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ON LOCAL PARETO OPTIMA OF REAL ANALYTIC MAPPINGS

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Abstract. This paper deals with the local problem of optimizing several analytic functions at the same time. We prove that the Milnor number of an isolated complete intersection singularity at a local Pareto optimal point is odd. Furthermore, high-order necessary and almost sufficient conditions are given, allowing one to recognize from the Newton diagram of an analytic mapping at the origin whether this point is a local Pareto optimum.

1. Introduction

Motivated by mathematical economics, we consider the problem of optimizing several analytic functions. More precisely, let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0), x \mapsto (f_1(x), f_2(x), \dots, f_k(x))$, be an analytic mapping defined in a neighborhood of the origin in $\mathbb{R}^n$ with $f(0) = 0$. The point $0 \in \mathbb{R}^n$ is said to be a *local Pareto optimum* (*strict local Pareto optimum*) for f if and only if there exists a neighborhood U of 0 in $\mathbb{R}^n$ such that for any $x \in U$, $f_i(x) \le 0$ for $i = 1, 2, \dots, k$, imply $f_i(x) = 0$ for $i = 1, 2, \dots, k$, $(x = 0)$. The problem is to find conditions for the origin in $\mathbb{R}^n$ to be a local Pareto optimum for f.

It is well-known that (see Vassiliev (1977)) if the Milnor number of an analytic function (i.e., in the case where $k = 1$) at a local optimal point is finite, then it is odd. The first objective of this paper is to establish a similar result for local Pareto optima. Namely, we prove in Section 2 that the Milnor number of an isolated complete intersection singularity at a local Pareto optimal point is odd.

The second objective of this paper concerns the well known first-order necessary and second-order sufficient conditions for a local Pareto optimum (see Smale (1973) and (1975), Wan (1975)). These low-order conditions are insufficient for the characterization of local Pareto optima for any generic class of mappings $f = (f_1, f_2, \dots, f_k)$ from $(\mathbb{R}^n, 0)$ onto $(\mathbb{R}^k, 0)$ with the first derivatives $Df_j(0) = 0$ and the second derivatives $D^2 f_j(0) = 0$ for all $j = 1, 2, \dots, k$. Thus, it is natural to ask the question: can one find certain high-order necessary and sufficient conditions for local Pareto optima? In order to handle those high-order criteria

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in a neat way, one is led to the notation of Newton diagrams (see Kouchnirenko (1976) and §3 below).

In Section 3 we will give high-order necessary and almost sufficient conditions which allow us to recognize from the Newton diagram of an analytic mapping at the origin whether this point is a local Pareto optimum. We also give a short, direct proof of the first-order necessary and second-order sufficient conditions for a local Pareto optimum. It must be noted that unlike the previous proofs (see Smale (1973) and (1975), Wan (1975)), the one presented below uses only the Curve Selection Lemma. A different approach, based on the notation of jets, can be found in Wan (1977), Hà Huy Vui (1980) and (1982).

2. Milnor number of a complete intersection

We first recall some basic facts about complete intersections with isolated singularity (see Looijenga (1984)). Let $f = (f_1, f_2, \dots, f_k)\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, with $1 \leq k < n$, be an analytic mapping defined in a neighborhood $U \subset \mathbb{C}^n$ of the origin such that $f(0) = 0$. Let $V := (f^{-1}(0), 0)$ be the germ of $f^{-1}(0)$ at $0 \in \mathbb{C}^n$. We say that V is a *germ of a complete intersection* with an isolated singularity at the origin if there is a positive number ϵ such that the holomorphic k-form $df_1(z) \wedge df_2(z) \wedge \dots \wedge df_k(z) \neq 0$ for any $z \in V \cap (B^{2n}_\epsilon - \{0\})$, where

$$B^{2n}_\epsilon := \{z \in \mathbb{C}^n \mid \|z\| \leq \epsilon\}.$$

In particular, $V \cap (B^{2n}_\epsilon - \{0\})$ is non-singular.

Taking $\epsilon_0 > 0$ sufficiently small, we may assume that any sphere $\mathbb{S}^{2n-1}_\epsilon := \partial B^{2n}_\epsilon$ $(0 < \epsilon \leq \epsilon_0)$ intersects V transversally. Let W be a sufficiently small neighborhood of $0 \in \mathbb{C}^n$, such that $\mathbb{S}^{2n-1}_{\epsilon_0}$ meets transversally with any fiber $f^{-1}(\delta), \delta \in W$. Let D_f be the set of the critical values of the restriction $f|_{f^{-1}(W) \cap B^{2n}_{\epsilon_0}}$. D_f is called the *discriminant locus* of f and it is well known that D_f is a hypersurface. Let $X^* := (f^{-1}(W) \cap B^{2n}_{\epsilon_0}) - f^{-1}(D_f)$. Then the mapping $f\colon X^* \to W - D_f$ is a C^∞-locally trivial fibration. This fibration is called the *Milnor fibration* of the mapping $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$.

