ACTA MATHEMATICA VIETNAMICA
Volume 30, Number 3, 2005, pp. 241-260

ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF THE FIRST INITIAL BOUNDARY VALUE PROBLEMS FOR SCHRÖDINGER SYSTEMS IN DOMAINS WITH CONICAL POINTS I

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Abstract. Some asymptotic formulas in a neighbourhood of a conical point for solutions of the first initial boundary value problem for strongly Schrödinger systems are given.

1. Introduction and notations

Boundary value problems for Schrödinger equations and Schrödinger systems in a finite cylinder $\Omega_T = \Omega \times (0,T)$ have been studied by many authors (see [3,8,9]). The existence, uniqueness and smoothness of generalized solutions of the first initial boundary value problem for strongly Schrödinger systems in an infinite cylinder $\Omega_\infty = \Omega \times (0,\infty)$ were given in [4,5]. The aim of this paper is to establish some theorems on the asymptotic expansions of generalized solutions of the problem in domains with conical points.

Let Ω be a bounded domain in $\mathbb{R}^n$. Its boundary $\partial\Omega$ is assumed to be an infinitely differentiable surface everywhere except for the coordinate origin, in a neighbourhood of which Ω coincides with the cone $K = \{x : x/|x| \in G\}$, where G is a smooth domain on the unit sphere S^{n-1}. We begin by recalling some notations and functional spaces which will be frequenly used in this paper :

• $\Omega_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T),\ \Omega_\infty = \Omega \times (0,\infty), S_\infty = \partial\Omega \times (0,\infty), x = (x_1, \dots, x_n) \in \Omega,\ u(x,t) = (u_1(x,t), \dots, u_s(x,t))$ is a vector complex function, $|D^\alpha u|^2 = \sum\limits_{i=1}^{s} |D^\alpha u_i|^2,\quad u_{t^j} = \big(\partial^j u_1/\partial t^j, \dots, \partial^j u_s/\partial t^j\big), |u_{t^j}|^2 = \sum\limits_{i=1}^{s} \big|\partial^j u_i/\partial t^j\big|^2$, $dx = dx_1 \dots dx_n,\ \ r = |x| = \sqrt{x_1^2 + \dots + x_n^2}$.

• $H^l_\beta(\Omega)$ - the space consisting of all functions $u(x) = (u_1(x), \dots, u_s(x))$ which have generalized derivatives $D^\alpha u_i$, $|\alpha| \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^l_\beta(\Omega)} = \sum_{|\alpha|=0}^{l} \int_\Omega r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 dx < +\infty.$$

Received November 16, 2004.
2000 *Mathematics Subject Classification.* 35B65, 35C20, 35G15.
Key words and phrases. Generalized solution, smoothness, asymptotic expression, first boundary value problem, strongly Schrödinger system, domain with conical points.

• $H^{l,k}(e^{-\gamma t},\Omega_\infty)$ - the space consisting of all functions $u(x,t)$ which have generalized derivatives $D^\alpha u_i, \dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \le l$, $1 \le j \le k$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l,k}(e^{-\gamma t},\Omega_\infty)} = \int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{l} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2 \Big) e^{-2\gamma t} dxdt < +\infty.$$

In particular,

$$\|u\|^2_{H^{l,0}(e^{-\gamma t},\Omega_\infty)} = \sum_{|\alpha|=0}^{l} \int\limits_{\Omega_\infty} |D^\alpha u|^2 e^{-2\gamma t} dxdt.$$

• $\overset{\circ}{H}{}^{l,k}(e^{-\gamma t},\Omega_\infty)$ - the closure in $H^{l,k}(e^{-\gamma t},\Omega_\infty)$ of the set consisting of all infinitely differentiable in Ω_∞ functions which belong to $H^{l,k}(e^{-\gamma t},\Omega_\infty)$ and vanish near S_∞.
• $H^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)$ - the space consisting of all functions $u(x,t)$ which have generalized derivatives $D^\alpha u_i, \dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \le l$, $1 \le j \le k$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)} = \int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2 \Big) e^{-2\gamma t} dxdt < +\infty.$$

• $H^{l}_\beta(e^{-\gamma t},\Omega_\infty)$ - the space consisting of all functions $u(x,t)$ which have generalized derivatives $D^\alpha (u_i)_{t^j}, |\alpha|+j \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l}_\beta(e^{-\gamma t},\Omega_\infty)} = \sum_{|\alpha|+j=0}^{l} \int\limits_{\Omega_\infty} r^{2(\beta+|\alpha|+j-l)} |D^\alpha u_{t^j}|^2 e^{-2\gamma t} dxdt < +\infty.$$

• $V^{l}_\beta(e^{-\gamma t},\Omega_\infty)$ - the space consisting of all functions $u(x,t)$ which have generalized derivatives $D^\alpha (u_i)_{t^j}, |\alpha|+j \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{V^{l}_\beta(e^{-\gamma t},\Omega_\infty)} = \sum_{|\alpha|+j=1}^{l} \int\limits_{\Omega_\infty} r^{2(\beta+|\alpha|+j-l)} |D^\alpha u_{t^j}|^2 e^{-2\gamma t} dxdt + \int\limits_{\Omega_\infty} |u|^2 e^{-2\gamma t} dxdt < \infty.$$

• Let X be a Banach space. Denote by $L^\infty(0,\infty;X)$ the space consisting of all measurable functions $u : (0,\infty) \longrightarrow X, t \longmapsto u(x,t)$, satisfying

$$\|u\|_{L^\infty(0,\infty;X)} = \operatorname*{ess\,sup}_{t>0} \|u(x,t)\|_X < +\infty.$$

Consider the differential operator of order $2m$

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p\big(a_{pq}(x,t) D^q\big),$$

where a_{pq} are $s \times s$-matrices of measurable bounded in $\overline{\Omega}_\infty$ complex functions, $a_{pq} = (-1)^{|p|+|q|}a^*_{qp}$. Suppose that a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0, \infty)$ if $|p| = |q| = m$, and for each $t \in [0, \infty)$ the operator $L(x, t, D)$ is uniformly elliptic in $\overline{\Omega}$ with ellipticity constant a_0 independent of time t, i.e., we have

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geq a_0|\xi|^{2m}|\eta|^2,$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{C}^s \setminus \{0\}$ and $(x, t) \in \overline{\Omega}_\infty$.

In this paper we study the following problem: Find a function $u(x, t)$ such that

$$(1.1)\qquad (-1)^{m-1}iL(x,t,D)u - u_t = f(x,t) \quad \text{in } \Omega_\infty,$$

$$(1.2)\qquad u|_{t=0} = 0,$$

$$(1.3)\qquad \frac{\partial^j u}{\partial \nu^j}\Big|_{S_\infty} = 0, \quad j = 0, \dots, m-1,$$

where ν is the outer unit normal to S_∞.

A function $u(x, t)$ is called a generalized solution of the problem (1.1)-(1.3) in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ if and only if $u(x, t)$ belongs to $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ and for each $T > 0$ the following equality holds

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{\Omega_T} a_{pq}D^q u\overline{D^p\eta}dxdt + \int_{\Omega_T} u\overline{\eta}_t dxdt = \int_{\Omega_T} f\overline{\eta}dxdt$$

for any test function $\eta \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$, $\eta(x, T) = 0$.

Put

$$B(u,u)(t) = \sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{\Omega} a_{pq}D^q u\overline{D^p u}dx, \;\; u(x,t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty).$$

For a.e. $t \in [0, \infty)$, the function $x \mapsto u(x, t)$ belongs to $\overset{\circ}{H}{}^m(\Omega)$. Hence by Garding's inequality [2, Th.5.I, p. 44], we have

Lemma 1.1. *There exist two constants μ_0 and λ_0 ($\mu_0 > 0, \lambda_0 \geq 0$) such that*

$$(-1)^m B(u,u)(t) \geq \mu_0\|u(x,t)\|^2_{H^m(\Omega)} - \lambda_0\|u(x,t)\|^2_{L_2(\Omega)}$$

for all $u(x, t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$.

