Volume 30, Number 2, 2005, pp. 169-179

OPTIMALITY CONDITIONS IN DC-CONSTRAINED OPTIMIZATION

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Abstract. This paper studies the necessary and sufficient optimality conditions associated with the problem of minimizing a DC-function (difference of two convex functions) subject to a DC-constraint.

1. Introduction

We consider the following DC-constrained minimization problem

$$(\mathcal{P}) \qquad \inf\Big\{f_1(x) - f_2(x) : h_1(x) - h_2(x) \notin -\text{ int } Y_+\Big\},$$

where $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are two extended real-valued functions and h_1, h_2 are two convex mappings defined from X and taking values in a topological vector real space Y equipped with a partial order induced by a convex cone $Y_+ \subset Y$. This model is versatile and various models arising from optimization, economics, operation research and others (see [4] and references therein) can be stated in the form $(\mathcal{P})$. So problem $(\mathcal{P})$ provides an unified frame work for obtaining various results of DC-optimization. Let us point out that this large class contains an important subclass of programming problems namely reverse convex optimization problems by taking $f_2 \equiv 0$ and $h_2 \equiv 0$.

In recent years significant advances have been made in the study of duality theory associated with constrained DC-optimization (see [5], [7], [6], [9], [10] and [13]).

Recently, the author [8] has developed sufficient optimality conditions for problem $(\mathcal{P})$ subject to a vector reverse convex constraint termed by $h_1(x) \notin -\text{ int } Y_+$ (with $h_2 \equiv 0$). He also stated, under the same above constraint, the necessary optimality conditions in the case where f_2 is supposed to be strictly Hadamard differentiable without convexity.

In the present work, our purpose is to study optimality conditions for the problem $(\mathcal{P})$, extending the recent result on reverse convex programming by Laghdir [8]. Let us point out that the same problem has been considered in [3] where the objective function takes vector values by using a scalarization method.

Received February 5, 2004; in revised form November 2, 2004.
1991 *Mathematics Subject Classification.* 90C48, 49N15, 90C26.
Key words and phrases. Optimality conditions, DC-function, DC-constraints.

The paper is organized as follows. In Section 2, we introduce some notations and preliminaries. Section 3 is devoted to extend the necessary conditions established by Laghdir [8], to the case where the objective function is DC. In Sections 4 and 5, we formulate the optimality conditions associated with problem $(\mathcal{P})$. The approach that we will adopt for getting our main results, is based on the use of an equivalent transformation of $(\mathcal{P})$ into a minimization problem given by

$$\inf\{F_1(x,y) - F_2(x,y) : H(x,y) \notin -\text{int } \mathrm{Y}_+\},$$

where F_1, F_2 and H are auxiliary convex functions on $X \times Y$ defined by means of the functions g_1, g_2, h_1 and h_2. This allows to derive the desired results by applying the recent results in [8] and the related necessary conditions proved in Section 3.

2. Definitions and notations

Throughout the paper, $(X, \|.\|)$ stands for a real normed vector space and X^* is its topological dual. Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be an extended-real-valued function and let $\bar{x}$ be any point where f is finite. f is said to be locally Lipschitzian around $\bar{x}$ if there exist two real numbers $k > 0$ and $\delta > 0$ such that

$$|f(x) - f(y)| \leq k \, \|x - y\|, \qquad \forall x, y \in \bar{x} + \delta\mathbb{B}_X,$$

where $\mathbb{B}_X$ denotes the closed unit ball of X. In [1], it was shown that when f is locally Lipschitzian, the Clarke's generalized directional of f at $\bar{x}$ defined by

$$v \longrightarrow f^0(\bar{x}, v) := \limsup_{\substack{x \to \bar{x} \\ t \to 0^+}} \frac{f(x + tv) - f(x)}{t},$$

is a finite sublinear function. The following set

$$\partial^c f(\bar{x}) := \{x^* \in X^* : \langle x^*, v\rangle \leq f^0(\bar{x}, v), \quad \forall v \in X\},$$

called the Clarke subdifferential of f at $\bar{x}$, is a nonempty convex $\sigma(X^*, X)$-compact subset of X^*. If f is convex and continuous at $\bar{x}$, then f is locally Lipschitzian and $f^{'}(\bar{x}, v) = f^0(\bar{x}, v)$ for any $v \in X$, where $v \longrightarrow f^{'}(\bar{x}, v)$ is the usual directional derivative defined by

$$v \longrightarrow f^{'}(\bar{x}, v) := \lim_{t \to 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t},$$

and therefore, $\partial^c f(\bar{x})$ is exact the subdifferential of f in the sense of the convex analysis, usually denoted by $\partial f(\bar{x})$.