Let $f^{-1}(\delta)$ be a generic fiber. It is known that $f^{-1}(\delta)$ has the homotopy type of a bouquet of spheres of dimension $n - k$ (see Milnor (1968) in the case where $k = 1$, and Hamm (1971) in the case where $k > 1$). The number of spheres in this bouquet is called the *Milnor number* at $0 \in \mathbb{C}^n$ of f and denoted by $\mu(f)$.

In the case where $k = 1$, according to Milnor (1968) and Palamodov (1967)

$$\mu(f) = \dim_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{C}^n,0} \Big/ \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \right),$$

where $\mathcal{O}_{\mathbb{C}^n,0}$ is the ring of germs of complex analytic functions at the origin.

If $n > k > 1$, then we have the following formula of Lê Dũng Tráng (1974) and Greuel (1975)

$$\mu(f') + \mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, I,$$

where $f' := (f_1, f_2, \dots, f_{k-1})$ and I is the ideal generated by $f_1, f_2, \dots, f_{k-1}$ and all $k \times k$ minors $\dfrac{\partial(f_1, f_2, \dots, f_k)}{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})}$ in $\mathcal{O}_{\mathbb{C}^n,0}$. We have

Theorem 1. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a real analytic germ defined in a neighborhood of the origin with $f(0) = 0$. Let $f_{\mathbb{C}}\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ be the complexification of f. Assume that $0 \in \mathbb{R}^n$ is a local Pareto optimum for f, and $f_{\mathbb{C}}$ is a germ of complete intersection with an isolated singularity at $0 \in \mathbb{C}^n$. Then we have*

$$\mu(f_{\mathbb{C}}) = 1 \bmod 2.$$

Proof. Since f is a germ of a real analytic mapping and $f_{\mathbb{C}}$ is a complete intersection with isolated singularity at $0 \in \mathbb{C}^n$, by Lemma 4.2 of Dutertre (2002), there exists an analytic germ $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that the mapping

$$(f_{\mathbb{C}}, g_{\mathbb{C}})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{k+1}, 0)$$

is also a complete intersection with isolated singularity at $0 \in \mathbb{C}^n$. Since f has $0 \in \mathbb{R}^n$ as a local Pareto optimum, the germ (f, g) also has $0 \in \mathbb{R}^n$ as its local Pareto optimum. Thus, by induction there are real analytic functions $g_1, g_2, \dots, g_{n-k}$ such that for any $j = 1, 2, \dots, n-k$, the mapping

$$(f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_j)\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^{k+j}, 0)$$

has $0 \in \mathbb{R}^n$ as a local Pareto optimum and the complexified mapping

$$(f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \dots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \dots, g_{j,\mathbb{C}})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{k+j}, 0)$$

defines a complete intersection with an isolated singularity at $0 \in \mathbb{C}^n$.

Let us consider the following mappings

$$\begin{aligned} \Phi &:= (f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_{n-k}) &:\quad& (\mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^{n-k}, 0), \\ \Phi_{\mathbb{C}} &:= (f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \dots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \dots, g_{n-k,\mathbb{C}}) &:\quad& (\mathbb{C}^n, 0) \to (\mathbb{C}^k \times \mathbb{C}^{n-k}, 0). \end{aligned}$$

According to Looijenga (1984), Proposition 5.12, we know that

$$\mu(\Phi_{\mathbb{C}}) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, J - 1, \tag{1}$$

where J is the ideal generated by $f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \dots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \dots, g_{n-k,\mathbb{C}}$ in $\mathcal{O}_{\mathbb{C}^n,0}$. Since $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / J$ is the number of complex points of $\Phi_{\mathbb{C}}^{-1}(\delta, \delta')$ in $\mathbb{C}^n$ for a generic $(\delta, \delta') \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, sufficiently close to $0 \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, and since $g_1, g_2, \dots, g_{n-k}$ are convergent series with real coefficients, the number of non-real points of $\Phi_{\mathbb{C}}^{-1}(\delta, \delta')$ is even. Thus, $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / J$ is equal to the number of real points of $\Phi_{\mathbb{C}}^{-1}(\delta, \delta')$ modulo 2.

