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STRONG LAW OF LARGE NUMBERS AND L^p-CONVERGENCE FOR DOUBLE ARRAYS OF INDEPENDENT RANDOM VARIABLES

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Abstract. For a double array of independent random variables $\{X_{mn}, m \geq 1, n \geq 1\}$, a strong law of large numbers and the L^p-convergence are established for the double sums $\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}, m \geq 1, n \geq 1$.

1. Introduction and notations

Pyke and Root [9] proved that if $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables with $E|X_1|^p < \infty \quad (1 \leq p < 2)$, then

$$\frac{E\left|\sum_{i=1}^{n} X_i - nEX_1\right|^p}{n} \to 0 \text{ as } n \to \infty.$$

By using an inequality due to von Bahr and Esseen [1], Chatterji [3] extended the result of Pyke and Root [9] to the case where $\{X_n, n \geq 1\}$ is dominated in distribution by a random variable X with $E|X|^p < \infty \quad (1 \leq p < 2)$. Later, using Burkholder's inequality (see [2]), Chow [4] strengthened the result of Chatterji [3] by relaxing the domination condition of [3] to uniform integrability.

The aim of this paper is to establish a version of the strong law of large numbers and the L^p-convergence for double arrays of independent random variables. From this, we obtain the result of Gut [6, Theorem 3.2]. We also generalize an earlier result of Smythe [11] for arrays of independent identically distributed random variables.

Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be an array of independent random variables. Our main result provides conditions for $\dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}}{m^\alpha n^\beta} \to 0$ almost surely (a.s.) and in L^p as $\max\{m, n\} \to \infty$, where $\alpha > 0, \beta > 0$.

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Random variables $\{X_{mn}, m \geq 1, n \geq 1\}$ are said to be *dominated in distribution* by a random variable X if for some constant C it holds

$$P\{|X_{mn}| > t\} \leq CP\{|X| > t\}, \quad t \geq 0, m \geq 1, n \geq 1.$$

For $a, b \in \mathbb{R}, \min\{a, b\}$ and $\max\{a, b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$. The number of divisors of a positive integer k will be denoted by d_k. Throughout this paper, the symbol C will denote a generic positive constant which is not necessarily the same one in each appearance. The logarithms are to basis 2.

2. Main results

We now present some lemmas which will be needed in the sequel.

Lemma 2.1. *Let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of random variables. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E|X_{mn}|^p < \infty \quad \text{for some} \quad p > 0, \tag{1}$$

then

$$X_{mn} \to 0 \text{ a.s. and in } L^p \quad \text{as} \quad m \vee n \to \infty. \tag{2}$$

Proof. The L^p-convergence follows immediately from (1). For an arbitrary $\varepsilon > 0$ and for all $k \geq 1$,

$$\begin{aligned} P\{\sup_{m\vee n\geq k} |X_{mn}| > \varepsilon\} &\leq \sum_{m\vee n\geq k} P\{|X_{mn}| > \varepsilon\} \\ &\leq \frac{1}{\varepsilon^p} \sum_{m\vee n\geq k} E|X_{mn}|^p \quad \text{(by Markov's inequality)} \\ &\to 0 \quad \text{as } k \to \infty \quad \text{(by (1))}. \end{aligned}$$

This proves the almost sure convergence. □

Lemma 2.2. *If $\{X_{kl}, \mathcal{F}_l, l \geq 1\}, k = 1, 2, \cdots, m$, are nonnegative submartingales, then $\{\max_{1\leq k\leq m} X_{kl}, \mathcal{F}_l, l \geq 1\}$ is a nonnegative submartingale.*

Proof. For $L > l \geq 1$,

$$E(\max_{1\leq k\leq m} X_{kL}|\mathcal{F}_l) \geq \max_{1\leq k\leq m} E(X_{kL}|\mathcal{F}_l) \geq \max_{1\leq k\leq m} X_{kl}.$$

□

The next lemma is due to von Bahr and Esseen [1].