Recall that the Fréchet subdifferential $\partial^F f(\bar{x})$ is the set of all $x^* \in X^*$ such that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \varepsilon\|x - \bar{x}\|, \qquad \forall x \in \bar{x} + \delta\mathbb{B}_X.$$

When f is convex then the Fréchet subdifferential coincides with the the subdifferential of convex analysis. Note that one always has

$$\partial^F f(\bar{x}) \subset \partial^c f(\bar{x}).$$

Let S be a nonempty closed subset of X. Consider the distance function $d_S : X \longrightarrow [0, +\infty[$ defined, by

$$d_S(x) := \inf_{y \in S} \|x - y\|, \quad \forall x \in X.$$

The Clarke normal cone to S at $\bar{x}$ is given by

$$N_S^c(\bar{x}) := \mathrm{cl}\big(\bigcup_{\lambda \geq 0} \lambda \partial^c d_S(\bar{x})\big),$$

where "cl" stands for weak star closure in X^*. When S is a convex subset, $N_S^c(\bar{x})$ coincides with the normal cone

$$N_S(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0, \quad \forall x \in X\},$$

in the sense of convex analysis.

Let us recall (see [11] and [12]) that a subset S is said to be epi-Lipschitzian at $\bar{x}$ ($\bar{x}$ is a cluster point of S) if there exist some neighborhood V of $\bar{x}$, $\lambda > 0$ and a nonempty open subset O such that

$$x + ty \in S, \quad \forall x \in S \cap V, \quad \forall y \in O, \quad \forall t \in (0, \lambda).$$

It was demonstrated in [12] that if S is epi-Lipschitzian and $\bar{x}$ is a boundary point of S then

$$N_{X \setminus S}^c(\bar{x}) = -N_S(\bar{x}).$$

Every nonempty open convex subset is epi-Lipschitzian.

3. Necessary conditions associated with the problem of minimizing a DC-function subject to a reverse convex constraint

Consider the following minimization problem

$$(\mathcal{P}_1) \qquad \inf \Big\{ f_1(x) - f_2(x) : x \in X \setminus S \Big\},$$

where $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ are two extended real-valued functions and S is a nonempty open convex subset of X.

Recently, necessary conditions for problem $(\mathcal{P}_1)$ are discussed in [8] in the case where $f_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is supposed only strictly Hadamard differentiable without convexity. In this section, our goal is to establish necessary conditions for problem $(\mathcal{P}_1)$ in the larger class of objective functions that can be written as a difference of two convex functions.

Proposition 3.1. *Assume that $(\mathcal{P}_1)$ admits a local minimum at $\bar{x}$, f_1 and f_2 are convex, finite and continuous at $\bar{x}$. Then*

(i) *For any boundary point $\bar{x}$ of to S, $\partial f_2(\bar{x}) \subset \partial f_1(\bar{x}) - N_S(\bar{x})$;*

(ii) *For any topological interior point $\bar{x}$ of $X \setminus S$, $\partial f_2(\bar{x}) \subset \partial f_1(\bar{x})$.*

Proof. (i) Since f_1 and f_2 are convex, finite and continuous at $\bar{x}$, it follows from a classical result (see [2]) that f_1 and f_2 are locally Lipschitzian at $\bar{x}$. By $k > 0$ we denote a common Lipschitz constant of f_1 and f_2. As $\bar{x}$ is a local minimum of $(\mathcal{P}_1)$, by Proposition 2.4.3 in Clarke [1], the function $x \longrightarrow f_1(x) - f_2(x) + kd_{X\backslash S}(x)$ attains its local minimum at $\bar{x}$; that is, there exists some $\delta > 0$ such that

$$f_1(\bar{x}) - f_2(\bar{x}) + kd_{X\backslash S}(\bar{x}) \leq f_1(x) - f_2(x) + kd_{X\backslash S}(x),$$

for any $x \in \bar{x} + \delta\mathbb{B}_X$. Setting

$$\begin{aligned} F(x) &:= f_1(x) + kd_{X\backslash S}(x) + f_2(\bar{x}), \\ G(x) &:= f_2(x) + f_1(\bar{x}), \end{aligned}$$