Therefore, using the transformation $u = e^{i\lambda_0 t}v$ if necessary, we can assume that the operator $L(x, t, D)$ satisfies

$$(1.4)\qquad (-1)^m B(u,u)(t) \geq \mu_0\|u\|^2_{H^m(\Omega)}$$

for all $u(x, t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$. This inequality is a basic tool for proving the existence and uniqueness of solutions of the problem under consideration.

2. Existence, uniqueness and smoothness of solutions

In this section we sumarize the known results on the existence, uniqueness and smoothness of generalized solutions of the problem (1.1)-(1.3).

Denote by m^* the number of multi-indexes which have order not exceeding m. Let μ_0 be the constant in (1.4). By using Theorems 3.1, 3.2 in [4] and induction we obtain the following result.

Theorem 2.1. *Let*

(i) $\sup\Big\{\Big|\dfrac{\partial a_{pq}}{\partial t}\Big| : (x,t) \in \overline{\Omega}_\infty, \quad 0 \le |p|, |q| \le m\Big\} = \mu < +\infty;$

$$\Big|\frac{\partial^k a_{pq}}{\partial t^k}\Big| \le \mu_1, \quad \mu_1 = \text{const} > 0, \quad \textit{for} \quad 2 \le k \le h+1;$$

(ii) $f_{t^k} \in L^\infty(0,\infty; L_2(\Omega))$, *for* $k \le h+1$;

(iii) $f_{t^k}(x,0) = 0$, *for* $k \le h$.

Then for every $\gamma > \gamma_0 = \dfrac{m^*\mu}{2\mu_0}$, *the problem* (1.1)-(1.3) *has exactly one generalized solution* $u(x,t)$ *in the space* $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$. *Moreover,* $u(x,t)$ *has derivatives with respect to* t *up to order* h *belonging to* $\overset{\circ}{H}{}^{m,0}(e^{-(2h+1)\gamma t}, \Omega_\infty)$ *and the following estimate holds*

$$\|u_{t^h}\|^2_{H^{m,0}(e^{-(2h+1)\gamma t}, \Omega_\infty)} \le C \sum_{k=0}^{h+1} \|f_{t^k}\|^2_{L^\infty(0,\infty; L_2(\Omega))},$$

where C *is a positive constant independent of* u *and* f.

From now on, for the sake of brevity we will write γ_h instead of $(2h+1)\gamma$ $(h = 1, 2, ...)$.

To study the smoothness with respect to (x,t) and establish asymptotic formulas of solutions of the problem (1.1)-(1.3), for simplicity we assume that the coefficients $a_{pq}(x,t)$ of the operator $L(x,t,D)$ are infinitely differentiable in $\overline{\Omega}_\infty$. Moreover, we also assume that $a_{pq}(x,t)$ and all its derivatives are bounded in $\overline{\Omega}_\infty$.

First, we recall two basic lemmas.

Lemma 2.1. [5] *Let* $f, f_t, f_{tt} \in L^\infty(0,\infty; L_2(K))$ *and* $f(x,0) = f_t(x,0) = 0$. *If* $u(x,t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ *is a generalized solution of the problem* (1.1)-(1.3) *in the space* $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ *such that* $u \equiv 0$ *whenever* $|x| > R = \text{const}$, *then* $u \in H^{2m,0}_m(e^{-\gamma_1 t}, K_\infty)$ *and the following estimate holds*

$$\|u\|^2_{H^{2m,0}_m(e^{-\gamma_1 t}, K_\infty)} \le C\Big[\|f\|^2_{L^\infty(0,\infty;L_2(K))} + \|f_t\|^2_{L^\infty(0,\infty;L_2(K))} + \|f_{tt}\|^2_{L^\infty(0,\infty;L_2(K))}\Big],$$

where $C = \text{const}$.

Denote by $L_0(0,t,D)$ the principal part of the operator $L(x,t,D)$ at origin 0. We consider the Dirichlet problem for the system

$$(-1)^{m-1}L_0(0,t,D)u = F(x,t), \quad x \in K. \tag{2.1}$$

Lemma 2.2. [5] *Let $u(x,t)$ be a generalized solution of the Dirichlet problem for the system* (2.1) *for a.e. $t \in [0,\infty)$ such that $u \equiv 0$ whenever $|x| > R =$ const, and $u(x,t) \in H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t}, K_\infty)$. Let $F \in H^{l,0}_{\beta}(e^{-\gamma t}, K_\infty)$. Then $u(x,t) \in H^{2m+l,0}_{\beta}(e^{-\gamma t}, K_\infty)$ and*

$$\|u\|^2_{H^{2m+l,0}_{\beta}(e^{-\gamma t},K_\infty)} \leq C\Big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},K_\infty)} + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},K_\infty)}\Big],$$

where $C =$ const.

Let ω be a local coordinate system on S^{n-1}. The principal part of the operator $L(x,t,D)$ at origin 0 can be written in the form

$$L_0(0,t,D) = r^{-2m}Q(\omega,t,rD_r,D_\omega), \;\; D_r = \frac{i\partial}{\partial r},$$

where Q is a linear operator with smooth coefficients. From now on the following spectral problem will play an important role

$$Q(\omega,t,\lambda,D_\omega)v(\omega) = 0, \quad \omega \in G, \tag{2.2}$$

$$D^j_\omega v(\omega) = 0, \quad \omega \in \partial G, \quad |j| = 0,\dots,m-1. \tag{2.3}$$

It is well known [7, p. 146] that for every $t \in [0,\infty)$ its spectrum is discrete.

Theorem 2.2. [5] *Let $u(x,t)$ be a generalized solution of the problem* (1.1)-(1.3) *in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ and let $f_{t^k} \in L^\infty(0,\infty;H^l_0(\Omega))$ for $k \leq 2m+l+1$, $f_{t^k}(x,0) = 0$ for $k \leq 2m+l$. In addition, supppose that the strip*

$$m - \frac{n}{2} \leq \text{ Im } \lambda \leq 2m+l-\frac{n}{2}$$

does not contain points of spectrum of the problem (2.2)-(2.3) *for every $t \in [0,\infty)$. Then $u(x,t) \in H^{2m+l}_0(e^{-\gamma_{2m+l}t},\Omega_\infty)$ and the following estimate holds*

$$\|u\|^2_{H^{2m+l}_0(e^{-\gamma_{2m+l}t},\Omega_\infty)} \leq C\sum_{k=0}^{2m+l+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;H^l_0(\Omega))},$$

where $C =$ const.

3. Asymptotic expansions of solutions

In this section we will study asymptotic expansions of generalized solutions of the problem (1.1)-(1.3) in the case the strip

$$m - \frac{n}{2} < \text{Im}\lambda < 2m+l-\frac{n}{2}$$

contains only one simple eigenvalue of the problem (2.2)-(2.3). From now on, for convenience we denote

$$L_{2,\gamma}[0,\infty) = \Big\{c(t) : c(t)e^{-\gamma t} \in L_2[0,\infty)\Big\}.$$

Lemma 3.1. *Let $u(x,t)$ be a generalized solution of the Dirichlet problem for the system (2.1) for a.e. $t \in [0,\infty)$ such that $u \equiv 0$ whenever $|x| > R = \mathrm{const}$, and let $u_{t^k} \in H_\beta^{2m+l,0}(e^{-\gamma t}, K_\infty)$, $F_{t^k} \in H_{\beta'}^{l,0}(e^{-\gamma t}, K_\infty)$ for $k \le h$, $\beta' < \beta \le m+l$. In addition, suppose that the straight lines*

$$\mathrm{Im}\lambda = -\beta + 2m + l - \frac{n}{2} \quad \text{and} \quad \mathrm{Im}\lambda = -\beta' + 2m + l - \frac{n}{2}$$

do not contain any point from the spectrum of the problem (2.2)-(2.3) for every $t \in [0,\infty)$, and in the strip

$$-\beta + 2m + l - \frac{n}{2} < \mathrm{Im}\lambda < -\beta' + 2m + l - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of the problem (2.2)-(2.3). Then the following representation holds

$$u(x,t) = c(t)r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t),$$

where ϕ is an infinitely differentiable function of (ω,t) that does not depend on the solution, $c_{t^k} \in L_{2,\gamma}[0,\infty)$ and $(u_1)_{t^k} \in H_{\beta'}^{2m+l,0}(e^{-\gamma t}, K_\infty)$ for $k \le h$.

