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A SHORT PROOF OF THE SNAKE THEOREM OF KARLIN

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Abstract. New results on surjective mappings on simplexes are given. Based on the results, a short proof of the snake theorem of Karlin is obtained.

1. Introduction

The snake theorem of Karlin (see [1], Chapter II) is one of the deepest theorems of the theory of T-systems. Two proofs on the existence of the snake are known in the literature: the first one of Karlin based essentially the Brouwer fixed-point theorem and the possibility of approximating of T-systems by differentiable T-systems, and the second one in [2] (Chapter IX) which is derived from a deep result on the best approximation due to M. G. Krein.

The aim of this paper is to give a short proof of the snake theorem. Our approach follows essentially the same line as the proof of Karlin, but it has two advantages:

(i) Based on our results on surjective mappings on simplexes (Theorem 2.1), the proof can neglect the approximation techniques used in the proof of Karlin and thus considerably simplifies the arguments.

(ii) The existence of a snake with a finite number of boundary curves (Corollary 3.1) follows at once from our proof.

2. The Brouwer fixed-point theorem and surjective mappings on simplexes

We denote respectively by Σ^n and B^n the n-dimensional simplex and its relative interior:

$$\Sigma^n = \{x = (x_0, x_1, \ldots, x_n) : x_i \geq 0 \quad i = 0, 1, \ldots, n, \quad \sum_0^n x_i = 1\},$$

$$B^n = \{x = (x_0, x_1, \ldots, x_n) : x_i > 0 \quad i = 0, 1, \ldots, n, \quad \sum_0^n x_i = 1\}.$$

Theorem 2.1. *The following conclusions are equivalent:*

1. *The simplex Σ^n has the fixed-point property, i.e., for every continuous mapping f from Σ^n into itself there is a point $x \in \Sigma^n$ such that $f(x) = x$.*

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2. *Let $f = (f_0, f_1, \dots, f_n)$ be a continuous mapping from Σ^n into itself such that:*

$$\text{(1)} \qquad \textit{For each } i = 0, 1, \dots, n, \ \textit{if } x_i = 0 \textit{ then } f_i(x) = 0.$$

Then f is surjective, i.e., $f(\Sigma^n) = \Sigma^n$.

3. *Let $f = (f_0, f_1, \dots, f_n)$ be a continuous mapping from B^n into itself such that:*

$$\text{(2)} \qquad \textit{For each } i = 0, 1, \dots, n, \ \textit{if } x_i \to 0 \textit{ then } f_i(x) \to 0.$$

Then f is surjective.

Proof. 1) $\Rightarrow$ 2). For every fixed point $k = (k_0, k_1, \dots, k_n) \in B^n$, put

$$h_i(x) = \frac{k_i^{-1} f_i(x)}{\sum\limits_{j=0}^{n} k_j^{-1} f_j(x)} \quad (i = 0, 1, \dots, n; \quad x \in \Sigma^n).$$

It is clear that $h = (h_0, h_1, \dots, h_n)$ is a continuous mapping from Σ^n into itself and also satisfies to the condition like (1), i.e.,

$$\text{(3)} \qquad \text{For each } i = 0, 1, 2, \dots, n, \ \text{ if } \ x_i = 0 \text{ then } h_i(x) = 0.$$

Therefore, as observed by Karlin (see [1], p. 68), it follows from the conclusion 1 and from the condition (3) that there exists a point $x \in \Sigma^n$ such that

$$h_0(x) = h_1(x) = \cdots = h_n(x) = \frac{1}{n+1}.$$

Consequently,

$$\frac{f_0(x)}{k_0} = \frac{f_1(x)}{k_1} = \cdots = \frac{f_n(x)}{k_n} = \frac{\sum\limits_0^n f_i(x)}{\sum\limits_0^n k_i} = \frac{1}{1} = 1.$$

In other words, $f(x) = k$ or equivalently, $B^n \subset f(\Sigma^n)$. On the other hand, since f is continuous, the image $f(\Sigma^n)$ is compact, hence $f(\Sigma^n) = \Sigma^n$.