we have $F(\bar{x}) = G(\bar{x})$ and

$$F(x) \geq G(x), \qquad \forall x \in \bar{x} + \delta\mathbb{B}_X.$$

Hence, by means of Fréchet subdifferential, we get

$$\partial^F G(\bar{x}) \subset \partial^F F(\bar{x}). \tag{3.1}$$

As

$$\begin{aligned} \partial^F G(\bar{x}) &= \partial^F f_2(\bar{x}), \\ \partial^F F(\bar{x}) &= \partial^F (f_1 + kd_{X\backslash S})(\bar{x}), \end{aligned}$$

and f_2 is convex, it follows from 3.1 that

$$\begin{aligned} \partial f_2(\bar{x}) &\subset \partial^F (f_1 + kd_{X\backslash S})(\bar{x}) \\ &\subset \partial^c (f_1 + k\partial^c d_{X\backslash S})(\bar{x}) \\ &\subset \partial^c f_1(\bar{x}) + k\partial^c d_{X\backslash S}(\bar{x}) \\ &\subset \partial f_1(\bar{x}) + N^c{}_{X\backslash S}(\bar{x}). \end{aligned}$$

Since S is an open convex subset, it follows from [12] that it is epi-Lipschitzian at $\bar{x}$ which is a boundary point to S. According to a result from [12], we have

$$N^c{}_{X\backslash S}(\bar{x}) = -N_S(\bar{x}).$$

Thus we get

$$\partial f_2(\bar{x}) \subset \partial f_1(\bar{x}) - N_S(\bar{x}).$$

(ii) If $\bar{x}$ is a topological interior point of $X\backslash S$ then $\bar{x}$ is indeed a local minimum of $(\mathcal{P}_1)$ without constraint and, therefore, $\partial f_2(\bar{x}) \subset f_1(\bar{x})$. ☐

Now, we apply the above Proposition 3.1 in order to derive necessary conditions related to the following reverse convex programming problem

$$(\mathcal{P}_2) \qquad \inf\,\{f_1(x) - f_2(x) : h(x) \notin -\operatorname{int} Y_+\},$$

where $h : X \longrightarrow Y \cup \{+\infty\}$ is a convex and proper mapping taking values in a topological vector real space equipped with a partial ordered induced by a convex cone Y_+ :

$$y_1 \leq_Y y_2 \Longleftrightarrow y_2 - y_1 \in Y_+.$$

By "intY_+" we denote the topological interior of the cone Y_+. The convexity of the mapping h is taken with respect to the partial order in the following sense

$$h(\alpha x_1 + (1-\alpha)x_2) \leq_Y \alpha h(x_1) + (1-\alpha)h(x_2),$$

for any $\alpha \in [0,1]$ and any $x_1, x_2 \in X$. Let us notice that the mapping h is authorized to take the value $+\infty$ supposed the greatest element adjoined to Y : $y \leq +\infty,\ \forall y \in Y$.

Throughout, we assume that the positive cone Y_+ is with nonempty topological interior and h is continuous. By Y_+^* we denote the polar positive cone of Y_+ defined as

$$Y_+^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall y \in Y_+\},$$

where the symbol $\langle\ ,\ \rangle$ denotes the bilinear pairing between Y and Y^* (resp. X and X^*).

Let us consider the subset S of X defined by

$$S := \{x \in X : h(x) \in -\ \text{int}\ Y_+\} = h^{-1}(\text{int}\ Y_+), \tag{3.2}$$

and the constraint qualification

$$\text{(C.Q.S)} \quad \exists a \in X \text{ such that } \ h(a) \in -\ \text{int}\ Y_+,$$

called usually the Slater condition. From convexity and continuity of the mapping h and the condition (C.Q.S), it follows that S is a nonempty convex open subset of X. By adopting the same reasoning used in [8] combined with Proposition 3.1, we get the related necessary conditions given by

Proposition 3.2. *Assume that $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ are convex, proper and continuous at $\bar{x}$, $h : X \longrightarrow Y \cup \{+\infty\}$ is continuous, Y_+ - convex, the Slater's condition* (C.Q.S) *is satisfied and $\bar{x}$ is a local minimum of $(\mathcal{P}_2)$. Then we have*

(i) *If $\bar{x}$ is a boundary point of S, then: $\forall x^* \in \partial f_2(\bar{x}), \exists y^* \in Y_+^*$ satisfying $x^* \in \partial f_1(\bar{x}) - \partial(y^* \circ h)(\bar{x})$ and $\langle y^*, h(\bar{x})\rangle = 0$.*

(ii) *If $\bar{x}$ is a topological interior point of $X \backslash S$, then $\partial f_2(\bar{x}) \subset \partial f_1(\bar{x})$.*

4. Necessary conditions associated with $(\mathcal{P})$

Now, coming back to our minimization problem $(\mathcal{P})$ and in order to state the related necessary conditions, we start with the following lemmas.