Lemma 2.3. *Let $\{X_i, 1 \leq i \leq n\}$ be random variables such that $E\{X_{k+1}|S_k\} = 0$ for $0 \leq k \leq n-1$, where $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$ for $1 \leq k \leq n$. Then*

$$E|S_n|^p \leq 2\sum_{i=1}^{n} E|X_i|^p \quad \text{for all} \quad 1 \leq p \leq 2.$$

Note that Lemma 2.3 holds when $\{X_i, 1 \le i \le n\}$ are independent random variables with $EX_i = 0$ for $1 \le i \le n$.

Lemma 2.4. *Let $\{X_{ij}, 1 \le i \le m, 1 \le j \le n\}$ be a collection of mn independent random variables. If $EX_{ij} = 0$ for all $1 \le i \le m, 1 \le j \le n$, then*

$$E\big(\max_{1\le k\le m, 1\le l\le n} |S_{kl}|^p\big) \le C \sum_{i=1}^{m}\sum_{j=1}^{n} E|X_{ij}|^p \quad \text{for all} \quad 0 < p \le 2, \tag{3}$$

where $S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij}$; the constant C is independent of m and n. In the case $0 < p \le 1$, the independence hypothesis and the hypothesis that $EX_{ij} = 0,\ 1 \le i \le m,\ 1 \le j \le n$ are superfluous.

Proof. If $E|X_{ij}|^p = \infty$ for some $1 \le i \le m$ and $1 \le j \le n$, then (3) is immediate. Thus, we can assume that $E|X_{ij}|^p < \infty,\ 1 \le i \le m,\ 1 \le j \le n$.

First, suppose that $1 < p \le 2$ and $m \wedge n \ge 2$. Set

$$Y_l = \max_{1\le k\le m} |S_{kl}|$$

and

$$\mathcal{F}_l = \sigma(X_{ij},\ 1 \le i \le m,\ 1 \le j \le l), \quad 1 \le l \le n.$$

For each $1 \le k \le m$ and $2 \le l \le n$, we have

$$\begin{aligned} E(S_{kl}|\mathcal{F}_{l-1}) =& E\,(S_{k,l-1} + X_{1l} + \cdots + X_{kl}|\mathcal{F}_{l-1}) \\ =& E(S_{k,l-1}|\mathcal{F}_{l-1}) + E(X_{1l}|\mathcal{F}_{l-1}) + \cdots + E(S_{kl}|\mathcal{X}_{l-1}) \\ =& S_{k,l-1} \text{ a.s.} \end{aligned}$$

So $\{S_{kl}, \mathcal{F}_l, 1 \le l \le n\}$ is a martingale for each $k = 1, \ldots, m$. As in Scalora [10], $\{|S_{kl}|, \mathcal{F}_l,\ 1 \le l \le n\}$ is a nonnegative submartingale for each $k = 1, 2, \ldots, m$. Then, by Lemma 2.2, $\{Y_l, \mathcal{F}_l, 1 \le l \le n\}$ is a nonnegative submartingale. By Doob's inequality (see, e.g., Chow and Teicher [5], p. 255),

$$E\left(\max_{1\le k\le m, 1\le l\le n} |S_{kl}|^p\right) = E\big(\max_{1\le l\le n}\big)^p \le \left(\frac{p}{p-1}\right)^p EY_n^p. \tag{4}$$

Set $\mathcal{G}_k = \sigma(X_{ij}, 1 \le i \le k,\ 1 \le j \le n), 1 \le k \le m$. Since $\{|S_{kn}|, \mathcal{G}_k, 1 \le k \le m\}$ is a submartingale, applying Doob's inequality once more, we have

$$\begin{aligned} EY_n^p &= E\big(\max_{1\le k\le m} |S_{kn}|\big)^p \\ &\le \big(\frac{p}{p-1}\big)^p E|S_{mn}|^p \\ &\le 2\big(\frac{p}{p-1}\big)^p \sum_{i=1}^{m}\sum_{j=1}^{n} E|X_{ij}|^p \quad (\text{ by Lemma 2.3}). \end{aligned} \tag{5}$$

The conclusion (3) follows immediately from (4) and (5).