On the other hand, since the map Φ has $0 \in \mathbb{R}^n$ as a local Pareto optimum, $\Phi^{-1}(\delta, \delta') \cap U$ is empty, where U is a sufficiently small neighborhood of the origin in $\mathbb{R}^n$, and $(\delta, \delta') \in \mathbb{R}^n_- := \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n \mid y_i < 0, i = 1, 2, \dots, n\}$.

Therefore

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, J = 0 \bmod 2.$$

Hence, it follows from (1) that

$$\mu(\Phi_{\mathbb{C}}) = 1 \bmod 2. \tag{2}$$

In what follows, we shall show that

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, I_j = 0 \bmod 2$$

for all $j = 1, 2, \ldots, n-k$, where I_j is the ideal generated by $f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}$, $g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \ldots, g_{j-1,\mathbb{C}}$, and all $(k+j)\times(k+j)$-minors $\dfrac{\partial(f_{1,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, \ldots, g_{j,\mathbb{C}})}{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_{k+j}})}$ in $\mathcal{O}_{\mathbb{C}^n,0}$.

Let J_j be the ideal generated by all $(k+j)\times(k+j)$-minors $\frac{\partial(f_{1,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, \ldots, g_{j,\mathbb{C}})}{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_{k+j}})}$ in $\mathcal{O}_{\mathbb{C}^n,0}$. (Whence $I_j = \langle f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \ldots, g_{j-1,\mathbb{C}}, J_j\rangle \mathcal{O}_{\mathbb{C}^n,0}$.) Let C_j be the germ of complex zeros of J_j. According to a result of Saito (1973), $\mathcal{O}_{\mathbb{C}^n,0} \,/\, J_j$ is a Cohen-Macaulay ring of dimension $k+j$, and so C_j is equidimensional of dimension $k+j$. Then a result about multiplicity from Serre (1989) gives the following relation

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, I_j = (\Phi_{j,\mathbb{C}}^{-1}(0), C_j)_0,$$

where $\Phi_{j,\mathbb{C}} := (f_{1,\mathbb{C}}, f_{2,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, g_{2,\mathbb{C}}, \ldots, g_{j-1,\mathbb{C}})$, and $(\Phi_{j,\mathbb{C}}^{-1}(0), C_j)_0$ is the intersection multiplicity of $\Phi_{j,\mathbb{C}}^{-1}(0)$ and C_j at $0 \in \mathbb{C}^n$. Let $\tilde{g}_j$ be a suitable perturbation of g_j, and let $\tilde{C}_j$ be the germ of complex zeros of the ideal, generated by all $(k+j)\times(k+j)$-minors $\dfrac{\partial(f_{1,\mathbb{C}}, \ldots, f_{k,\mathbb{C}}, g_{1,\mathbb{C}}, \ldots, g_{j-1,\mathbb{C}}, \tilde{g}_{j,\mathbb{C}})}{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_{k+j}})}$. Then the intersection multiplicity $(\Phi_{j,\mathbb{C}}^{-1}(0), C_j)_0$ is equal to the number of the intersection points of $\Phi_{j,\mathbb{C}}^{-1}(\delta)$ and $\tilde{C}_j$, where δ is generic and sufficiently close to $0 \in \mathbb{C}^{k+j-1}$. Hence, we can assume, without loss of generality, that $\delta \in \mathbb{R}_-^{k+j-1}$ and $\tilde{C}_j$ intersects $\Phi_{j,\mathbb{C}}^{-1}(\delta)$ transversally at regular points. By Lemma 3.7 of Dutertre (2002), $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, I_j$ is equal modulo 2 to the number of real non-degenerate critical points of the restriction $\tilde{g}_{j,\mathbb{C}}|_{\Phi_{j,\mathbb{C}}^{-1}(\delta)\cap B_\epsilon^{2n}}$. But, $0 \in \mathbb{R}^n$ is a local Pareto optimum of $\Phi_j = (f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_{j-1})$ and $\delta \in \mathbb{R}_-^{k+j-1}$, therefore the number of real non-degenerate critical points of $\tilde{g}_{j,\mathbb{C}}|_{\Phi_{j,\mathbb{C}}^{-1}(\delta)\cap B_\epsilon^{2n}}$ is evidently equal to 0. Hence

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} \,/\, I_j = 0 \bmod 2. \tag{3}$$