Proof. From Theorem 3.2 in [10, p. 37] it follows that

$$u(x,t) = c(t)r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t), \tag{3.1}$$

where $\phi(\omega,t)$ is the eigenfunction of the problem (2.2)-(2.3) which corresponds to the eigenvalue $\lambda(t)$, $u_1 \in H_{\beta'}^{2m+l,0}(e^{-\gamma t}, K_\infty)$, and

$$c(t) = i\int_K F(x,t)r^{-i\overline{\lambda(t)}+2m-n}\psi(x,t)dx,$$

where ψ is the eigenfunction of the problem conjugating to the problem (2.2)-(2.3) and which corresponds to the eigenvalue $\overline{\lambda(t)}$. Since

$$\mathrm{Im}\overline{\lambda(t)} > \beta' - 2m - l + \frac{n}{2}$$

and

$$F \in H_{\beta'}^{l,0}(e^{-\gamma t}, K_\infty),$$

we have $c(t) \in L_{2,\gamma}[0,\infty)$. Hence the assertion is proved for $h = 0$.

Assume that the assertion is true for $0, 1, \dots, h-1$. Denoting u_{t^h} by v. From (2.1) we obtain

$$(-1)^{m-1}L_0(0,t,D)v = F_{t^h} + (-1)^m \sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}}, \tag{3.2}$$

where

$$L_{0t^k} = \sum_{|p|=|q|=m} \frac{\partial^k a_{pq}(0,t)}{\partial t^k} D^p D^q.$$

Put $S_0(\omega,t) = r^{-i\lambda(t)}\phi(\omega,t)$. Since $\phi(\omega,t) \in C^\infty(\omega,t)$ [1], from (3.1) it follows that

$$\begin{aligned}\sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}} = \sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)\big[(cS_0)_{t^{h-k}}\big] \\ + \sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)(u_1)_{t^{h-k}}.\end{aligned}$$

Using the induction hypothesis we obtain

$$\sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}} = F_1 - \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}} L_0(0,t,D)(S_0)_{t^k}, \tag{3.3}$$

where $F_1 \in H^{l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$. From (3.2) and (3.3) we see that

$$(-1)^{m-1}L_0(0,t,D)v = F_2 - (-1)^m \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}} L_0(0,t,D)(S_0)_{t^k},$$

where $F_2 \in H^{l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$. Hence by the arguments used in the proof of the case $h = 0$ we can find

$$u_{t^h} = v = \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^k} + d(t)S_0 + u_2, \tag{3.4}$$

where $d(t) \in L_{2,\gamma}[0,\infty)$, $u_2 \in H^{2m+l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$. From this equality it follows that

$$\begin{aligned}S_{0,1} &= u_{t^h} - \sum_{k=2}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^k} - (h-1)c_{t^{h-1}}(S_0)_t \\ &= c_{t^{h-1}}(S_0)_t + dS_0 + u_2.\end{aligned}$$

Now differentiate the equality (3.1) $(h-1)$ times by t. As a result we obtain

$$u_{t^{h-1}} = \sum_{k=0}^{h-1} \binom{h-1}{k} c_{t^{h-k-1}}(S_0)_{t^k} + (u_1)_{t^{h-1}}. \tag{3.5}$$

We rewrite (3.5) in the form

$$S_{0,2} = u_{t^{h-1}} - \sum_{k=1}^{h-1} \binom{h-1}{k} c_{t^{h-k-1}}(S_0)_{t^k} = c_{t^{h-1}}S_0 + (u_1)_{t^{h-1}}.$$

Then

$$\begin{aligned}(S_{0,2})_t &= u_{t^h} - \sum_{k=1}^{h-1} \binom{h-1}{k} \left[c_{t^{h-k}}(S_0)_{t^k} + c_{t^{h-k-1}}(S_0)_{t^{k+1}}\right] \\ &= u_{t^h} - \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^k} + c_{t^{h-1}}(S_0)_t.\end{aligned}$$

From this equality and (3.4) we obtain

$$(S_{0,2})_t = c_{t^{h-1}}(S_0)_t + dS_0 + u_2.$$

Put $S_1 = S_0^{-1}(u_1)_{t^{h-1}}$, $S_2 = S_0^{-1}u_2 - S_0^{-2}(S_0)_t(u_1)_{t^{h-1}}$. It is easy to check that

$$S_0^{-1}S_{0,2} = c_{t^{h-1}} + S_1, \quad (S_0^{-1}S_{0,2})_t = d + S_2.$$

It follows that

$$\begin{aligned}I(t) &= c_{t^{h-1}}(t) - c_{t^{h-1}}(0) - \int_0^t d(\tau)d\tau \\ &= \int_0^t S_2(x,\tau)d\tau - S_1(x,t) + S_1(x,0).\end{aligned}$$

Since $(u_1)_{t^{h-1}} \in H^{2m+l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$, $u_2 \in H^{2m+l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$; so $S_1, S_2 \in H^{0,0}_{-\frac{n}{2}}$ $(e^{-\gamma t}, K_\infty)$. Therefore $I(t) \in H^0_{-\frac{n}{2}}(K)$, i.e., $I(t) \equiv 0$. Hence $c_{t^h} = d \in L_{2,\gamma}[0,\infty)$, and $(u_1)_{t^h} = u_2 \in H^{2m+l,0}_{\beta'}(e^{-\gamma t}, K_\infty)$. The proof is completed. □

Lemma 3.2. *Let $u(x,t)$ be a generalized solution of the Dirichlet problem for the system (2.1) for a.e. $t \in [0,\infty)$ such that $u \equiv 0$ whenever $|x| > R =$ const, and let $u_{t^k} \in H^{2m,0}_{m-\mu}(e^{-\gamma t}, K_\infty)$, $F_{t^k} \in H^{0,0}_{m-\mu-1}(e^{-\gamma t}, K_\infty)$ for $k \le 2m-1$, $0 \le \mu \le m-1$. In addition, suppose that the straight lines*

$$\mathrm{Im}\lambda = m + \mu - \frac{n}{2} \quad \textit{and} \quad \mathrm{Im}\lambda = m + \mu + 1 - \frac{n}{2}$$

do not contain any point from the spectrum of the problem (2.2)-(2.3) for every $t \in [0,\infty)$, and in the strip

$$m + \mu - \frac{n}{2} < \mathrm{Im}\lambda < m + \mu + 1 - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of the problem (2.2)-(2.3). Then the representation

$$u(x,t) = c(x,t)r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\,\lambda(t)}(e^{-\gamma t}, K_\infty)$ and $u_1(x,t) \in H^{2m}_{m-\mu-1}(e^{-\gamma t}, K_\infty)$, holds.

Proof. From Lemma 3.1 it follows that

$$u(x,t) = c(t)r^{-i\lambda(t)}\varphi(\omega,t) + u_1(x,t), \tag{3.6}$$

where $\varphi(\omega,t)$ is the eigenfunction of the problem (2.2)-(2.3) which corresponds to the eigenvalue $\lambda(t)$, $c_{t^k} \in L_{2,\gamma}[0,\infty)$ and $(u_1)_{t^k} \in H^{2m,0}_{m-\mu-1}(e^{-\gamma t}, K_\infty)$ for $k \leq 2m-1$.