2) $\Rightarrow$ 3). Suppose that $f = (f_0, f_1, \cdots, f_n)$ is continuous mapping from B^n into itself, satisfying the property (2). Let P be the centre of the simplex Σ^n, i.e.,

$$P = \left(\frac{1}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1} \right).$$

For every $0 < t < \frac{1}{n+1}$, put

$$\Sigma_t^n = \{x = (x_0, x_1, \dots, x_n) : x_i \geq t \text{ for all } i = 0, 1, \dots, n; \sum_0^n x_i = 1\}$$

and call Π^t the homothetic, with centre P, which transforms the simplex Σ^n onto the simplex Σ_t^n. Note that Π^t is a homomorphism satisfying the following property

$$\text{(4)} \qquad \text{For each } i = 0, 1, \dots, n, \ x_i = 0 \text{ if and only if } \Pi_i^t(x) = t.$$

Now we define a continuous mapping $T^t = (T^t_0, T^t_1, \ldots, T^t_n)$ from Σ^n into Σ^n_t such that

$$\text{For each } i = 0, 1, \ldots, n, \text{ if } x_i \le t \text{ then } T^t_i(x) = t \tag{5}$$

and

$$T^t(x) = x \quad \text{for all} \quad x \in \Sigma^n_t.$$

In fact, for each $i = 0, 1, \ldots, n$, T^t_i can be defined explicitly by the formula

$$T^t_i(x) = \begin{cases} t & \text{if } x_i \le t, \\ t + \dfrac{(x_i - t)}{\sum\limits_{x_j > t} (x_j - t)}(1 - (n+1)t) & \text{if } x_i > t. \end{cases}$$

It follows from (2) that for every $0 < t < \dfrac{1}{n+1}$ there exists $s \in (0, \dfrac{1}{n+1})$ such that

$$\text{For each } i = 0, 1, \ldots, n, \text{ if } x_i \le s \text{ then } f_i(x) \le t.$$

If we put $g = (\Pi^t)^{-1} \circ T^t \circ f \circ \Pi^s$, then g is a continuous mapping from Σ^n into itself such that for each $i = 0, 1, 2, \ldots, n$ if $x_i = 0$ then $g_i(x) = 0$. Therefore, we deduce from the conclusion 2 that $g(\Sigma^n) = \Sigma^n$.

But (4) gives again $(T^t \circ f)(\Sigma^n_s) = \Sigma^n_t$ and it follows at once from (5) that

$$\Sigma^n_t \subset f(\Sigma^n_s) \subset f(B^n).$$

The above assertion is true for every $0 < t < \dfrac{1}{n+1}$. Hence $f(B^n) = B^n$.

3) $\Rightarrow$ 1). Let $f = (f_0, f_1, \ldots, f_n)$ be a continuous mapping from Σ^n into itself. For $t > 0$ let us define the mapping $f^t = (f^t_0, f^t_1, \ldots, f^t_n)$ from B^n into itself as follows

$$f^t_i(x) = \frac{x_i (f_i(x) + t)^{-1}}{\sum\limits_{j=0}^{n} x_j (f_j(x) + t)^{-1}}, \quad i = 0, 1, \ldots, n; x \in B^n.$$

Clearly, f^t is continuous and it satisfies the condition (2), i.e., for each $i = 0, 1, 2, \cdots, n$ if $x_i \to 0$ then $f^t_i(x) \to 0$. It should be noted that the mapping $x \mapsto \sum\limits_0^n x_j (f_j(x) + t)^{-1}$ (from Σ^n into R) is continuous and strictly positive on the simplex Σ^n, Σ^n is a compact set. Hence

$$\min_{x \in \Sigma^n} \left(\sum_0^n x_j (f_j(x) + t)^{-1} \right) > 0.$$

It follows from the conclusion 3 that $f^t(B^n) = B^n$, thus there exists a point $x^t \in B^n$ such that

$$f^t_0(x^t) = f^t_1(x^t) = \cdots = f^t_n(x^t) = \frac{1}{n+1}.$$

Consequently,

$$\frac{x_0^t}{f_0(x^t)+t} = \frac{x_1^t}{f_1(x^t)+t} = \cdots = \frac{x_n^t}{f_n(x^t)+t} = \frac{\sum\limits_0^n x_i^t}{\sum\limits_0^n f_i(x^t)+(n+1)t}$$
$$= \frac{1}{1+(n+1)t}.$$

