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SHARP WEIGHTED INEQUALITY FOR MULTILINEAR COMMUTATOR OF THE LITTLEWOOD-PALEY OPERATOR

ZHANG MINGJUN AND LIU LANZHE

Abstract. In this paper, we prove the sharp inequality for multilinear commutator related to Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1-4]). Let T be the Calderón-Zygmand singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [3]) states that commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n))$ for $1 < p < \infty$. In [6-8], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp inequality for multilinear commutator related to the Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

2. Multilinear commutator

First let us introduce some notations (see [4], [8], [9]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q let

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx$$

and the sharp function of f is defined by

$$f^\#(x) = \sup_{x\in Q}\frac{1}{|Q|}\int\limits_Q |f(y) - f_Q|dy.$$

It is well-known that (see [4])

$$f^\#(x) = \sup_{x\in Q}\inf_{c\in C}\frac{1}{|Q|}\int\limits_Q |f(y) - C|dy.$$

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We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define

$$||b||_{BMO} = ||b||_{L^\infty}.$$

It has been known that (see [9])

$$||b - b_{2^kQ}||_{BMO} \le Ck||b||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q} \big|Q\big|^{-1} \int_Q |f(y)|dy.$$

We write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $b_j \in BMO(j = 1, \cdots, m)$, set

$$||\vec{b}||_{BMO} = \prod_{j=1}^{m} ||b_j||_{BMO}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \backslash \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and

$$||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}.$$

We denote the Muckenhoupt weights by A_1 (see [4]), that is

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

In this paper, we will study some multilinear commutators defined as follows.

Definition 1. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integral functions on R^n. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x + y) - \psi(x)| \le C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The Littlewood-Paley multilinear commutator is defined by

$$g_\psi^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m}(b_j(x) - b_j(y))\right] \psi_t(x - y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set

$$F_t(f)(x) = \int_{R^n} \psi_t(x - y)f(y)dy.$$

We also define

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley g function (see [10]).

Let H be the space $H = \{h : ||h|| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2}\}$. Then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)||$$

and

$$g_\psi^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $g_\psi^{\vec{b}}$ is just the m order commutator (see [1], [5]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-3] [5-8]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

3. Main results

We state our main results as follows.

Theorem 1. *Let $b_j \in BMO$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(g_\psi^{\vec{b}}(f))^\#(x) \le C||\vec{b}||_{BMO}\left(M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_\psi^{\vec{b}_{\sigma^c}}(f))(x)\right).$$

Theorem 2. *Let $b_j \in BMO$ for $j = 1, \cdots, m$. Then $g_\psi^{\vec{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

To prove these theorems, we need the following lemma.

Lemma 3 (see [10]). Let $w \in A_1$ and $1 < p < \infty$. Then g_ψ is bounded on $L^p(w)$.

Proof of Theorem 1. It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |g_\psi^{\vec{b}}(f)(x) - C_0|dx \le C\left(||b||_{BMO} M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_\psi^{\vec{b}_{\sigma^c}}(f)(x))\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$.

Case $m = 1$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x).$$

Then

$$\begin{aligned}
&\left|g_\psi^{b_1}(f)(x) - g_\psi(((b_1)_{2Q} - b_1)f_2)(x_0)\right| \\
&= \left|\left\|F_t^{b_1}(f)(x)\right\| - \left\|F_t(((b_1)_{2Q} - b_1)f_2)(x_0)\right|\right| \\
&\le \left\|F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)\right\| \\
&\le \left\|(b_1(x) - (b_1)_{2Q})F_t(f)(x)\right\| + \left\|F_t((b_1 - (b_1)_{2Q})f_1)(x)\right\| \\
&+ \left\|F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)\right\| \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by the Hölder inequality with exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q A(x)dx &= \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}||g_\psi(f)(x)|dx \\
&\le C\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |g_\psi(f)(x)|^r dx\right)^{1/r} \\
&\le C||b_1||_{BMO}M_r(g_\psi(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose p such that $1 < p < r$. By the boundedness of g_ψ on $L^p(R^n)$ and the Hölder inequality, we obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q B(x)dx &= \frac{1}{|Q|}\int_Q g_\psi((b_1 - (b_1)_{2Q})f_1)(x)dx \\
&\le \left(\frac{1}{|Q|}\int_{R^n} [g_\psi((b_1 - (b_1)_{2Q})f\chi_{2Q})(x)]^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n} (|b_1(x) - (b_1)_{2Q}||f_1(x)|)^p dx\right)^{1/p} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q} |f(x)|^r dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |b_1 - (b_1)_{2Q}|^{rp/(r-p)}dx\right)^{(r-p)/rp} \\
&\le C||b_1||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by the Minkowski inequality, we obtain