Let K' be a domain such that $K' \subseteq K$ and $\varphi(\omega,t) \neq 0$ in K'. Consider in K' a linear differential operator of the form

$$D_1 = \frac{1-\varphi^2_\omega}{-i\varphi}\frac{\partial}{\partial r} + \frac{\lambda(t)\varphi_\omega}{r}\frac{\partial}{\partial \omega}.$$

Then

$$r^{i\lambda(t)+1}D_1 u = \lambda(t)c(t) + r^{i\lambda(t)+1}D_1 u_1. \tag{3.7}$$

Put

$$c_1(x,t) = r^{i\lambda(t)+1}D_1 u, \quad c_0(t) = \lambda(t)c(t).$$

Since $u_1 \in H^{2m,0}_{m-\mu-1}(e^{-\gamma t}, K_\infty)$, it follows from (3.7) that

(3.8)
$$\int\limits_{K'_\infty} r^{2(m-\mu+\mathrm{Im}\lambda(t)-2m-1)}|c_0-c_1|^2 e^{-2\gamma t}dxdt = \int\limits_{K'_\infty} r^{2(m-\mu)}|D_1 u_1|^2 e^{-2\gamma t}dxdt < \infty.$$

We have $c_1 \in V^{2m-1}_{m-\mu-2+\mathrm{Im}\ \lambda(t)}(e^{-\gamma t}, K'_\infty)$. Indeed, in variable x the operator D_1 has the form

$$D_1 = \sum_{i=1}^{n} \phi_i(\omega,t)\frac{\partial}{\partial x_i},$$

where $\phi_i(\omega,t) \in C^\infty(\omega,t)$. Since $(u_1)_{t^k} \in H^{2m,0}_{m-\mu-1}(e^{-\gamma t}, K'_\infty)$ for $k \leq 2m-1$,

$$\sum_{0\leq|\alpha|\leq 2m}\int\limits_{K'_\infty} r^{2(-m-\mu-1+|\alpha|)}|D^\alpha(u_1)_{t^k}|^2 e^{-2\gamma t}dxdt < \infty$$

for $k \leq 2m-1$. Hence it follows that

(3.9)
$$\sum_{1\leq|\alpha|+k\leq 2m-1}\int\limits_{K'_\infty} r^{2(-\mu-1+\mathrm{Im}\lambda(t)+k+|\alpha|-m)}|D^\alpha(r^{i\lambda(t)+1}D_1u_1)_{t^k}|^2 e^{-2\gamma t}dxdt < \infty.$$

Since $(c_0)_{t^k} \in L_{2,\gamma}[0,\infty), k \leq 2m-1$, and $\mathrm{Im}\lambda(t) > m+\mu-\dfrac{n}{2}$ we have

$$\sum_{1\leq k\leq 2m-1}\int\limits_{K'_\infty} r^{2(-\mu-1+\mathrm{Im}\lambda(t)+k-m)}|(c_0)_{t^k}|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.10}$$

From (3.9) and (3.10) we obtain

(3.11)
$$\sum_{1\le|\alpha|+k\le 2m-1}\int_{K'_\infty} r^{2(-\mu-1+\mathrm{Im}\lambda(t)+k+|\alpha|-m)}|D^\alpha(c_1)_{t^k}|^2 e^{-2\gamma t}dxdt$$
$$= \sum_{1\le|\alpha|+k\le 2m-1}\int_{K'_\infty} r^{2(-\mu-1+\mathrm{Im}\lambda(t)+k+|\alpha|-m)}|D^\alpha(r^{i\lambda(t)+1}D_1u_1)_{t^k}|^2 e^{-2\gamma t}dxdt$$
$$+ \sum_{1\le k\le 2m-1}\int_{K'_\infty} r^{2(-\mu-1+\mathrm{Im}\lambda(t)+k-m)}|(c_0)_{t^k}|^2 e^{-2\gamma t}dxdt < \infty.$$

Since $u \in H^{2m,0}_{m-\mu}(e^{-\gamma t}, K_\infty)$ and $-\mathrm{Im}\lambda(t) > -m-\mu-1+\dfrac{n}{2}$, it holds

$$\int_{K'_\infty} |c_1|^2 e^{-2\gamma t}dxdt = \int_{K'_\infty} |r^{i\lambda(t)+1}D_1u|^2 e^{-2\gamma t}dxdt \tag{3.12}$$
$$\le C\int_{K'_\infty} r^{2(-m-\mu+n/2)}|Du|^2 e^{-2\gamma t}dxdt$$
$$\le C\int_{K'_\infty} r^{2(1-m-\mu)}|Du|^2 e^{-2\gamma t}dxdt \le C\|u\|^2_{H^{2m,0}_{m-\mu}(e^{-\gamma t},K'_\infty)} < \infty,$$

where $C = \text{const}$. From (3.11) and (3.12) we deduce that
$$c_1(x,t) \in V^{2m-1}_{m-\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K'_\infty).$$

From (3.8) it follows that the function $c_1(x,t)$ can be extended to an element of $V^{2m-1}_{m-\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$ (we denote the extended function also by $c_1(x,t)$) and

$$\int_{K_\infty} r^{(m-\mu+\mathrm{Im}\lambda(t)-2m-1)}|c_0-c_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.13}$$

By Lemma 2 in [6], there exists a function $\widetilde{c}_1(x,t)$ such that
$$\widetilde{c}_1(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$$

and
$$\|\widetilde{c}_1\|_{V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t},K_\infty)} \le C\|c_1\|_{V^{2m-1}_{m-\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t},K_\infty)},$$

$$\int_{K_\infty} r^{2(m-\mu+\mathrm{Im}\lambda(t)-2m-1)}|c_1-\widetilde{c}_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$\int_{K_\infty} r^{2(m-\mu+\mathrm{Im}\lambda(t)-2m-1)}|c_0-\widetilde{c}_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.15}$$

Put

$$u_2 = \frac{1}{\lambda(t)}\Big[c_0 - \widetilde{c}_1\Big] r^{-i\lambda(t)}\varphi(\omega, t) + u_1. \tag{3.16}$$

By the property $\widetilde{c}_1 \in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$ and by (3.15), we have

$$\Big[c_0 - \widetilde{c}_1\Big] r^{-i\lambda(t)} \in H^{2m}_{m-\mu-1}(e^{-\gamma t}, K_\infty).$$

From (3.6) and (3.16) we get

$$u(x,t) = \frac{1}{\lambda(t)}\widetilde{c}_1(x,t) r^{-i\lambda(t)}\varphi(\omega, t) + u_2(x,t). \tag{3.17}$$

Put

$$c_2(x,t) = \frac{1}{\lambda(t)}\widetilde{c}_1(x,t)\varphi(\omega,t).$$

From (3.17) it follows that

$$u(x,t) = c_2(x,t) r^{-i\lambda(t)} + u_2(x,t),$$

where $c_2(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$.

We will prove that $u_2 \in H^{2m}_{m-\mu-1}(e^{-\gamma t}, K_\infty)$. On one hand, since

$$\widetilde{c}_1(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty),$$

we have

$$\sum_{1\le|\alpha|+k\le 2m, |\alpha|\neq 0} \int_{K_\infty} r^{2(-m-\mu-1+\mathrm{Im}\lambda(t)+|\alpha|+k)} |D^\alpha(c_0 - \widetilde{c}_1)_{t^k}|^2 e^{-2\gamma t} dxdt$$
$$= \sum_{1\le|\alpha|+k\le 2m, |\alpha|\neq 0} \int_{K_\infty} r^{2(-m-\mu-1+\mathrm{Im}\lambda(t)+|\alpha|+k)} |D^\alpha(\widetilde{c}_1)_{t^k}|^2 e^{-2\gamma t} dxdt < \infty.$$

On the other hand, since

$$c_2 \in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$$

and

$$u_{t^k} \in H^{2m,0}_{m-\mu}(e^{-\gamma t}, K_\infty) \quad \text{for} \quad k \le 2m-1,$$

we have

$$\sum_{1\le k\le 2m} \int_{K_\infty} r^{2(-m-\mu-1+k)} |(u_2)_{t^k}|^2 e^{-2\gamma t} dxdt$$
$$= \sum_{1\le k\le 2m} \int_{K_\infty} r^{2(-m-\mu-1+k)} |(u - c_2 r^{-i\lambda(t)})_{t^k}|^2 e^{-2\gamma t} dxdt < \infty.$$