It follows that

$$f_i(x^t) = \big(1+(n+1)t\big)x_i^t - t \quad \text{ for all } i = 0,1,\dots,n.$$

Since Σ^n is a compact set and f is continuous, there exists an accumulation point $\bar{x}$ of the set $\{x^{\frac{1}{m}} \mid m = 1,2,3,\dots\}$. Then we have $f_i(\bar{x}) = \bar{x}_i$ for all $i = 0,1,\dots,n$, i.e., $f(\bar{x}) = \bar{x}$. □

Remark. The conclusion 1 in Theorem 2.1 is nothing else than the Brouwer fixed point theorem, whereas the conclusion 2 is implicity showed in Karlin's proof of the snake theorem.

3. A short proof of the existence of the snake

Let $\{u_i\}_0^n$ be $(n+1)$ continuous functions defined on the closed interval $[a,b]$. We recall that $\{u_i\}_0^n$ is called a T-system (Tchebycheff system) of order n on $[a,b]$ if every non-zero polynomial u (i.e., u is a non-trivial linear combination of the $u_i's$) has no more than n zeros in $[a,b]$. This condition is equivalent to the fact that the determinant

$$\det\begin{pmatrix} u_0 & u_1 & \cdots & u_n \\ t_0 & t_1 & \cdots & t_n \end{pmatrix} = \det\big(u_i(t_j)\big)_{(n+1)\times(n+1)}$$

keeps a fixed sign on the set of all partitions

$$a \le t_0 < t_1 < \cdots < t_n \le b \;\text{ of }\; [a,b].$$

The given T-system is called a T_+ or T_- system according to the above mentioned sign is positive or negative.

We formulate the snake theorem as follows.

Theorem 3.1 (Karlin [1]). *Let $\{u_i\}_0^n$ be a T-system and f and g two continuous functions on $[a,b]$ such that there exists a polynomial $v(t)$ with the property that $f(t) > v(t) > g(t)$ for all $t \in [a,b]$. Then there exists a unique polynomial $P(t)$ satisfying the properties:*

(i) *$f(t) \ge P(t) \ge g(t)$ for all $t \in [a,b]$.*

(ii) *There exist $(n+1)$ points $a \le t_0 < t_1 \cdots < t_n \le b$ such that*

$$P(t_i) = \begin{cases} f(t_i) & i = 0,2,4,\dots \\ g(t_i) & i = 1,3,5,\dots \end{cases}$$

$P(t)$ is usually called the snake and f, g the gates.

Proof of the existence of the snake. The proof of the unicity of the snake can be proceeded exactly as in the original proof of Karlin. We give only the proof of the existence of the snake.

We can assume without loss of generality that the given system $\{u_i\}_0^n$ is a T_+-system and that $v \equiv 0$. (It should be noted that v is also a polynomial.)

For every x belonging to the simplex

$$(b-a)B^n = \left\{ x = (x_0, x_1, \ldots, x_n) : x_i > 0 \text{ for } i = 0, 1, \ldots, n; \sum_0^n x_i = b - a \right\}$$

we put

$$u_x(t) = \lambda \det \begin{pmatrix} u_0 & u_1 & \cdots & u_n \\ t & s_1 & \cdots & s_n \end{pmatrix},$$

where $\{s_i\}_1^n$ is a partition of the interval $[a, b]$. We can choose λ and $\{s_i\}_i^n$ so that $u_x(t)$ can be described as follows: $u_x(t)$ is a polynomial (for the given T_+-system) of the form

$$u_x(t) = \sum_0^n a_i(x) u_i(t)$$

and furthermore it satisfies to the following properties:

(i) $u_x(t)$ vanishes at each of the points

$$s_i = a + \sum_0^{i-1} x_j, \quad i = 1, 2, \cdots, n,$$

(ii) $u_x(a) > 0$ and $\sum\limits_0^n a_i^2(x) = 1$.