Now, we easily get the proof of the theorem. In fact, it follows from the formula of Lê Dũng Tráng (1974) and Greuel (1975) and from (3) that

$$\mu(f_{1,\mathbb{C}},\dots,f_{k,\mathbb{C}},g_{1,\mathbb{C}},\dots,g_{n-k-1,\mathbb{C}})+\mu(f_{1,\mathbb{C}},\dots,f_{k,\mathbb{C}},g_{1,\mathbb{C}},\dots,g_{n-k,\mathbb{C}})$$
$$=\dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^n,0}\,/\,I_{n-k}=0 \bmod 2,$$
$$\dots$$
$$\mu(f_{1,\mathbb{C}},\dots,f_{k,\mathbb{C}})+\mu(f_{1,\mathbb{C}},\dots,f_{k,\mathbb{C}},g_{1,\mathbb{C}})$$
$$=\dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^n,0}\,/\,I_1=0 \bmod 2.$$

All these equalities and (2) imply that

$$\mu(f_{1,\mathbb{C}},\dots,f_{k,\mathbb{C}})=1 \bmod 2.$$

The theorem is proved. □

Example 1. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ $(n\ge 2)$ be distinct numbers. Then the mapping

$$f\colon(\mathbb{R}^n,0)\to(\mathbb{R}^2,0),\quad x\mapsto(f_1:=x_1^2+x_2^2+\cdots+x_n^2,\,f_2:=\lambda_1x_1^2+\lambda_2x_2^2+\cdots+\lambda_nx_n^2)$$

is a complete intersection with isolated singularity, and f has a local Pareto optimum at $0\in\mathbb{R}^n$. Consider the ideal $I\subset\mathcal{O}_{\mathbb{C}^n,0}$ generated by f_1 and all 2×2-minors $\dfrac{\partial(f_1,f_2)}{\partial(x_i,x_j)}, 1\le i\le j\le n$. Then it is easy to check that

$$I=\left(x_1^2+x_2^2+\cdots+x_n^2,x_ix_j:1\le i\le j\le n\right).$$

This ideal contains all homogeneous polynomials of degree 3 so that if we denote by $\mathfrak{m}$ the maximal ideal in $\mathcal{O}_{\mathbb{C}^n,0}$, then $I+\mathfrak{m}^4\supset\mathfrak{m}^3$. It then follows from Nakayama's lemma that $I\supset\mathfrak{m}^3$. This implies that $\mathcal{O}_{\mathbb{C}^n,0}/I$ is generated by the residue classes of $1,x_1,\dots,x_n,x_1^2,\dots,x_{n-1}^2$. Consequently,

$$\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I)=2n.$$

On the other hand, it is clear that the dimension of $\mathcal{O}_{\mathbb{C}^n,0}\,/\,(\dfrac{\partial f_1}{\partial x_1},\dots,\dfrac{\partial f_1}{\partial x_n})$ is 1. Therefore,

$$\mu(f_1,f_2)=2n-1=1 \bmod 2.$$

3. Necessary and sufficient conditions for local Pareto optima

In this section, we give the high-order necessary and sufficient conditions for a local Pareto optimum. First let us recall the definition of the Newton polyhedron of mappings in the real space $\mathbb{R}^n$ (see, for example, Kouchnirenko, 1976). Let $\mathbb{N}\subset\mathbb{R}_+\subset\mathbb{R}$ be the sets of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let $f_i:=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(i)x^\alpha, i=1,2,\dots,k$. Let us write

$$\operatorname{supp}(f):=\cup_{i=1}^k\{\alpha\in\mathbb{N}^n\mid a_\alpha(i)\neq 0\}.$$

Then the *Newton polyhedron* $\Gamma_+(f)$ of f is the convex hull in $\mathbb{R}^n_+$ of the set $\cup_{\alpha\in\operatorname{supp}(f)}(\alpha+\mathbb{R}^n_+)$. For any $m\in\mathbb{R}^n_+, m\neq 0$, we consider a *supporting hyperplane*

$\{\alpha \in \mathbb{R}^n \mid \langle m, \alpha \rangle = \nu(m)\}$ such that

$$\langle m, \alpha \rangle \geq \nu(m), \qquad \text{for all } \alpha \in \Gamma_+(f).$$