Therefore

$$\sum_{1\le|\alpha|+k\le 2m}\int_{K_\infty} r^{2(-m-\mu-1+|\alpha|+k)}|D^\alpha(u_2)_{t^k}|^2 e^{-2\gamma t}dxdt \tag{3.18}$$
$$\le C_1\sum_{1\le|\alpha|+k\le 2m}\int_{K_\infty} r^{2(-m-\mu-1+\mathrm{Im}\lambda(t)+|\alpha|+k)}|D^\alpha(c_0-\widetilde{c}_1)_{t^k}|^2 e^{-2\gamma t}dxdt$$
$$+C_2\|u_1\|^2_{H^{2m,0}_{m-\mu-1}(e^{-\gamma t},K_\infty)}<\infty,$$

where $C_i=\text{const},\ i=1,2$. Since $u_1\in H^{2m,0}_{m-\mu-1}(e^{-\gamma t},K_\infty)$, from (3.15) we deduce that

$$\int_{K_\infty} r^{2(-m-\mu-1)}|u_2|^2e^{-2\gamma t}dxdt \tag{3.19}$$
$$\le C\int_{K_\infty} r^{2(-m-\mu-1)}\Big(|c_0-\widetilde{c}_1|^2r^{2\mathrm{Im}\lambda(t)}+|u_1|^2\Big)e^{-2\gamma t}dxdt$$
$$=C\int_{K_\infty} r^{2(-m-\mu+\mathrm{Im}\lambda(t)-2m-1)}|c_0-\widetilde{c}_1|^2e^{-2\gamma t}dxdt$$
$$+C\int_{K_\infty} r^{2(-m-\mu-1)}|u_1|^2e^{-\gamma t}dxdt<\infty,\quad C=\text{const}.$$

From (3.18) and (3.19) it follows that

$$\sum_{0\le|\alpha|+k\le 2m}\int_{K_\infty} r^{2(-m-\mu-1+|\alpha|+k)}|D^\alpha(u_2)_{t^k}|^2 e^{-2\gamma t}dxdt<\infty,$$

i.e., $u_2\in H^{2m}_{m-\mu-1}(e^{-\gamma t},K_\infty)$. The proof of the lemma is completed. □

Proposition 3.1. *Let $u(x,t)$ be a generalized solution of the problem* (1.1)-(1.3) *in the spaces $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ such that $u\equiv 0$ whenever $|x|>R=\text{const}$, and let $f_{t^k}\in L^\infty(0,\infty;L_2(K))$ for $k\le 2m+1$, $f_{t^k}(x,0)=0$ for $k\le 2m$. Assume that in the strip $m-\dfrac{n}{2}\le \mathrm{Im}\lambda\le m+\mu+1-\dfrac{n}{2}, 0\le\mu\le m-1$, there exists only one simple eigenvalue $\lambda(t)$ of the problem* (2.2)-(2.3) *such that*

$$m+\mu-\frac{n}{2}<\mathrm{Im}\lambda(t)<m+\mu+1-\frac{n}{2}.$$

Then the representation

$$u(x,t)=c(x,t)r^{-i\lambda(t)}+u_1(x,t),$$

where $c(x,t)\in V^{2m}_{m-\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma_{2m}t},K_\infty)$ and $u_1\in H^{2m}_{m-\mu-1}(e^{-\gamma_{2m}t},K_\infty)$, holds

Proof. We distinguish the following cases:

Case 1: $\mu\le 1$. Rewrite the system (1.1) in the form

$$(-1)^{m-1}L_0(0,t,D)u=F, \tag{3.20}$$

where $F(x,t) = -i(u_t + f) + (-1)^{m-1}\Big[L_0(0,t,D) - L(x,t,D)\Big]u$. From Theorem 2.1 and Lemma 2.1 it follows that $F \in H^{0,0}_{m-\mu}(e^{-\gamma_1 t}, K_\infty)$. Since the strip

$$m - \frac{n}{2} < \mathrm{Im}\lambda < m + \mu - \frac{n}{2}$$

does not contain any point belonging to the spectrum of the problem (2.2)-(2.3) for every $t \in [0,\infty)$, from Theorem 2.1 and the results on elliptic problems [11,12], we deduce that $u \in H^{2m,0}_{m-\mu}(e^{-\gamma_1 t}, K_\infty)$.

Since $f_{t^k} \in L^\infty(0,\infty; L_2(K))$ for $k \le 2m+1$, $f_{t^k}(x,0) = 0$ for $k \le 2m$, it follows from Theorem 2.1 and Lemma 2.1 that $u_{t^k} \in H^{2m,0}_{m-\mu}(e^{-\gamma_{k+1} t}, K_\infty)$ for $k \le 2m-1$. Therefore

$$F_{t^k} \in H^{0,0}_{m-\mu}(e^{-\gamma_{k+1} t}, K_\infty), \quad k \le 2m-1. \tag{3.21}$$

Put $v = u_{t^k}$. By (3.20) we have

$$(-1)^{m-1} L_0(0,t,D)v = F_{t^k}(x,t) + L_{0t^k}, \tag{3.22}$$

where

$$L_{0t^k} u_{t^k} = \sum_{s=1}^{k} \binom{k}{s} \sum_{|p|=|q|=m} \frac{\partial^k a_{pq}(0,t)}{\partial t^k} D^p D^q u_{t^{k-s}}.$$

Let $u_{t^j} \in H^{2m,0}_{m-\mu}(e^{-\gamma_{j+1} t}, K_\infty), j \le k-1$. Then

$$\sum_{s=1}^{k} \binom{k}{s} \sum_{|p|=|q|=m} \frac{\partial^k a_{pq}(0,t)}{\partial t^k} D^p D^q u_{t^{k-s}} \in H^{0,0}_{m-\mu}(e^{-\gamma_k t}, K_\infty).$$

Hence from (3.21) and (3.22) it follows that

$$(-1)^{m-1} L_0(0,t,D)v = F_1,$$

where $F_1 \in H^{0,0}_{m-\mu}(e^{-\gamma_{k+1} t}, K_\infty)$. Then $v \in H^{2m,0}_{m-\mu}(e^{-\gamma_{k+1} t}, K_\infty)$, i.e.,

$$u_{t^k} \in H^{2m,0}_{m-\mu}(e^{-\gamma_{k+1} t}, K_\infty), \quad k \le 2m-1. \tag{3.23}$$

By Theorem 2.1, $(u_t + f)_{t^k} \in H^{0,0}_{m-\mu-1}(e^{-\gamma_{k+1} t}, K_\infty), k \le 2m-1$. On the other hand,

$$\Big[L_0(0,t,D) - L(x,t,D)\Big] = \sum_{|\alpha|=2m} [b_\alpha(x,t) - b_\alpha(0,t)]D^\alpha + \sum_{|\alpha|\le 2m-1} b_\alpha(x,t) D^\alpha, \tag{3.24}$$

and $|b_\alpha(x,t) - b_\alpha(0,t)| \le C|x|, C = \text{const}$. Hence from (3.23) it follows that

$$F_{t^k}(x,t) \in H^{0,0}_{m-\mu-1}(e^{-\gamma_{k+1} t}, K_\infty), k \le 2m-1. \tag{3.25}$$

By Lemma 3.2, from (3.23) and (3.25) we obtain

$$u(x,t) = c(x,t) r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\,\lambda(t)}(e^{-\gamma_{2m} t}, K_\infty)$, $u_1(x,t) \in H^{2m}_{m-\mu-1}(e^{-\gamma_{2m} t}, K_\infty)$.