Next, if $1 < p \le 2$ and $m \wedge n = 1$ then (3) is obtained similarly as in the case $m \wedge n \ge 2$.

Finally, if $0 < p \le 1$, then we have

$$
\begin{aligned}
E\left(\max_{1\le k\le m,1\le l\le n} |S_{kl}|^p\right) &\le E\left(\max_{1\le k\le m,1\le l\le n} \sum_{i=1}^{k}\sum_{j=1}^{l} |X_{ij}|^p\right)\\
&= E\left(\sum_{i=1}^{m}\sum_{j=1}^{n} |X_{ij}|^p\right)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} E|X_{ij}|^p,
\end{aligned}
$$

which establishes (3). □

Lemma 2.5. *Let $\{X_{mn}, m \ge 1, n \ge 1\}$ be a double array of random variables. Suppose that $\{X_{mn}, m \ge 1, n \ge 1\}$ is dominated in distribution by a random variable X. If*

$$
E\big(|X|^p \log^+ |X|\big) < \infty \quad \textit{for some} \quad p > 0, \tag{6}
$$

then

(i) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\left(|X_{mn}|^q I\left(|X_{mn}|\le (mn)^{\frac{1}{p}}\right)\right)}{(mn)^{\frac{q}{p}}} < \infty$ *for all* $q > p$,

(ii) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\left(|X_{mn}|^r I\left(|X_{mn}| > (mn)^{\frac{1}{p}}\right)\right)}{(mn)^{\frac{r}{p}}} < \infty$ *for all* $0 < r < p$.

Proof. Let F be the distribution function of X. By using the fact that

$$
\sum_{k=j}^{\infty} \frac{d_k}{k^{\frac{q}{p}}} = O\left(\frac{\log j}{j^{\frac{q}{p}-1}}\right),
$$

we obtain

$$
\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\left(|X_{mn}|^q I(|X_{mn}| \le (mn)^{\frac{1}{p}})\right)}{(mn)^{\frac{q}{p}}} &\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(mn)^{\frac{q}{p}}} \int_0^{(mn)^{\frac{1}{p}}} x^q dF(x)\\
&= C\sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{q}{p}}} \int_0^{k^{\frac{1}{p}}} x^q dF(x)\\
&= C\sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{q}{p}}} \sum_{j=1}^{k} \int_{(j-1)^{\frac{1}{p}}}^{j^{\frac{1}{p}}} x^q dF(x)\\
&= C\sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \frac{d_k}{k^{\frac{q}{p}}} \int_{(j-1)^{\frac{1}{p}}}^{j^{\frac{1}{p}}} x^q dF(x)
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{\log j}{j^{\frac{q}{p}-1}}\int_{(j-1)^{\frac{1}{p}}}^{j^{\frac{1}{p}}} x^q dF(x)\\
&\le C\sum_{j=2}^{\infty}\int_{(j-1)^{\frac{1}{p}}}^{j^{\frac{1}{p}}} x^p \log x dF(x)\\
&\le CE\big(|X|^p \log^+ |X|\big),
\end{aligned}$$

which proves (i). Noting that

$$\sum_{k=1}^{n}\frac{d_k}{k^{\frac{r}{p}}} = O\big(\frac{\log n}{n^{\frac{r}{p}-1}}\big) \quad (0 < r < p),$$

we can obtain (ii) by the same method. □

We are now in a position to establish the main result which provides conditions for almost sure convergence and L^p-convergence for double sum of independent random variables. This theorem in the particular case $\alpha = \beta = 1$ and $p = 2$ is the two-dimensional version of Kolmogorov's theorem (see, e.g., Chow and Teicher [5], pp. 121).