Lemma 4.1. *If we set, for any $y \in Y$,*

$$E_y := \{x \in X :\ h_1(x) - y \notin -\ \text{int}Y_+ \text{ and } h_2(x) - y \in -Y_+\},$$

and we suppose that dom $h_2 = X$, *then we have*

$$\{x \in X :\ h_1(x) - h_2(x) \notin -\text{int}Y_+\} = \bigcup_{y \in Y} E_y.$$

Proof. Let $x \in X$ be such that $h_1(x) - h_2(x) \notin -\ \text{int}\ Y_+$. By putting $y = h_2(x)$ we obtain $x \in E_y$. Conversely, let $x \in \underset{y\in Y}{\cup} E_y$, there exists some $y \in Y$ satisfying

$h_1(x)-y \notin -$ int Y_+ and $h_2(x)-y \in -Y_+$. If we suppose $h_1(x)-h_2(x) \in -\text{int } Y_+$, then we get

$$h_1(x) - y = h_1(x) - h_2(x) + h_2(x) - y \in -\text{int } Y_+ - Y_+ \subset -\text{int } Y_+,$$

which contradicts the fact that $h_1(x) - y \notin -$ int Y_+. □

Lemma 4.2. *If we assume that the mapping* $h : X \longrightarrow Y \cup \{+\infty\}$ *is* Y_+*-convex, continuous and the cone* Y_+ *is closed then we have under the Slater condition* (C.Q.S) *that*

$$\overline{\{x \in X : h(x) \notin -Y_+\}} = \{x \in X : h(x) \notin -\text{int}Y_+\}.$$

Here the closure is taken with respect to the norm topology in X.

Proof. By considering the subset S defined in (3.2), it was proved in [8] that

$$\bar{S} = \{x \in X : h(x) \in -Y_+\}.$$

By virtue of convexity and continuity of the mapping h and tha Slater's condition, the subset S is nonempty, convex and open and therefore it follows from a classical result of convex analysis [2] that

$$S = \text{ int } \bar{S} = \text{ int } \{x \in X : h(x) \in -Y_+\}. \tag{4.1}$$

Passing to the complementary of (4.1), we obtain

$$\overline{\{x \in X : h(x) \notin -Y_+\}} = \{x \in X : h(x) \notin - \text{ int } Y_+\}.$$

□

Remark 4.1. Under the same assumptions of the above Lemma 4.2, a boundary point $\bar{x}$ of the feasible set $\{x \in X : h(x) \notin -$ int $Y_+\}$ is characterized by $h(\bar{x}) \in -Y_+$ and $h(\bar{x}) \notin -$ int Y_+.

Now, let us consider the following auxiliary minimization problem

$$(P_3) \qquad \begin{cases} \inf F_1(x,y) - F_2(x,y) \\ H(x,y) \notin - \text{ int } Y_+, \end{cases}$$

where $F_1, F_2 : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $H : X \times Y \longrightarrow Y \cup \{+\infty\}$ are given by

$$\begin{cases} F_1(x,y) := f_1(x) + \delta_{-Y_+}(h_2(x) - y), \\ F_2(x,y) := f_2(x), \\ H(x,y) := h_1(x) - y. \end{cases}$$

Here $\delta_{-Y_+} : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ stands for the indicator function defined by $\delta_{-Y_+}(y) = 0$ if $y \in -Y_+$ and $\delta_{-Y_+}(y) = +\infty$ otherwise.