Case 2: $\mu = m_0 + \mu_0, 0 < \mu_0 \leq 1, m_0 \in \mathbb{Z}_+$. Let $m_0 = 0$. By the arguments used in Case 1 we have

$$u_{t^k} \in H^{2m,0}_{m-\mu_0}(e^{-\gamma_{k+1}t}, K_\infty), \quad F_{t^k}(x,t) \in H^{0,0}_{m-\mu_0-1}(e^{-\gamma_{k+1}t}, K_\infty), \tag{3.26}$$

for $k \leq 2m-1$. Assume that (3.26) is true for $\mu = m_0 - 1 + \mu_0$, i.e.

$$u_{t^k} \in H^{2m,0}_{m-m_0-\mu_0+1}(e^{-\gamma_{k+1}t}, K_\infty), F_{t^k}(x,t) \in H^{0,0}_{m-m_0-\mu_0}(e^{-\gamma_{k+1}t}, K_\infty), \tag{3.27}$$

for $k \leq 2m-1$. Let $k = 0$. From (3.27) it follows that

$$F(x,t) \in H^{0,0}_{m-m_0-\mu_0}(e^{-\gamma_1 t}, K_\infty).$$

Since in the strip

$$m + m_0 + \mu_0 - 1 - \frac{n}{2} \leq \mathrm{Im}\lambda \leq m + m_0 + \mu_0 - \frac{n}{2}$$

there are no points belonging to the spectrum of the problem (2.2)-(2.3) for every $t \in [0, \infty)$, by the arguments analogous to those used in Case 1, we obtain

$$u \in H^{2m,0}_{m-m_0-\mu_0}(e^{-\gamma_1 t}, K_\infty).$$

Hence it follows from (3.24) that

$$F(x,t) \in H^{0,0}_{m-m_0-\mu_0-1}(e^{-\gamma_1 t}, K_\infty).$$

By induction on k and the arguments analogous to those used in the proof of (3.23) and (3.25), we obtain

$$u_{t^k} \in H^{2m,0}_{m-m_0-\mu_0}(e^{-\gamma_{k+1}t}, K_\infty), F_{t^k}(x,t) \in H^{0,0}_{m-m_0-\mu_0-1}(e^{-\gamma_{k+1}t}, K_\infty), k \leq 2m-1,$$

i.e., (3.26) is true for $\mu = m_0 + \mu_0$. Hence

(3.28)
$$u_{t^k} \in H^{2m,0}_{m-\mu}(e^{-\gamma_{k+1}t}, K_\infty), \; F_{t^k}(x,t) \in H^{0,0}_{m-\mu-1}(e^{-\gamma_{k+1}t}, K_\infty), \quad k \leq 2m-1.$$

Since $m + \mu - n/2 < \mathrm{Im}\lambda(t) < m + \mu + 1 - n/2$, from (3.28) and Lemma 3.2 it follows that

$$u(x,t) = c(x,t)r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m}_{m-\mu-1+\mathrm{Im}\ \lambda(t)}(e^{-\gamma_{2m}t}, K_\infty)$, $u_1(x,t) \in H^{2m}_{m-\mu-1}(e^{-\gamma_{2m}t}, K_\infty)$. The proof is completed. □

Lemma 3.3. *Let $u(x,t)$ be a generalized solution of the Dirichlet problem for the system (2.1) for a.e. $t \in [0,\infty)$ such that $u \equiv 0$ whenever $|x| > R = \mathrm{const}$, and let $u_{t^k} \in H^{2m+l}_\mu(e^{-\gamma t}, K_\infty), k \leq 1$, $F \in H^l_{\mu-1}(e^{-\gamma t}, K_\infty), 0 \leq \mu \leq 1$. In addition, suppose that the straight lines*

$$\mathrm{Im}\lambda = -\mu + 2m + l - \frac{n}{2} \quad \text{and} \quad \mathrm{Im}\lambda = -\mu + 2m + l + 1 - \frac{n}{2}$$

do not contain any point from the spectrum of the problem (2.2)-(2.3) for every $t \in [0, \infty)$, and in the strip

$$-\mu + 2m + l - \frac{n}{2} < \mathrm{Im}\lambda < -\mu + 2m + l + 1 - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ *of the problem* (2.2)-(2.3). *Then the representation*

$$u(x,t) = c(x,t)r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m+l}_{\mu-1+\mathrm{Im}\ \lambda(t)}(e^{-\gamma t}, K_\infty)$ *and* $u_1(x,t) \in H^{2m+l}_{\mu-1}(e^{-\gamma t}, K_\infty)$*, holds.*

Proof. We will use the same symbols as in the proof of Lemma 3.2. Repeating the arguments used in the proof of Lemma 3.2, we obtain

$$u(x,t) = c(t)r^{-i\lambda(t)}\varphi(\omega,t) + u_1(x,t), \tag{3.29}$$

where $\varphi(\omega,t)$ is the eigenfunction of the problem (2.2) - (2.3) which corresponds to the eigenvalue $\lambda(t)$, $c(t) \in L_{2,\gamma}[0,\infty)$, $u_1(x,t) \in H^{2m+l,0}_{\mu-1}(e^{-\gamma t}, K_\infty)$.

We have

$$r^{i\lambda(t)+1}D_1 u = \lambda(t)c(t) + r^{i\lambda(t)+1}D_1 u_1.$$

Put

$$c_1(x,t) = \frac{1}{\lambda(t)} r^{i\lambda(t)+1} D_1 u.$$

Since $u_1 \in H^{2m+l,0}_{\mu-1}(e^{-\gamma t}, K_\infty)$, from (3.29) it follows that

$$\int\limits_{K'_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)-2m-l-1)}|c-c_1|^2 e^{-2\gamma t}dxdt$$
$$= \int\limits_{K'_\infty} |\lambda(t)|^{-2} r^{2(\mu-2m-l)}|D_1u_1|^2 e^{-2\gamma t}dxdt < \infty.$$

Since $u_t \in H^{2m+l}_{\mu}(e^{-\gamma t}, K'_\infty)$,

$$\sum_{0\le|\alpha|+k\le 2m+l} \int\limits_{K'_\infty} r^{2(\mu+k+|\alpha|-2m-l)}|D^\alpha u_{t^{k+1}}|^2 e^{-2\gamma t}dxdt < \infty.$$

Therefore

$$\sum_{0\le|\beta|+1+k\le 2m+l} \int\limits_{K'_\infty} r^{2(\mu+k+|\beta|+1-2m-l)}|D^\beta D_1 u_{t^{k+1}}|^2 e^{-2\gamma t}dxdt < \infty.$$

Hence

$$\sum_{0\le|\beta|+s\le 2m+l, s\ge 1} \int\limits_{K'_\infty} r^{2(\mu+s+|\beta|-2m-l)}|D^\beta D_1 u_{t^s}|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.30}$$

Since $u \in H^{2m+l}_{\mu}(e^{-\gamma t}, K'_\infty)$, we have

$$\sum_{0\le|\beta|\le 2m+l} \int\limits_{K'_\infty} r^{2(\mu+1+|\beta|-2m-l)}|D^\beta D_1 u|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.31}$$

Since

$$D^\alpha (c_1)_{t^k} = \sum_{|\beta|\le|\alpha|, s\le k} d_{s,\beta}(t) r^{i\lambda(t)+1-|\alpha|+|\beta|} \ln^{k-s} r D^\beta D_1 u_{t^s},$$

where $d_{s,\beta}(t) \in C^\infty[0,\infty)$, from (3.30) and (3.31) we obtain

(3.32)

$$\begin{aligned}
&\sum_{1\le|\alpha|+k\le 2m+l-1, k\ge 1} \int_{K'_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)+|\alpha|+k-2m-l-1)} |D^\alpha (c_1)_{t^k}|^2 e^{-2\gamma t} dxdt \\
&\le C_1 \sum_{1\le|\alpha|+k\le 2m+l-1, k\ge 1} \sum_{|\beta|\le|\alpha|, s\le k} \int_{K'_\infty} r^{2(\mu+|\beta|+k-2m-l)} |D^\beta D_1 u_{t^s}|^2 e^{-2\gamma t} dxdt \\
&\le C_2 \sum_{|\beta|+s\le 2m+l-1} \int_{K'_\infty} r^{2(\mu+|\beta|+s-2m-l)} |D^\beta D_1 u_{t^s}|^2 e^{-2\gamma t} dxdt \\
&\quad + C_3 \sum_{|\beta|\le 2m+l-1} \int_{K'_\infty} r^{2(\mu+1+|\beta|-2m-l)} |D^\beta D_1 u|^2 e^{-2\gamma t} dxdt < \infty,
\end{aligned}$$

where $C_i =$ const, $i = 1,2,3$.