These conditions determine $\nu(m)$ uniquely, while $\Gamma_+(f)$ is given by the system of inequalities[1]

$$\langle m, \alpha \rangle \geq \nu(m), \quad m \in \mathbb{R}^n_+.$$

A *face* of the boundary of the Newton polyhedron $\Gamma_+(f)$ is an intersection of $\Gamma_+(f)$ with some supporting hyperplane. The *Newton diagram* $\Gamma(f)$ of f is the union of the compact faces of the Newton polyhedron $\Gamma_+(f)$. The mapping f is called *convenient* if the Newton diagram $\Gamma(f)$ of f meets all coordinate axes. For each face $\gamma \in \Gamma(f)$, the restrictions

$$f_{i,\gamma}(x) := \sum_{\alpha \in \gamma} a_\alpha(i) x^\alpha, \quad i = 1, 2, \ldots, k,$$

are called the *quasi-homogeneous components* of f with respect to γ.

Let $\{\alpha \in \mathbb{R}^n \mid \langle m, \alpha \rangle = \nu(m)\}$ be the supporting hyperplane of a given face $\gamma \in \Gamma(f)$. The following lemma indicates a convenient way to determine $f_{i,\gamma}$ from f_i.

Lemma 1. *Let $x \in \mathbb{R}^n, x \neq 0$. We have*

$$f_i(t^m \bullet x) \;=\; t^{\nu(m)} f_{i,\gamma}(x) + o(t^{\nu(m)}) \qquad \text{as} \qquad t \to 0,$$

where $t^m \bullet x := (t^{m_1} x_1, t^{m_2} x_2, \ldots, t^{m_n} x_n)$.

Proof. By definition, $\langle m, \alpha \rangle \geq \nu(m)$ for all $\alpha \in \Gamma_+(f)$ with equality if and only if $\alpha \in \gamma$. Moreover, by the definition of the quasi-homogeneous components with respect to the face γ, we get

$$f_{i,\gamma}(t^m \bullet x) = t^{\nu(m)} f_{i,\gamma}(x).$$

From this the lemma follows. □

Theorem 2. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a real analytic mapping defined in a neighborhood of the origin with $f(0) = 0$.*

(i) *If 0 is a local Pareto optimum for f, then*

$$\max_{i=1,2,\ldots,k} f_{i,\gamma}(x) \geq 0$$

for all $\gamma \in \Gamma(f)$ and $x \in \mathbb{R}^n$.

(ii) *Suppose that f is convenient. If for any $\gamma \in \Gamma(f)$ we have*

$$\max_{i=1,2,\ldots,k} f_{i,\gamma}(x) > 0$$

everywhere except in the coordinate planes, then 0 is a strict local Pareto optimum for f.

[1] The system of inequalities is infinite; however, there exists a finite number of inequalities of which the remaining inequalities are a consequence.

Proof. (i) Suppose on the contrary that there exist $\gamma \in \Gamma(f)$ and $x^0 \in \mathbb{R}^n$ such that

$$\max_{i=1,2,\dots,k} f_{i,\gamma}(x^0) < 0.$$

Then, it follows from Lemma 1 that

$$f_i(t^m \bullet x^0) < 0, i = 1, 2, \dots, k, \quad \text{for all } 0 < t \ll 1.$$

Thus 0 is not a local Pareto optimum for f, which contradicts the hypothesis.

(ii) We now suppose that for any $\gamma \in \Gamma(f)$ we have $\max_{i=1,2,\dots,k} f_{i,\gamma}(x) > 0$ everywhere except in the coordinate planes. We will prove that 0 is a strict local Pareto optimum for f. Indeed, suppose that contrary to our claim, in any neighborhood of 0 there are points of the set where the functions $f_i, i = 1, 2, \dots, k$, are non-positive. Then, by the Curve Selection Lemma (see Milnor, 1968), there exists an analytic curve $\varphi\colon [0, \epsilon) \to \mathbb{R}^n, t \mapsto \varphi(t)$, such that

(a) $f_i[\varphi(t)] \le 0, i = 1, 2, \dots, k$, for $t \in [0, \epsilon)$;

(b) $\varphi(t) = 0$ if and only if $t = 0$.