Indeed, the coefficients $\{a_i(x)\}_0^n$ of $u_x(t)$ can be computed explicitly as follows

$$a_i(x) = \frac{(-1)^i B_i}{\sqrt{\sum\limits_0^n B_j^2}},$$

where

$$B_i = \det \begin{pmatrix} u_0 & \cdots & u_{i-1} & u_{i+1} & \cdots & u_n \\ s_1 & \cdots & s_i & s_{i+1} & \cdots & s_n \end{pmatrix}.$$

It is clear that

(6) $\quad a_0(x), a_1(x), \ldots, a_n(x)$ are continuous functions in $(b-a)B^n$.

(Note that in the original proof, Karlin constructed the polynomials $u_x(t)$ on the simplex $(b-a)\Sigma^n$, therefore he has to suppose first that the $\{u_i\}_0^n$ is an ET-system, i.e., a T-system such that the functions $u_0, u_1, \ldots, u_n$ are n times differentialbe in $[a, b]$ and satisfy some other constraints.) Now for $i = 0, 2, 4, \ldots$ we define

$$d_i(x) = \min\{d : d > 0, \;\; d\, f(t) \geq u_x(t) \text{ for all } t \in [s_i, s_{i+1}]\},$$

where $s_0 = a, s_{n+1} = b$. For $i = 1, 3, \dots$, we put

$$d_i(x) = \min\{d : d > 0; u_x(t) \geq d\, g(t) \text{ for all } t \in [s_i, s_{i+1}]\}. \tag{7}$$

On one hand, $d_0(x), d_1(x), \dots, d_n(x)$ are positive. On the other hand, it follows from the remark following (6) that

$$d_0(x), d_1(x), \dots, d_n(x) \text{ are continuous in } (b-a)B^n. \tag{8}$$

Now let $\{x^{(m)}\}_1^\infty$ be a sequence in $(b-a)B^n$ converging to a point $x \in (b-a)\Sigma^n$ and let $\{u_{x^{(m)}}(t)\}_1^\infty$ be a sequence converging (in $C[a,b]$ equipped with $\|.\|$-sup) to a polynomial $R(t)$ of the form

$$R(t) = \sum_0^n a_i(x) u_i(t).$$

Then the polynomial $R(t)$ vanishes at each of the points

$$s_i = a + \sum_0^{i-1} x_j \quad (i = 1, 2, \dots, n).$$

Note that the points s_i could be multiple zeros. Furthermore, if we put $s_0 = a$, $s_{n+1} = b$, and

$$d_i(R) = \begin{cases} \min\{d : d \geq 0, d\, f(t) \geq R(t) & \text{for all } t \in [s_i, s_{i+1}]\}, \quad i = 0, 2, \dots, \\ \min\{d : d \geq 0, R(t) \geq d\, g(t) & \text{for all } t \in [s_i, s_{i+1}]\}, \quad i = 1, 3, \dots, \end{cases}$$

then we have $d_i(R) = 0$ if and only if $x_i = 0$ (for each $i = 0, 1, \dots, n$) and for each $i = 0, 1, \dots, n$ the sequence $d_i(x^{(m)})$ converges to $d_i(R)$ as $m \to +\infty$. It follows that

$$\text{For each } i = 0, 1, \dots, n \text{ if } x_i \to 0 \text{ then } d_i(x) \to 0 \tag{9}$$