For $k = 0$, since $u_1 \in H^{2m+l,0}_{\mu-1}(e^{-\gamma t}, K'_\infty)$ we have

$$\begin{aligned}
&\sum_{1\le|\alpha|\le 2m+l-1} \int_{K'_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)+|\alpha|-2m-l-1)} |D^\alpha c_1|^2 e^{-2\gamma t} dxdt \\
&\le C \sum_{1\le|\beta|\le 2m+l} \int_{K'_\infty} r^{2(\mu+|\beta|-2m-l-1)} |D^\beta u_1|^2 e^{-2\gamma t} dxdt \\
&\le C \|u_1\|^2_{H^{2m+l,0}_{\mu-1}(e^{-\gamma t}, K'_\infty)}, \quad C = \text{const}.
\end{aligned} \tag{3.33}$$

From (3.32) and (3.33) it follows that

$$\sum_{1\le|\alpha|+k\le 2m+l-1} \int_{K'_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)+|\alpha|+k-2m-l-1)} |D^\alpha (c_1)_{t^k}|^2 e^{-2\gamma t} dxdt < \infty. \tag{3.34}$$

Since $-\mathrm{Im}\lambda(t) > \mu - 1 - 2m - l + n/2$,

$$\begin{aligned}
\int_{K'_\infty} |c_1|^2 e^{-2\gamma t} dxdt &\le C \int_{K'_\infty} r^{2(1-\mathrm{Im}\lambda(t))} |D_1 u|^2 e^{-2\gamma t} dxdt \\
&\le \int_{K'_\infty} r^{2(\mu-2m-l+n/2)} |D_1 u|^2 e^{-2\gamma t} dxdt \\
&\le C \int_{K'_\infty} r^{2(\mu-2m-l+1)} |D_1 u|^2 e^{-2\gamma t} dxdt \\
&\le C \|u\|^2_{H^{2m+l}_\mu (e^{-\gamma t}, K'_\infty)} < \infty.
\end{aligned} \tag{3.35}$$

From (3.34) and (3.35) we deduce that $c_1 \in V^{2m+l-1}_{\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K'_\infty)$. Hence it follows that the function $c_1(x,t)$ can be extended to an element of $V^{2m+l-1}_{\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$ (we denote the extended function by $c_1(x,t)$) and

$$\int\limits_{K_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)-2m-l-1)}|c-c_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.36}$$

By Lemma 2 in [6], there exists a function $\widetilde{c}_1(x,t) \in V^{2m+l}_{\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t}, K_\infty)$ such that

$$\|\widetilde{c}_1\|_{V^{2m+l}_{\mu-1+\mathrm{Im}\lambda(t)}(e^{-\gamma t},K_\infty)} \le C\|c_1\|_{V^{2m+l-1}_{\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t},K_\infty)},$$

$$\int\limits_{K_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)-2m-l-1)}|c_1-\widetilde{c}_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.37}$$

From (3.36) and (3.37) we have

$$\int\limits_{K_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)-2m-l-1)}|c-\widetilde{c}_1|^2 e^{-2\gamma t}dxdt < \infty. \tag{3.38}$$

Put

$$u_2 = [c-\widetilde{c}_1]r^{-i\lambda(t)}\varphi(\omega,t) + u_1. \tag{3.39}$$

From (3.29) we have

$$u(x,t) = \widetilde{c}_1(x,t)r^{-i\lambda(t)}\varphi(\omega,t) + u_2(x,t). \tag{3.40}$$

We will prove that $u_2(x,t) \in H^{2m+l}_{\mu-1}(e^{-\gamma t}, K_\infty)$. Since $u_{t^k} \in H^{2m+l}_{\mu}(e^{-\gamma t}, K_\infty)$, it follows that

(3.41)

$$\begin{aligned}
&\sum_{1\le k+|\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu-1+k+|\alpha|-2m-l)}|D^\alpha(u_2)_{t^k}|^2 e^{-2\gamma t}dxdt\\
&\le \sum_{1\le k+|\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu-1+k+|\alpha|-2m-l)}|D^\alpha u_{t^k}|^2 e^{-2\gamma t}dxdt\\
&\quad + C_1\sum_{1\le k+|\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu-1+k+|\alpha|-2m-l)}|D^\alpha(r^{-i\lambda(t)}\widetilde{c}_1)_{t^k}|^2 e^{-2\gamma t}dxdt\\
&\le C_1\sum_{1\le k+|\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu-1+k+|\alpha|-2m-l)}|D^\alpha(r^{-i\lambda(t)}\widetilde{c}_1)_{t^k}|^2 e^{-2\gamma t}dxdt\\
&\quad + C_2\sum_{1\le k+|\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu+k+|\alpha|-2m-l)}|D^\alpha(u_t)_{t^k}|^2 e^{-2\gamma t}dxdt\\
&\quad + C_3\sum_{1\le |\alpha|\le 2m+l}\int\limits_{K_\infty} r^{2(\mu+|\alpha|-2m-l)}|D^\alpha u|^2 e^{-2\gamma t}dxdt
\end{aligned}$$

$$\leq C_4\Big[\|u_t\|^2_{H^{2m+l}_{\mu}(e^{-\gamma t},K_\infty)} + \|u\|^2_{H^{2m+l}_{\mu}(e^{-\gamma t},K_\infty)} + \|\widetilde{c}_1\|^2_{V^{2m+l-1}_{\mu-2+\mathrm{Im}\lambda(t)}(e^{-\gamma t},K_\infty)}\Big],$$

where $C_i = \mathrm{const},\ i = 1,2,3,4.$

Let $|\alpha| = k = 0$. From (3.38) and (3.39) we obtain

$$\int\limits_{K_\infty} r^{2(\mu-1-2m-l)}|u_2|^2 e^{-2\gamma t}dxdt \tag{3.42}$$

$$\leq C_5 \int\limits_{K_\infty} r^{2(\mu+\mathrm{Im}\lambda(t)-1-2m-l)}|c-\widetilde{c}_1|^2 e^{-2\gamma t}dxdt$$

$$+ C_6 \int\limits_{K_\infty} r^{2(\mu-1-2m-l)}|u_1|^2 e^{-2\gamma t}dxdt < \infty,$$

where $C_i = \mathrm{const},\ i = 5,6$. From (3.41) and (3.42) we obtain

$$\sum_{0\leq k+|\alpha|\leq 2m+l} \int\limits_{K_\infty} r^{2(\mu-1+k+|\alpha|-2m-l)}|D^\alpha(u_2)_{t^k}|^2 e^{-2\gamma t}dxdt.$$

Put $c_2 = \widetilde{c}_1(x,t)\varphi(\omega,t)$. Then (3.40) implies that

$$u(x,t) = c_2(x,t)r^{-i\lambda(t)} + u_2(x,t),$$

where $c_2 \in V^{2m+l}_{\mu-1+\mathrm{Im}\ \lambda(t)}(e^{-\gamma t},K_\infty)$, $u_2 \in H^{2m+l}_{\mu-1}(e^{-\gamma t},K_\infty)$. The lemma is proved. □

Proposition 3.2. *Let $u(x,t)$ be a generalized solution of the problem* (1.1)-(1.3) *in the spaces $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ such that $u \equiv 0$ whenever $|x| > R = \mathrm{const}$, and let $f_{t^k} \in L^\infty(0,\infty;H^l_0(K))$ for $k \leq l+2m+1$, $f_{t^k}(x,0) = 0$ for $k \leq l+2m$. Assume that in the strip $m-\dfrac{n}{2} \leq \mathrm{Im}\lambda \leq 2m+l-\dfrac{n}{2}$, there exists only one simple eigenvalue $\lambda(t)$ of the problem* (2.2)-(2.3) *such that*

$$2m+l-1-\frac{n}{2} < \mathrm{Im}\lambda(t) < 2m+l-\frac{n}{2}.$$

Then the representation

$$u(x,t) = c(x,t)r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m+l}_{\mathrm{Im}\lambda(t)}(e^{-\gamma_{2m+l}t},K_\infty)$ and $u_1(x,t) \in H^{2m+l}_0(e^{-\gamma_{2m+l}t},K_\infty)$, holds.