Without loss of generality, we may assume that this curve lies entirely in the coordinate planes $\{x_j = 0\}, j = l + 1, l + 2, \dots, n$, where $1 \le l \le n$, and does not lie in the remaining coordinate planes. Then we can write

$$\varphi(t) := \begin{cases} x_1(t) = x_1^0 t^{m_1} + \text{ higher order terms in } t, \\ x_2(t) = x_2^0 t^{m_2} + \text{ higher order terms in } t, \\ \qquad \cdots \\ x_l(t) = x_l^0 t^{m_l} + \text{ higher order terms in } t, \\ x_{l+1}(t) = x_{l+2}(t) = \cdots = x_n(t) = 0, \end{cases}$$

for $t \in [0, \epsilon)$, where $x_j^0, j = 1, 2, \dots, l$, are non-zero real numbers and $\min\limits_{j=1,2,\dots,l} m_j > 0$. We consider the set Γ' obtained by intersecting the Newton diagram $\Gamma(f)$ and the subspace $A := \{\alpha_j = 0, j = l + 1, l + 2, \dots n\}$. If f is convenient, then its restriction to the subspace A will again be convenient. Consequently, Γ' is the Newton diagram of the restriction $f|_A$. Let γ (resp., $\nu(m)$) be the set of minimal solutions (resp., the minimal value) of the following programming problem

$$\min_{\alpha \in \Gamma'} \langle m, \alpha \rangle,$$

where m is the column vector $(m_1, m_2, \dots, m_l, 0, 0, \dots, 0)^t$. Then γ is some face of the diagrams Γ' and $\Gamma(f)$. Let $x^0 := (x_1^0, x_2^0, \dots, x_l^0, 1, 1, \dots, 1)$. By assumption,

$$\max_{i=1,2,\dots,k} f_{i,\gamma}(x^0) > 0. \tag{4}$$

On the other hand, from the fact that $f_i[\varphi(t)] \le 0, i = 1, 2, \dots, k$, on the curve φ, it follows that on the curve $\bar{\varphi}\colon [0, \epsilon) \to \mathbb{R}^n$, which is defined by

$$\bar{\varphi} := \begin{cases} \bar{x}_1(t) = x_1^0 t^{m_1},\ \bar{x}_2(t) = x_2^0 t^{m_2},\ \dots,\ \bar{x}_l(t) = x_l^0 t^{m_l}, \\ \bar{x}_{l+1}(t) = \bar{x}_{l+2}(t) = \cdots = \bar{x}_n(t) = 0, \end{cases}$$

one has the relations $f_i[\bar{\varphi}(t)] \leq 0, i = 1, 2, \ldots, k$. However, by Lemma 1, for $i = 1, 2, \ldots, k$,

$$\begin{aligned} 0 \geq f_i[\bar{\varphi}(t)] &= f_{i,\gamma}[\bar{\varphi}(t)] + o(t^{\nu(m)}) \\ &= t^{\nu(m)} f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_l^0, 0, 0, \ldots, 0) + o(t^{\nu(m)}) \\ &= t^{\nu(m)} f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_l^0, 1, 1, \ldots, 1) + o(t^{\nu(m)}), \end{aligned}$$

which contradicts (4). (The last equality follows from the independence of the quasi-homogeneous component $f_{i,\gamma}$ in the variables $x_{l+1}, x_{l+2}, \ldots, x_n$.) □

Remark 1. Theorem 2 has been proved by Vassiliev (1977) in the case where $k = 1$.

Example 2. (i) Consider the following real analytic mapping

$$f(x, y) := (f_1(x, y) := y^6 + x^3y^2, f_2(x, y) := x^8 + 2x^3y^2 + xy^4)\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0).$$

The Newton diagram $\Gamma(f)$ of f consists of the three line segments AB, BC and CD, where A, B, C and D are the points of coordinates $(0, 6), (1, 4), (3, 2)$ and $(8, 0)$, respectively. Choose $\gamma = \{C(3, 2)\}$-the vertex of $\Gamma(f)$. We have

$$f_{1,\gamma}(x, y) = x^3y^2, \qquad f_{2,\gamma}(x, y) = 2x^3y^2.$$

Hence, $\max_{i=1,2} f_{i,\gamma}(x, y) < 0$ for all (x, y) such that $x < 0, y \neq 0$. Therefore, by Theorem 2 (i), 0 is not a local Pareto optimum for f.

(ii) Let k be a positive integer number. Let

$$\begin{aligned} f\colon (\mathbb{R}^n, 0) &\to (\mathbb{R}^2, 0), \\ x &\mapsto (f_1(x), f_2(x)), \end{aligned}$$

be an analytic mapping which is defined by

$$\begin{aligned} f_1(x) &:= x_1^{2k} + x_2^{2k} + \cdots + x_{n-1}^{2k} - x_n^{2k+1} + \sum_{j>2k+1} H_j(x), \\ f_2(x) &:= x_n^{2k+1} + \sum_{j>2k+1} G_j(x), \end{aligned}$$

where H_j, G_j are homogeneous polynomials of degree j. Then, it is easy to check that the Newton diagram $\Gamma(f)$ of f is the convex hull of the following points:

$$A_1(2k, 0, \ldots, 0), A_2(0, 2k, \ldots, 0), \ldots, A_{n-1}(0, 0, \ldots, 2k, 0), A_n(0, 0, \ldots, 0, 2k+1).$$

Let γ be a face of $\Gamma(f)$. There are two cases to be considered.