Theorem 2.1. *Let $\{X_{mn}, m \ge 1, n \ge 1\}$ be a double array of independent random variables with $EX_{mn} = 0, m \ge 1, n \ge 1$. If*

$$(7) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E|X_{mn}|^p}{m^{\alpha p}n^{\beta p}} < \infty \quad \text{for some} \quad 0 < p \le 2 \quad \text{and} \quad \alpha > 0, \beta > 0$$

then

$$(8) \qquad \frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} X_{ij}}{m^{\alpha}n^{\beta}} \to 0 \text{ a.s. and in } L^p \text{ as } \quad m \vee n \to \infty.$$

Proof. Since

$$\begin{aligned}
(9) \qquad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} E\left|\frac{\sum\limits_{i=1}^{2^k}\sum\limits_{j=1}^{2^l} X_{ij}}{2^{\alpha k}2^{\beta l}}\right|^p &\le C\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sum\limits_{i=1}^{2^k}\sum\limits_{j=1}^{2^l} E|X_{ij}|^p}{(2^{\alpha k}2^{\beta l})^p} \quad \text{(by Lemma 2.4)}\\
&\le C\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\frac{E|X_{ij}|^p}{(i^{\alpha}j^{\beta})^p}\\
&< \infty \quad \text{(by (7))},
\end{aligned}$$

Lemma 2.1 ensures that

$$(10) \qquad \frac{\sum\limits_{i=1}^{2^k}\sum\limits_{j=1}^{2^l} X_{ij}}{2^{\alpha k}2^{\beta l}} \to 0 \text{ a.s. and in } L^p \text{ as } \quad k \vee l \to \infty.$$

Set

$$S_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij},\quad m\ge 1,n\ge 1$$

and

$$T_{kl}=\max_{2^k\le m<2^{k+1},2^l\le n<2^{l+1}}\left|\frac{S_{mn}}{m^\alpha n^\beta}-\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|,\quad k\ge 1,l\ge 1.$$

By Lemma 2.4, for all $k\ge 1$ and $l\ge 1$, we have

$$\begin{aligned}E|T_{kl}|^p\le & C\left(E\left|\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|^p+E\big(\max_{2^k\le m<2^{k+1},2^l\le n<2^{l+1}}\left|\frac{S_{mn}}{m^\alpha n^\beta}\right|^p\big)\right)\\ \le & C\left(E\left|\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|^p+\frac{1}{2^{\alpha k}2^{\beta l}}E\big(\max_{1\le m\le 2^{k+1},1\le n\le 2^{l+1}}|S_{mn}|^p\big)\right)\\ \le & CE\left|\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|^p+C\frac{\sum\limits_{i=1}^{2^{k+1}}\sum\limits_{j=1}^{2^{l+1}}E|X_{ij}|^p}{2^{(k+1)\alpha p}2^{(l+1)\beta p}}\end{aligned}$$

whence $\sum\limits_{k=1}^{\infty}\sum\limits_{l=1}^{\infty}ET_{kl}^p<\infty$ by (9). Then by Lemma 2.1,

$$T_{kl}\to 0 \text{ a.s. and in } L^p \text{ as } \quad k\vee l\to\infty. \tag{11}$$

Note that for $2^k\le m<2^{k+1}$ and $2^l\le n<2^{l+1}$ it holds

$$\frac{|S_{mn}|}{m^\alpha n^\beta}\le\left|\frac{S_{mn}}{m^\alpha n^\beta}-\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|+\left|\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|\le T_{kl}+\left|\frac{S_{2^k2^l}}{2^{\alpha k}2^{\beta l}}\right|,$$

so the conclusion (8) follows from (10) and (11). □

Remark 2.1. The argument used for proving in Theorem 2.1 reveals that if $0<p\le 1$, then the independence hypothesis and the hypothesis that the random variables $\{X_{mn},m\ge 1,n\ge 1\}$ have mean 0 are not needed for the validity of the conclusion of the theorem.