Proof. Rewrite the system (1.1) in the form

$$(-1)^{m-1}L_0(0,t,D)u = F(x,t),$$

where $F(x,t) = -i(u_t+f) + (-1)^{m-1}[L_0(0,t,D) - L(x,t,D)]u$.

Since $f_{t^k} \in L^\infty(0,\infty;H^l_0(K))$ for $k \leq l+2m+1$, $f_{t^k}(x,0) = 0$ for $k \leq l+2m$, and the strip

$$m-\frac{n}{2} \leq \mathrm{Im}\lambda \leq 2m+l-1-\frac{n}{2}$$

does not contain any point from the spectrum of the problem (2.2)-(2.3) for every $t \in [0, \infty)$, from Theorem 2.2 it follows that

$$u_{t^k} \in H_0^{2m+l-1}(e^{-\gamma_{2m+l}t}, K_\infty), \quad k \leq 1. \tag{3.43}$$

Since $[i(u_t + f)]_{t^k} \in L^\infty(0, \infty; H_0^l(K)), k \leq 1$, from (3.43) and the arguments used in the proof of Lemma 2.2 [5] we obtain

$$u_{t^k} \in H_1^{2m+l}(e^{-\gamma_{2m+l}t}, K_\infty), \quad k \leq 1.$$

Hence from (3.24) it follows that

$$[L_0(0, t, D) - L(x, t, D)]u \in H_0^l(e^{-\gamma_{2m+l}t}, K_\infty).$$

Therefore

$$F(x, t) \in H_0^l(e^{-\gamma_{2m+l}t}, K_\infty).$$

Since $2m + l - 1 - \dfrac{n}{2} < \mathrm{Im}\lambda(t) < 2m + l - \dfrac{n}{2}$, the straight lines

$$\mathrm{Im}\lambda = -1 + 2m + l - \frac{n}{2} \ \text{ and } \ \mathrm{Im}\lambda = 2m + l - \frac{n}{2}$$

do not contain points of spectrum of problem (2.2)-(2.3) for every $t \in [0, \infty)$, and in the strip

$$-1 + 2m + l - \frac{n}{2} < \mathrm{Im}\lambda < 2m + l - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of the problem (2.2)-(2.3). By Lemma 3.3,

$$u(x, t) = c(x, t)r^{-i\lambda(t)} + u_1(x, t),$$

where $c(x, t) \in V_{\mathrm{Im}\lambda(t)}^{2m+l}(e^{-\gamma_{2m+l}t}, K_\infty), u_1 \in H_0^{2m+l}(e^{-\gamma_{2m+l}t}, K_\infty)$. The proposition is proved. $\square$

From Propositions 3.1, 3.2 and the arguments used in the proof of Theorem 3.1 in [5], we obtain the following results.

Theorem 3.1. *Let $u(x, t)$ be a generalized solution of the problem* (1.1)-(1.3) *in the spaces $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$, and let $f_{t^k} \in L^\infty(0, \infty; L_2(\Omega))$ for $k \leq 2m + 1$, $f_{t^k}(x, 0) = 0$ for $k \leq 2m$. Assume that in the strip $m - \dfrac{n}{2} < \mathrm{Im}\lambda < m + \mu + 1 - \dfrac{n}{2}, 0 \leq \mu \leq m - 1$, there exists only one simple eigenvalue $\lambda(t)$ of the problem* (2.2)-(2.3) *such that*

$$m + \mu - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + \mu + 1 - \frac{n}{2}.$$

Then the representation

$$u(x, t) = c(x, t)r^{-i\lambda(t)} + u_1(x, t),$$

where $c(x, t) \in V_{m-\mu-1+\mathrm{Im}\lambda(t)}^{2m}(e^{-\gamma_{2m}t}, \Omega_\infty)$ and $u_1 \in H_{m-\mu-1}^{2m}(e^{-\gamma_{2m}t}, \Omega_\infty)$, holds.

Theorem 3.2. *Let $u(x,t)$ be a generalized solution of the problem* (1.1)-(1.3) *in the spaces $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$, and let $f_{t^k} \in L^\infty(0,\infty;H_0^l(\Omega))$ for $k \le l+2m+1$, $f_{t^k}(x,0)=0$ for $k \le l+2m$. Assume that in the strip $m-\dfrac{n}{2} \le \mathrm{Im}\lambda \le 2m+l-\dfrac{n}{2}$ there exists only one simple eigenvalue $\lambda(t)$ of the problem* (2.2)-(2.3) *such that*

$$2m+l-1-\frac{n}{2} < \mathrm{Im}\lambda(t) < 2m+l-\frac{n}{2}.$$

Then the representation

$$u(x,t) = c(x,t)r^{-i\lambda(t)} + u_1(x,t),$$

where $c(x,t) \in V^{2m+l}_{\mathrm{Im}\lambda(t)}(e^{-\gamma_{2m+l}t},\Omega_\infty)$ and $u_1 \in H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)$, holds.

References

[1] V. M. Eni, *On the stability of number of solution of an analytic operator - function and pertubation of eigenvalues and eigenvectors*, DAN SSSR **173** (1967), N° 6, 1251-1254 (in Russian).

[2] G. Fichera, *Existence Theorems in Elasticity Theory,* Mir, Moscow 1974.

[3] N. M. Hung, *The first initial boundary value problem for Schrödinger systems in non-smooth domains*, Diff. Urav. **34** (1998), 1546-1556 (in Russian).

[4] N. M. Hung and C. T. Anh, *On the solvability of the first initial boundary value problem for Schrödinger systems in infinite cylinders*, Vietnam J. Math. **32** (2004), 41-48.

[5] N. M. Hung and C. T. Anh, *On the smoothness of solutions of the first initial boundary value problem for Schrödinger systems in domains with conical points*, Vietnam J. Math. **33** (2005), 135-147.

[6] V. A. Kondratiev, *Singularities of a solution of the Dirichlet problem for second elliptic equations in a neighbourhood of an edge*, Diff. Urav. **13** (1977), 2026 - 2032 (in Russian).

[7] V. A. Kozlov, V. G. Mazya, J. Rossmann, *Elliptic Boundary Value Problems in Domains with Points Singularities,* Math. Surveys and Monographs, Vol. **52**, Amer. Math. Society, 1997.

[8] O. A. Ladyzhenskaya, *On the non-stationary operator equations and its application to linear problems of mathematical physics*, Mat. Sbornik. **45** (1958) 123-158 (in Russian).

[9] O. A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics,* Nauka, Moscow, 1973 (in Russian).

[10] V. G. Mazya, B. A. Plamenevsky, *On the coefficients in the asymptotic of solutions of the elliptic boundary problem in domains with conical points*, Math Nachr. **76** (1977), 29-60 (in Russian).

[11] V. G. Mazya, B. A. Plamenevsky, *Elliptic boundary value problems on manifolds with singularities*, Problems Math. Anal. LGU, 1977, 85-145 (in Russian).

[12] S. A. Nazarov, B. A. Plamenevsky, *Elliptic Problems in Domains with Piecewise - Smooth Boundary,* Nauka, Moscow 1991 (in Russian).

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