Case 1: $A_n \notin \gamma$. We have

$$f_{1,\gamma}(x) = \sum_{\{j | A_j \in \gamma\}} x_j^{2k}, \qquad f_{2,\gamma}(x) = 0.$$

Therefore, $\max_{i=1,2} f_{i,\gamma}(x) \geq 0$, with equality if and only if $x_j = 0$ for all j with $A_j \in \gamma$.

Case 2: $A_n \in \gamma$. In this case, we have

$$f_{1,\gamma}(x) = \sum_{\{j|A_j\in\gamma, j\neq n\}} x_j^{2k} - x_n^{2k+1}, \qquad f_{2,\gamma}(x) = x_n^{2k+1}.$$

Thus, $\max_{i=1,2} f_{i,\gamma}(x) \geq |x_n|^{2k+1} \geq 0$. In particular, the equality $\max_{i=1,2} f_{i,\gamma}(x) = 0$ implies that $x_n = 0$.

Combining cases (1) and (2), we get the following inequality

$$\max_{i=1,2} f_{i,\gamma}(x) > 0,$$

everywhere except in the coordinate planes. Hence, we can apply the sufficient condition (ii) in Theorem 2 and obtain that f has 0 as a strict local Pareto optimum.

The following result has been proved by Smale (1973) and (1975), and Wan (1975); (the proofs were simplified later in Geldrop (1980), see also Wan (1977), Hà Huy Vui (1980) and (1982)). We will prove it in a quite different way, using the Curve Selection Lemma (see Milnor, 1968).

Theorem 3. *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a real analytic mapping defined in a neighborhood of the origin with $f(0) = 0$.*

(i) *If 0 is a local Pareto optimum for f, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$, not all zero, such that*

$$\sum_{i=1}^{k} \lambda_i Df_i(0) = 0. \tag{5}$$

(ii) *Let be given $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ not all zero satisfying (5). If the bilinear symmetric form $\left[\sum_{i=1}^{k} \lambda_i D^2 f_i(0)\right]$ is positive definite on the linear subspace*

$$\{v \in \mathbb{R}^n \mid \langle \lambda_i Df_i(0), v\rangle = 0, \text{ for all } i\},$$

then 0 is a strict local Pareto optimum for f.

Proof. (i) It is clear that we only have to consider the case $Df_i(0) \neq 0, i = 1, 2, \ldots, k$. Let γ be the set of minimal solutions of the following linear programming problem

$$\min_{\alpha\in\Gamma_+(f)} \langle m, \alpha\rangle,$$

where m is the column vector $(1, 1, \ldots, 1)^t$. Then γ is some face of the polyhedron with the vertices $e^{(j)} := (0, 0, \ldots, \overset{j}{1}, 0 \ldots, 0)$ for $j = 1, 2, \ldots, n$. This leads to the fact that the quasi-homogeneous components of f with respect to γ is defined by

$$f_{i,\gamma}(x) = \langle Df_i(0), x\rangle, \quad i = 1, 2, \ldots, k.$$

By Theorem 2, $\max_{i=1,2,\ldots,k} f_{i,\gamma}(x) \geq 0$ on $\mathbb{R}^n$. Hence, the set

$$\{x \in \mathbb{R}^n \mid \langle Df_i(0), x\rangle < 0, \quad i = 1, 2, \ldots, k\}$$

is empty. It follows from Farkas's lemma that this relation is equivalent to (5).

(ii) Suppose, by contradiction, that in any neighborhood of 0 there are points of the set where the functions $f_i, i = 1, 2, \ldots, k$, are non-positive. Then, by the Curve Selection Lemma (see Milnor, 1968), there exists an analytic curve $\varphi\colon [0, \epsilon) \to \mathbb{R}^n, t \mapsto \varphi(t)$, such that

(a) $f_i[\varphi(t)] \leq 0, i = 1, 2, \ldots, k$, for $t \in [0, \epsilon)$;

(b) $\varphi(t) = 0$ if and only if $t = 0$.