$$C(x) = ||F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)||$$

$$
\begin{aligned}
&= \left[\int_0^\infty \left(\int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)||\psi_t(x-y)-\psi_t(x_0-y)|dy\right)^2 \frac{dt}{t}\right]^{1/2} \\
&= \int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)| \left(\int_0^\infty \frac{1}{t}|\psi_t(x-y)-\psi_t(x_0-y)|^2 dt\right)^{1/2} dy \\
&\leq C \int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)| \left(\int_0^\infty \frac{|x_0-x|^{2\varepsilon}\cdot tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right)^{1/2} dy \\
&\leq C \int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)| \frac{|x_0-x|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}} dy \\
&\leq C \sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |b_1(y)-(b_1)_{2Q}||f(y)| \frac{|x_0-x|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}} dy \\
&\leq C \sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1} \int_{2^{k+1}Q} |b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C \sum_{k=1}^\infty 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |f(x)|^r dx\right)^{1/r} \times \\
&\times \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |b_1-(b_1)_{2Q}|^{rp/(r-p)} dx\right)^{(r-p)/rp} \\
&\leq C \sum_{k=1}^\infty k2^{-k\varepsilon}||b_1||_{BMO} M_r(f)(\tilde{x}) \\
&\leq C||b_1||_{BMO} M_r(f)(\tilde{x}).
\end{aligned}
$$

Thus

$$
\frac{1}{|Q|}\int_Q C(x)dx \leq C||b_1||_{BMO} M_r(f)(\tilde{x}).
$$

Case $m \geq 2$. We have known that, for $b = (b_1, \cdots, b_m)$,

$$
\begin{aligned}
F_t^{\vec{b}}(f)(x) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x)-b_j(y))\right] \psi_t(x-y)f(y)dy \\
&= \int_{R^n} (b_1(x)-(b_1)_{2Q})-(b_1(y)
\end{aligned}
$$

$$
\begin{aligned}
&- (b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})-(b_m(y)-(b_m)_{2Q})\psi_t(x-y)f(y)dy\\
&=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_\sigma\psi_t(x-y)f(y)dy\\
&=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\\
&+(-1)^mF_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-b(x))_{\sigma^c}\psi_t(x-y)f(y)dy\\
&=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\\
&+(-1)^mF_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|g_\psi^{\vec{b}}(f)(x)-g_\psi(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m))f_2)(x_0)|\\
=&\ \Big|\big\|f_t^{\vec{b}}(f)(x)\big\|-\big\|f_t(((b_1)_{2Q}-b_1)\cdots((b_m)2Q-b_m)f_2)(x_0)\big\|\Big|\\
\le&\ \Big\|f_t^{\vec{b}}-F_t(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m)f_2)(x_0)\Big\|\\
\le&\ \Big\|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\Big\|\\
+&\ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big\|(\vec{b}(x)-(b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)\Big\|\\
+&\ \Big\|F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)\Big\|\\
+&\ \Big\|F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)\\
-&\ F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)\Big\|\\
=&\ I_1(x)dx+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, by the Hölder inequality with exponent $1/p_1+\cdots+1/p_m+1/r=1$, where $1<p_j<\infty,\ j=1,\cdots,m$, we get

$$
\frac{1}{|Q|}\int_Q I_1(x)dx\le\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||g_\psi(f)(x)|dx
$$

$$\leq \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{1/p_1} \times \cdots \times$$

$$\times \left(\frac{1}{|Q|}\int_Q |b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m} \left(\frac{1}{|Q|}\int_Q |g_\psi(f)(x)|^r dx\right)^{1/r}$$

$$\leq C||\vec{b}||_{BMO} M_r(g_\psi(f))(\tilde{x}).$$

For $I_2(x)$, by the Minkowski and Hölder inequalities, we get

$$\frac{1}{|Q|}\int_Q I_2(x)dx = \frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||dx$$