and, for all $t > 0$ and $i = 0, 1, \dots, n$,

$$\inf\{d_i(x) : x \in (b-a)B^n \quad x_i \geq t\} > 0. \tag{10}$$

Since for every $x = (x_0, x_1, \dots, x_n) \in (b-a)B^n$ it holds

$$\max_{0 \leq i \leq n} x_i \geq \frac{b-a}{n+1},$$

we can deduce from (10) that

$$\inf\left\{\sum_0^n d_i(x),\ x \in (b-a)B^n\right\} > 0. \tag{11}$$

Put

$$f_i(x) = \frac{d_i(x)}{\sum_0^n d_j(x)}(b-a) \quad (i = 0, 1, \dots, n;\ x \in (b-a)B^n).$$

It follows from (8) (9) and (11) that the mapping $f^* = (f_0, f_1, \dots, f_n)$ from $(b-a)B^n$ into itself is continuous and satisfies to the condition (2). By Theorem 2.1, f^* is surjective; hence there is a point $x \in (b-a)B^n$ such that

$$f_0(x) = f_2(x) = \cdots = f_n(x) = \frac{b-a}{n+1}.$$

Therefore, $d_0(x) = d_1(x) = \cdots = d_n(x) = d > 0$ and $P(t) = \frac{1}{d}u_x(t)$ is the desired polynomial. □

Corollary 3.1. *In the snake theorem, if instead of the functions f and g we consider $(n+1)$ continuous functions $F_0, F_1, \dots, F_n$ satisfying*

$$F_i(t) > v(t) \text{ for all } i = 0, 2, 4, \dots \text{ and } t \in [a,b],$$
$$F_i(t) < v(t) \text{ for all } i = 1, 3, 5, \dots \text{ and } t \in [a,b],$$

then there exists a polynomial P such that

(i) *There exist n points $(a <) < s_1 < s_2 < \cdots < s_n (< b)$ such that $P(s_i) = v(s_i)$ for all $i = 1, 2, 3, \dots$;*

(ii) *For each $i = 0, 1, 2, 3, \dots, n$,*

$$F_i(t) \geq P(t) \geq v(t) \text{ for all } t \in [s_i, s_{i+1}] \text{ if } i \text{ is even},$$
$$F_i(t) \leq P(t) \leq v(t) \text{ for all } t \in [s_i, s_{i+1}] \text{ if } i \text{ is odd},$$

and there exists $t_i \in (s_i, s_{i+1})$ such that $F_i(t_i) = P(t_i)$, where $s_0 = a, s_{n+1} = b$.

Proof. We can assume that $v \equiv 0$. Now we replace the definition (7) of $d_i(x)$, $i = 0, 1, \cdots, n$, by the following

$$d_i(x) = \begin{cases} \min\{d : d > 0, dF_i(t) \geq u_x(t) \text{ for all } t \in [s_i, s_{i+1}]\}, & i = 0, 2, 4, \dots, \\ \min\{d : d > 0, u_x(t) \geq dF_i(t) \text{ for all } t \in [s_i, s_{i+1}]\}, & i = 1, 3, 5, \dots, \end{cases}$$

i.e.,

$$d_i(x) = \max_{s_i \leq t \leq s_{i+1}} \frac{u_x(t)}{F_i(t)} \quad \text{for all } \ i = 0, 1, 2, \dots, n.$$

The same arguments as in the above proof work and the assertion (8)-(11) are true for the case under consideration. Hence there exists a polynomial $P(t)$ vanishing at each of the points $(a <) s_1 < s_2 < \cdots < s_n (< b)$ and satisfying

$$\max_{s_i \leq t \leq s_{i+1}} \frac{P(t)}{F_i(t)} = 1 \quad \text{for all} \quad i = 0, 1, 2, \dots, n.$$

Note that we have no more the unicity of the snake as in the original snake theorem where only two gates are considered. □

References

[1] S. Karlin, W. J. Studden, *Tchebycheff Systems with Applications in Analysis and Statistics*, Interscience Publishers, 1966.

[2] M. G. Krein and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems* (Mockva 1973, A.M.S. translation 1977).

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