Proposition 4.1. *Assume that* dom $h_2 = X$. *Then we have*

(i)

$$\inf_{h_1(x)-h_2(x)\notin-\text{int}Y_+} f_1(x) - f_2(x) = \inf_{H(x,y)\notin-\text{int}Y_+} F_1(x,y) - F_2(x,y).$$

(ii) *If* $\bar{x}$ *is a local minimum of problem* $(\mathcal{P})$ *then* $(\bar{x},\ h_2(\bar{x}))$ *is a local minimum of* $(\mathcal{P}_3)$. *If, furthermore,* h_2 *is continuous we have the equivalence.*

Proof. (i) According to Lemma 4.1 we have

$$\begin{aligned}&\inf_{h_1(x)-h_2(x)\notin \text{ int } Y_+} f_1(x) - f_2(x)\\ &= \inf_{h_1(x)-y\notin \text{ int } Y_+} \{f_1(x) - f_2(x) + \delta_{-Y_+}(h_2(x) - y)\}\\ &= \inf_{H(x,y)\notin - \text{ int } Y_+} F_1(x,y) - F_2(x,y).\end{aligned}$$

(ii) If $\bar{x}$ is a local minimum of problem $(\mathcal{P})$ then there exists some neighborhood V of $\bar{x}$ such that

$$f_1(\bar{x}) - f_2(\bar{x}) \leq f_1(x) - f_2(x), \quad \forall x \in V \ \cap \ C,$$

where

$$C := \{x \in X : h_1(x) - h_2(x) \notin - \text{ int } Y_+\}.$$

By setting

$$E := \{(x,y) \in X \times Y : H(x,y) \notin - \text{ int } Y_+\},$$

and $W := (V \times Y) \ \cap \ E$ we argue for any $(x,y) \in W$ as follows.

If $h_2(x) - y \in -Y_+$ then we claim that $h_1(x) - h_2(x) \notin - \text{ int } Y_+$. First, let us note that $x \in V$ and $h_1(x) - y \notin - \text{ int } Y_+$. Suppose the contrary, i.e., $h_1(x) - h_2(x) \in - \text{ int } Y_+$. Then

$$h_1(x) - y = h_1(x) - h_2(x) + h_2(x) - y \in - \text{ int } Y_+ - Y_+ \subset - \text{ int } Y_+,$$

which contradicts the fact that $h_1(x) - y \notin -Y_+$. Therefore we obtain

$$\begin{aligned}F_1(\bar{x}, h_2(\bar{x})) - F_2(\bar{x}, h_2(\bar{x})) &\leq f_1(x) - f_2(x)\\ &= f_1(x) + \delta_{-Y_+}(h_2(x) - y) - f_2(x)\\ &= F_1(x,y) - F_2(x,y),\end{aligned}$$

and thus we get finally

$$F_1(\bar{x}, h_2(\bar{x})) - F_2(\bar{x}, h_2(\bar{x})) \leq F_1(x,y) - F_2(x,y),$$

for any $(x,y) \in W$, which yields that $(\bar{x}, h_2(\bar{x}))$ is a local minimum of problem $(\mathcal{P}_3)$.

Conversely, if $(\bar{x}, h_2(\bar{x}))$ is a local minimum of problem $(\mathcal{P}_3)$, then there exists some neighborhood O of $\bar{x}$ and some neighborhood U of $h_2(\bar{x})$ such that

$$f_1(\bar{x}) - f_2(\bar{x}) \leq F_1(x,y) - F_2(x,y), \quad \forall (x,y) \in (O \times U) \cap E.$$

If we set $V := O \cap h_2^{-1}(U)$, which is a neighborhood of $\bar{x}$ since h_2 is continuous at $\bar{x}$, then for any $x \in V \cap C$ we have $(x, h_2(x)) \in (O \times U) \cap E$ and hence it follows that

$$f_1(\bar{x}) - f_2(\bar{x}) \leq F_1(x, h_2(x)) - F_2(x, h_2(x)) \quad \forall x \in V \cap C,$$

which means

$$f_1(\bar{x}) - f_2(\bar{x}) \leq f_1(x) - f_2(x), \quad \forall c \in V \cap C.$$

Thus $\bar{x}$ is a local minimum of $(\mathcal{P})$. $\square$

Before stating the necessary conditions for problem $\mathcal{P}$, we will need the following lemma.

Lemma 4.3. *For any* $(\bar{x}, \bar{y}) \in X \times Y$, *we have*

(i) $\partial F_2(\bar{x}, \bar{y}) = \partial f_2(\bar{x}) \times \{0\}$.

(ii) $\partial F_1(\bar{x}, h_2(\bar{x})) = \underset{y^* \in -Y_+^*}{\cup} \partial(f_1 - y^* \circ h_2)(\bar{x}) \times \{-y^*\}$.

(iii) $\partial(y^* \circ H)(\bar{x}, \bar{y}) = \partial(y^* \circ h_1)(\bar{x}) \times \{-y^*\}$.