Without loss of generality, we can suppose that this curve lies entirely in the coordinate planes $\{x_j = 0\}, j = l+1, l+2, \ldots, n$, where $1 \leq l \leq n$, and does not lie in the remaining coordinate planes. Then we can write

$$\varphi(t) := \begin{cases} x_1(t) = x_1^0 t^{m_1} + \text{ higher order terms in } t, \\ x_2(t) = x_2^0 t^{m_2} + \text{ higher order terms in } t, \\ \quad \cdots \\ x_l(t) = x_l^0 t^{m_l} + \text{ higher order terms in } t, \\ x_{l+1}(t) = x_{l+2}(t) = \cdots = x_n(t) = 0, \end{cases}$$

for $t \in [0, \epsilon)$, where $x_j^0, j = 1, 2, \ldots, l$, are non-zero real numbers and

$$\nu := \min_{j=1,2,\ldots,l} m_j > 0. \tag{6}$$

From the fact that $f_i[\varphi(t)] \leq 0, i = 1, 2, \ldots, k$, on the curve φ, it follows that on the curve

$$\bar{\varphi} := \begin{cases} \bar{x}_1(t) = x_1^0 t^{m_1},\ \bar{x}_2(t) = x_2^0 t^{m_2},\ \ldots,\ \bar{x}_l(t) = x_l^0 t^{m_l}, \\ \bar{x}_{l+1}(t) = \bar{x}_{l+2}(t) = \cdots = \bar{x}_n(t) = 0, \end{cases}$$

for sufficiently small $t > 0$ one has the inequalities

$$f_i[\bar{\varphi}(t)] \leq 0, \quad i = 1, 2, \ldots, k. \tag{7}$$

On the other hand, we have

$$\begin{aligned} f_i(x) &= \langle Df_i(0), x\rangle + o(\|x\|), \\ f_i(x) &= \langle Df_i(0), x\rangle + \left[D^2 f_i(0)\right](x, x) + o(\|x\|^2). \end{aligned}$$

Replacing x by $\bar{\varphi}(t)$, for $0 < t \ll 1$, we get

$$f_i[\bar{\varphi}(t)] = \langle Df_i(0), \bar{\varphi}(t)\rangle + o(t^\nu), \tag{8}$$

$$f_i[\bar{\varphi}(t)] = \langle Df_i(0), \bar{\varphi}(t)\rangle + \left[D^2 f_i(0)\right](\bar{\varphi}(t), \bar{\varphi}(t)) + o(t^{2\nu}), \tag{9}$$

for $i = 1, 2, \ldots, k$.

We now define the vector $w := (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ componentwise by

$$w_j := \begin{cases} x_j^0 & \text{if } m_j = \nu, \\ 0 & \text{if } m_j > \nu. \end{cases}$$

Then it is clear that $w \neq 0$. Using (6) and (8), we obviously have, for $0 < t \ll 1$,

$$f_i[\bar{\varphi}(t)] = \langle Df_i(0), w\rangle t^\nu + o(t^\nu), \quad \text{for all } i = 1, 2, \ldots, k.$$

This relation and (7) imply that

$$\langle Df_i(0), w\rangle \leq 0, \quad \text{for all } i = 1, 2, \dots, k.$$

Therefore, it follows from (5) that

$$\langle \lambda_i Df_i(0), w\rangle = 0, \quad \text{for all } i = 1, 2, \dots, k.$$

In other words, $w \in \{v \in \mathbb{R}^n \mid \langle \lambda_i Df_i(0), v\rangle = 0, \text{ for all } i\}$.

Moreover, from (5) and (9) we get

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i f_i[\bar{\varphi}(t)] &= \left[\sum_{i=1}^{k} \lambda_i D^2 f_i(0)\right] (\bar{\varphi}(t), \bar{\varphi}(t)) + o(t^{2\nu}) \\ &= \left[\sum_{i=1}^{k} \lambda_i D^2 f_i(0)\right] (w, w) t^{2\nu} + o(t^{2\nu}). \end{aligned}$$

Hence, by (7) we obtain

$$\left[\sum_{i=1}^{k} \lambda_i D^2 f_i(0)\right] (w, w) \leq 0,$$

which contradicts the fact that the bilinear symmetric form $\left[\sum_{i=1}^{k} \lambda_i D^2 f_i(0)\right]$ is positive definite on the linear subspace $\{v \in \mathbb{R}^n \mid \langle \lambda_i Df_i(0), v\rangle = 0, \text{ for all } i\}$. $\square$

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