Corollary 2.1. *Let $\{X_{mn},m\ge 1,n\ge 1\}$ be a double array of independent random variables. Suppose that $\{X_{mn},m\ge 1,n\ge 1\}$ are dominated in distribution by a random variable X. If*

$$E(|X|^p\log^+|X|)<\infty \quad \textit{for some} \quad 1\le p<2, \tag{12}$$

then

$$\frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}(X_{ij}-EX_{ij})}{(mn)^{\frac{1}{p}}}\to 0 \textit{ a.s. and in } L^p \textit{ as } \quad m\vee n\to\infty. \tag{13}$$

Proof. For $m\ge 1$ and $n\ge 1$, set

$$X'_{mn}=X_{mn}I\big(|X_{mn}|\le (mn)^{\frac{1}{p}}\big)$$

and

$$X''_{mn}=X_{mn}I\big(|X_{mn}|>(mn)^{\frac{1}{p}}\big).$$

By Lemma 2.5,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(X'_{mn}-EX'_{mn})^2}{(mn)^{\frac{2}{p}}}\le\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E(X'_{mn})^2}{(mn)^{\frac{2}{p}}}<\infty$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E|X''_{mn}-EX''_{mn}|^r}{(mn)^{\frac{r}{p}}}\le C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E|X''_{mn}|^r}{(mn)^{\frac{r}{p}}}<\infty$$

for all $0<r<p$. Then by Theorem 2.1,

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-EX'_{ij})}{(mn)^{\frac{1}{p}}}\to 0 \text{ a.s. and in } L^2 \text{ as } \quad m\vee n\to\infty, \tag{14}$$

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X''_{ij}-EX''_{ij})}{(mn)^{\frac{1}{p}}}\to 0 \text{ a.s. as } \quad m\vee n\to\infty. \tag{15}$$

Since $E|X|^p<\infty$ (by (12)),

$$\begin{aligned} E|X''_{mn}|^p &\le C\int_{(mn)^{\frac{1}{p}}}^{\infty}x^{p-1}P\{|X_{mn}|>x\}dx \\ &\le C\int_{(mn)^{\frac{1}{p}}}^{\infty}x^{p-1}P\{|X|>x\}dx \\ &\to 0 \quad \text{as} \quad m\vee n\to\infty. \end{aligned} \tag{16}$$

It implies that

$$\begin{aligned} \frac{E\big|\sum_{i=1}^{m}\sum_{j=1}^{n}(X''_{ij}-EX''_{ij})\big|^p}{mn} &\le C\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}E|X''_{ij}-EX''_{ij}|^p}{mn} \quad \text{(by Lemma 2.4)} \\ &\le C\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}E|X''_{ij}|^p}{mn} \\ &\to 0 \quad \text{as} \quad m\vee n\to\infty \quad (\text{ by (16)}). \end{aligned} \tag{17}$$

Combining (14), (15) and (17) we get (13). □

Remark 2.2. The generalization to d-dimensional arrays of random variables can be obtained by the same method under the condition $E\big(|X|^p(\log^+ X|)^{d-1}\big)<\infty$.

Remark 2.3. A part of Corollary 2.1 is due to Smythe [11] who proved that if $\{X_k, k\in\mathbb{N}^d\}$ be a d-dimensional array of independent identically distributed random variables with zero mean, $E\big(|X_k|(\log^+|X_k|)^{d-1}\big)<\infty$, then $\frac{\sum_{j\le k}X_j}{|k|}\to 0$ a.s. as $|k|\to\infty$, where $k=(k_1,k_2,\dots,k_d)\in\mathbb{N}^d$, $|k|=k_1k_2\cdots k_d$.

Remark 2.4. When $\{X_{mn}, m \geq 1, n \geq 1\}$ are pairwise independent random variables which are dominated in distribution by a random variable X, the almost sure convergence of $\dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - EX_{ij})}{(mn)^{\frac{1}{p}}}$ was obtained by Hong and Hwang [7, Theorem 2.3] under a stronger condition that $E\big(|X|^p(\log^+|X|^3)\big) < \infty \quad (1 < p < 2)$. More general results were proved by Hong and Volodin [8].

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