Proof. (i) We have $(x^*, y^*) \in \partial F_2(\bar{x}, \bar{y})$ if and only if

$$\langle x^*, x - \bar{x}\rangle + \langle y^*, y - \bar{y}\rangle \leq F_2(x, y) - F_2(\bar{x}, \bar{y}), \quad \forall (x, y) \in X \times Y,$$

which means

$$\langle x^*, x - \bar{x}\rangle + \langle y^*, y - \bar{y}\rangle \leq f_2(x) - f_2(\bar{x}), \quad \forall (x, y) \in X \times Y,$$

and hence

$$x^* \in \partial f_2(\bar{x}) \text{ and } y^* = 0.$$

(ii) We have $(x^*, y^*) \in \partial F_1(\bar{x}, h_2(\bar{x}))$ if and only if

$$\langle x^*, x - \bar{x}\rangle + \langle y^*, y - h_2(\bar{x})\rangle \leq f_1(x) - f_1(\bar{x}) + \delta_{-Y_+}(h_2(x) - y), \quad \forall (x, y) \in X \times Y.$$

By setting $z := h_2(x) - y$, we get $(x^*, y^*) \in \partial F_1(\bar{x}, h_2(\bar{x}))$ if and only if

$$\langle x^*, x - \bar{x}\rangle + \langle -y^*, z\rangle \leq (f_1 - y^* \circ h_2)(x) + (f_1 - y^* \circ h_2)(\bar{x}) + \delta_{-Y_+}(z), \quad \forall (x, z) \in X \times Y.$$

Accordingly,

$$x^* \in \partial(f_1 - y^* \circ h_2)(\bar{x}) \text{ and } -y^* \in Y_+^*,$$

therefore we obtain

$$\partial F_1(\bar{x}, h_2(\bar{x})) = \underset{y^* \in -Y_+^*}{\cup} \partial(f_1 - y^* \circ h_2)(\bar{x}) \times \{-y^*\}.$$

(iii) $(x^*, z^*) \in \partial(y^* \circ H)(\bar{x}, \bar{y})$ if and only if

$$\langle x^*, x - \bar{x}\rangle + \langle z^*, y - \bar{y}\rangle \leq (y^* \circ H)(x, y) - (y^* \circ H)(\bar{x}, \bar{y}), \quad (x, y) \in X \times Y.$$

or, equivalently,

$$\langle x^*, x - \bar{x}\rangle + \langle z^*, y - \bar{y}\rangle \leq (y^* \circ h_1)(x) - \langle y^*, y\rangle - (y^* \circ h_1)(\bar{x}) + \langle y^*, \bar{y}\rangle, \quad (x, y) \in X \times Y.$$

Accordingly,

$$x^* \in \partial(y^* \circ h_1)(\bar{x}) \text{ and } z^* = -y^*,$$

hence

$$\partial(y^* \circ H)(\bar{x}, \bar{y}) = \partial(y^* \circ h_1)(\bar{x}) \times \{-y^*\}.$$

□

Now, we are able to provide necessary conditions associated with problem $(\mathcal{P})$.

Proposition 4.2. *Assume that* $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *are convex, continuous and proper functions,* $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ *are* Y_+*-convex, continuous and proper,* $\mathrm{dom} h_2 = X$, *there exists some* $a \in \mathrm{dom} h_1$ *such that* $h_1(a) \in -\mathrm{int} Y_+$ *and* $\bar{x}$ *is a local minimum of problem* $(\mathcal{P})$. *Then we have*

(i) *If* $h_1(\bar{x}) - h_2(\bar{x}) \in -Y_+$ *and* $h_1(\bar{x}) - h_2(\bar{x}) \notin \mathrm{int} Y_+$ *then* $\forall x_2^* \in \partial f_2(\bar{x}), \exists y^* \in Y_+^*$ *such that*

$$x_2^* \in \partial(f_1 + y^* \circ h_2)(\bar{x}) - \partial(y^* \circ h_1)(\bar{x})$$

and

$$(y^* \circ h_1)(\bar{x}) = (y^* \circ h_2)(\bar{x}).$$

(ii) *If* $h_1(\bar{x}) - h_2(\bar{x}) \notin -Y_+$, *then* $\partial f_2(\bar{x}) \subset \partial f_1(\bar{x})$.