$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} \frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||g_\psi^{\vec{b}_{\sigma^c}}(f)(x)|dx$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} \left(\frac{1}{|2Q|}\int_{2Q} |(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'} \left(\frac{1}{|Q|}\int_Q |g_\psi^{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||\vec{b}_\sigma||_{BMO} M_r(g_\psi^{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $I_3(x)$, choose $1 < p < r$, $1 < q_j < \infty$, $j = 1, \cdots, m$ such that $1/q_1 + \cdots + 1/q_m + p/r = 1$. By the boundedness of $g_\psi(f)(x)$ on $L^p(R^n)$ and the Hölder inequality, we get

$$\frac{1}{|Q|}\int_Q I_3(x)dx = \frac{1}{|Q|}\int_Q ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)||dx$$

$$\leq \left(\frac{1}{|Q|}\int_{R^n} |g_\psi((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p}$$

$$\leq C\left(\frac{1}{|Q|}\int_{R^n} |b_1(x)-(b_1)_{2Q}|^p\cdots|b_m-(b_m)_{2Q}|^p|f\chi_{2Q}|^p dx\right)^{1/p}$$

$$\leq C\left(\frac{1}{|2Q|}\int_{2Q} |f(x)|^r dx\right)^{1/r} \left(\frac{1}{|2Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}|^{pq_1}dx\right)^{1/pq_1} \times\cdots\times$$

$$
\begin{aligned}
\times \quad & \left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{pq_m}dx\right)^{1/pq_m} \\
\leq \quad & C||\vec{b}||_{BMO}M_r(f_1)(\tilde{x}).
\end{aligned}
$$

For $I_4(x)$, choose $1<p_j<\infty,\ j=1,\cdots,m$ such that $1/p_1+\cdots+1/p_m+1/r=1$. We obtain, by the Hölder inequality,

$$
\begin{aligned}
I_4(x) = & \Big\|F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x) \\
& - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)\Big\| \\
= & \left(\int_0^\infty\left|\int_{R^n}\left[\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right]f\chi_{(2Q)^c}(y)(\psi_t(x-y)-\psi_t(x_0-y))dy\right|^2\frac{dt}{t}\right)^{1/2} \\
\leq & C\int_{R^n}\left|\prod_{j=1}^m(b_j(y)-b(b_j)_{2Q})\right|\left|f\chi_{(2Q)^c}(y)\right| \\
\times & \left(\int_0^\infty\frac{|\psi_t(x-y)-\psi_t(x_0-y)|^2dt}{t}dt\right)^{1/2}dy \\
\leq & C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|\left(\int_0^\infty\frac{|x-x_0|^{2\varepsilon}tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right)^{1/2}dy \\
\leq & C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{(n+\varepsilon)}}dy \\
\leq & C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x-x_0|^{\varepsilon}|x_0-y|^{-(n+\varepsilon)}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy \\
\leq & C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy \\
\leq & C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(x)|^rdx\right)^{1/r}\times \\
\times & \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(x)-(b_1)_{2Q}|^{p_1}dx\right)^{1/p_1}\times\cdots\times
\end{aligned}
$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}$$
$$\leq C\sum_{k=1}^{\infty}k^m 2^{-km}\prod_{j=1}^{m}||b_j||_{BMO}M_r(f)(\tilde{x})$$
$$\leq C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).$$

Thus

$$\frac{1}{|Q|}\int_Q I_4(x)dx \;=\; C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).$$

This completes the proof of the theorem. □

Proof of Theorem 2. We choose $1 < r < p$ as in Theorem 1. Using Lemma 3, we similarly get the conclusion. □

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Department of Mathematics
Hunan University
Changsha, 410082
P.R. of China

E-mail address: lanzheliu@263.net