i}\int_\gamma (zI - e^A)^{-1} dz$$

(see [16], [17]), where γ is a contour enclosing σ_1 and disjoint from σ_2. Since $T(t)$ commutes with the resolvent $(\lambda I - A)^{-1}$, we see that P commutes with $T(t)$.

Let

$$U(t) = PT(t), \quad V(t) = (I - P)T(t).$$

We have $\sigma(U(1)) \subset \{\lambda \in C : |\lambda < 1\}$. Hence $r_\sigma(U(1)) < 1$. Thus we need only to prove that $\|U(t)\| \leqslant Me^{-\omega t}$ for all $t > 0, M, \omega > 0$. By assumption, $(T(t))_{t\geqslant 0}$

is a eventually compact semigroup. So σ_2 has finitely many elements. Using the spectral maping theorem we have

$$\|V(t)\| \leqslant m < +\infty, \quad \forall t \in R.$$

Thus $T(t)$ satisfies the conditions of Lemma 2.1 and the assertion (i) is proved.

(ii) It suffices to observe that every bounded linear operator on R^n is compact. The theorem is proved. □

Theorem 2.4. *Suppose that $X = H$, H is Hilbert spase, and A is a compact, self-adjoint linear operator in H. If all the solutions of* (1) *are bounded, then* (1) *and* (5) *are asymptotically equivalent.*

Proof. Since $A \in \mathcal{L}(X)$ is self-adjoint and compact, we have that

$$\sigma(A) = \{\lambda_k : \lim_{k\to\infty} \lambda_k = 0, \lambda_k \in R, k = 1, 2, \dots\}.$$

Let $\{e_k\}_1^\infty$ be an orthonomal basis of H consisting eigen-vectors of A. Then, we can write A in the form

$$A = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}.$$

Moreover, all the solutions of (1) are bounded. Hence,

$$T(t) = \operatorname{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}, \dots\} \quad \text{and} \quad \operatorname{Re}\lambda_k \leqslant 0, \quad k = 1, 2, \dots$$

We deduce that $(T(t))_{t\geqslant 0}$ is a uniformly bounded C_0-semigroup. Using Theorem 2.1 we can show that there exists a possitive constant M_0 satisfying

$$\|T(t - t_0) - N(t, t_0)\| \leqslant 2M_0, \quad \forall t \in R^+.$$

Choosing $\xi = \sum_{i=1}^{\infty} \xi_i e_i \in H$ and denoting $P_n\xi = \sum_{i=1}^{n} \xi_i e_i$, we see that

$$\lim_{n\to\infty} (I - P_n)\xi = 0, \quad \forall \xi \in H.$$

Let $x(t) = T(t-t_0)\xi$ and $y(t) = N(t, t_0)\xi$ be solutions of (1) and (5) respectively. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|(I - P_n)\xi\| < \frac{\varepsilon}{4M_0}, \quad \forall n > n_0.$$

Taking $\lambda_0 = \max\{\lambda_1, \lambda_2, \dots, \lambda_{n_0}\}$ we have $\lambda_0 < 0$, which implies that there exists a number $t^* > t_0$ such that for any $t > t^*$ we get

$$\|[T(t - t_0) - N(t, t_0)]P_{n_0}\xi\| < \frac{\varepsilon}{2}\cdot$$

Thus, for any $t > t^*$ and $n > n_0$ we have

$$\begin{aligned}\|y(t) - x(t)\| &= \|T(t - t_0)\xi - N(t, t_0)\xi\| \\ &\leqslant \|T(t - t_0)P_{n_0}\xi - N(t, t_0)P_{n_0}\xi\| \\ &\quad + \|T(t - t_0)(I - P_{n_0})\xi - N(t, t_0)(I - P_{n_0})\xi\| \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon,\end{aligned}$$

i.e., $\lim_{t\to\infty} \|x(t) - y(t)\| = 0$. This implies that (1) and (5) are asymptotically equivalent. The theorem is proved. $\square$

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