Proof. First of all, let us observe that the condition

$$\exists a \in \text{ dom } h_1 : \quad h_1(a) \in - \text{ int } Y_+,$$

may be transformed, in the product space $X \times Y$ by means the mapping $H : X \times Y \longrightarrow Y \cup \{+\infty\}$, into

$$(\text{C.Q.S}_1) \quad \exists a \in \text{ dom } h_1 : \quad H(a,0) \in \text{ int } Y_+,$$

which is indeed the Slater's condition linked to problem $(\mathcal{P}_3)$.

If $\bar{x}$ is a local minimum of problem $(\mathcal{P})$, then according to Proposition 4.1, $(\bar{x}, h_2(\bar{x}))$ is a local minimum of problem (P_3) and therefore it follows from Proposition 3.2 that

(i) If $(\bar{x}, h_2(\bar{x}))$ is a boundary point of the set $\{(x,y) \in X \times Y : H(x,y) \notin - \text{ int } Y_+\}$ which means according to Remark 4.1 that $h_1(\bar{x}) - h_2(\bar{x}) \in -Y_+$ and $h_1(\bar{x}) - h_2(\bar{x}) \notin - \text{ int } Y_+$, then for any $(x^*, p^*) \in \partial F_2(\bar{x}, h_2(\bar{x}))$ there exist some $(x_1^*, p_1^*) \in \partial F_1(\bar{x}, h_2(\bar{x})), y^* \in Y_+^*$ and $(x_2^*, p_2^*) \in \partial(y^* \circ H)(\bar{x}, h_2(\bar{x}))$ such that $x^* = x_!^* - x_2^*,\ p^* = p_1^* - p_2^*$ and $\langle y^*, H(\bar{x}, h_2(\bar{x}))\rangle = 0$. By virtue of Lemma 4.3, we get $p^* = 0,\ p_1^* = p_2^* = -y^*,\ x^* \in \partial f_2(\bar{x}),\ x_1^* \in \partial(f_1 + y^* \circ h_2)(\bar{x}),\ x_2^* \in \partial(y^* \circ h_1)(\bar{x})$ and $(y^* \circ h_1)(\bar{x}) = (y^* \circ h_2)(\bar{x})$.

(ii) If $(\bar{x}, h_2(\bar{x}))$ is a topological interior point of the set

$$\{(x,y) \in X \times Y : H(x,y) \notin - \text{ int } Y_+\},$$

i.e.,

$$h_1(\bar{x}) - h_2(\bar{x}) \notin -Y_+,$$

then from Proposition 3.2 we deduce that

$$\partial F_2(\bar{x}, h_2(\bar{x})) \subset \partial F_1(\bar{x}, h_2(\bar{x})).$$

In other words, we have

$$\partial f_2(\bar{x}) \times \{0\} \subset \bigcup_{z^* \in -Y_+^*} \partial(f_1 - z^* \circ h_2)(\bar{x}) \times \{-z^*\},$$

which yields

$$\partial f_2(\bar{x}) \subset f_1(\bar{x}).$$

□

5. Sufficient conditions associated with $(\mathcal{P})$

Before stating the sufficient conditions related to problem $(\mathcal{P})$, first we need to recall a recent result due to Laghdir [8] expressing the sufficient optimality conditions associated with problem $(\mathcal{P}_2)$ given by

Proposition 5.1. [8] *Suppose that* $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *are convex, proper and lower semicontinuous functions,* $h : X \longrightarrow Y \cup \{+\infty\}$ *is proper, continuous and* Y_+*-convex,* $\bar{x} \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$ *satisfying* $h(\bar{x}) \in -Y_+$ *and* $h(\bar{x}) \notin -\mathrm{int} Y_+$, *and the Slater condition* (C.Q.S) *is satisfied. If for any* $y^* \in Y_+^*$ *satisfying* $\langle y^*, h(\bar{x})\rangle = 0$ *and*

$$\partial_\epsilon f_2(\bar{x}) + \partial(y^* \circ h)(\bar{x}) \subset \partial_\epsilon f_1(\bar{x}), \quad \forall \epsilon > 0, \tag{5.1}$$

then $\bar{x}$ *is a global minimum of* $(\mathcal{P}_2)$. *Here*

$$\partial_\epsilon f(\bar{x}) := \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - \epsilon, \quad \forall x \in X\},$$

denotes the ϵ*-subdifferential of the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *at* $\bar{x}$.

Now, we are in a position to state sufficient conditions related to problem $(\mathcal{P})$.

Proposition 5.2. *Assume that* $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *are convex, proper and lower semicontinuous,* $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ *are proper, continuous and* Y_+*-convex,* $\bar{x} \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$ *satisfying* $h_1(\bar{x}) - h_2(\bar{x}) \in -Y_+$ *and* $h_1(\bar{x}) - h_2(\bar{x}) \notin -\mathrm{int} Y_+$, *and there exists some* $a \in \mathrm{dom} h_1$ *such that* $h_1(a) \in \mathrm{int} Y_+$. *If for any* $y^* \in Y_+^*$ *satisfying* $(y^* \circ h_1)(\bar{x}) = (y^* \circ h_2)(\bar{x})$ *and*

$$\partial_\epsilon f_2(\bar{x}) + \partial(y^* \circ h_1)(\bar{x}) \subset \partial_\epsilon (f_1 + y^* \circ h_2)(\bar{x}), \quad \forall \epsilon > 0, \tag{5.2}$$

then $\bar{x}$ *is global minimum of* $(\mathcal{P})$.

Proof. For obtaining our desired result, it suffices to check that problem $(\mathcal{P}_3)$ satisfies all assumptions of Proposition 5.1 and therefore we get that $(\bar{x}, h_2(\bar{x}))$ is a global minimum of problem $(\mathcal{P}_3)$ which asserts, thanks to Proposition 4.1, that $\bar{x}$ is a global minimum of problem (P). For this, let us note that the mapping $(x, y) \longrightarrow \delta_{-Y_+}(h_2(x) - y)$ is proper, convex and lower semicontinuous since its epigraph $Epi\ h_2 \times \mathbb{R}^+$ is nonempty, convex and closed. This allows to ensure that the function $(x, y) \longrightarrow F_1(x, y)$ is proper, convex and lower semicontinuous. Obviously, F_2 is proper, convex and lower semicontinous and H is proper, continuous and Y_+-convex. The condition $h_1(\bar{x}) - h_2(\bar{x}) \in -Y_+$ and $h_1(\bar{x}) - h_2(\bar{x}) \notin -\ \mathrm{int}\ Y_+$ means, by virtue of Remark 4.1, that $(\bar{x}, h_2(\bar{x}))$ is a boundary point of the set $\{(x, y) \in X \times Y : H(x, y) \notin -\ \mathrm{int}\ Y_+\}$. Notice also that the condition:

$$\exists a \in\ \mathrm{dom}\ h_1 : \quad h_1(a) \in -\ \mathrm{int}\ Y_+,$$

translates the Slater condition linked to problem $(\mathcal{P}_3)$, i.e., $H(a, 0) \in -\ \mathrm{int}\ Y_+$. Now, it remains to check that

$$\partial_\epsilon F_2(\bar{x}, h_2(\bar{x})) + \partial(y^* \circ H)(\bar{x}, h_2(\bar{x})) \subset \partial_\epsilon F_1(\bar{x}, h_2(\bar{x})), \quad \forall \epsilon > 0,$$

and this is obtained easily by combining conditions (5.1) and the following expressions obtained in a similar way as in Lemma 4.3:

$$\begin{aligned} \partial_\epsilon F_2(\bar{x}, h_2(\bar{x})) &= \partial_\epsilon f_2(\bar{x}) \times \{0\}, \\ \partial(y^* \circ H)(\bar{x}, h_2(\bar{x})) &= \partial(y^* \circ h_1)(\bar{x}) \times \{-y^*\}, \\ \partial_\epsilon F_1(\bar{x}, h_2(\bar{x})) &= \bigcup_{y^* \in -Y_+^*} \partial_\epsilon (f_1 - y^* \circ h_2)(\bar{x}) \times \{-y^*\}. \end{aligned}$$

This completes the proof. $\square$

Remark 5.1. In the case where $Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$ we have $Y_+^* = \mathbb{R}_+$ and $(\mathcal{P})$ becomes

$$\inf \{f_1(x) - f_2(x) : h_1(x) - h_2(x) \geq 0\}.$$

Keeping in mind that $\partial(\lambda h_i)(\bar{x}) = \lambda \partial h_i(\bar{x})$ $(i = 1, 2)$ for any $\lambda > 0$ and $\partial(0.h_i)(\bar{x}) = \{0\}$, by involving the following convention

$$(y^* \circ h_i)(x) := \begin{cases} y^*(h_i(x)) & \text{if } x \in \text{dom h}_\text{i}, \\ \sup\limits_{y \in Y} \langle y^*, y \rangle & \text{otherwise,} \end{cases}$$

we derive easily from Proposition 4.2 and Proposition 5.2 optimality conditions for the above scalar minimization problem.

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