ACTA MATHEMATICA VIETNAMICA
Volume 30, Number 2, 2005, pp. 153-168

SOME STRONG COMPARISON PRINCIPLES AND CONVERGENCE THEOREMS IN THE CAPACITY AND THE DIRICHLET PROBLEM IN THE CLASS $\mathcal{F}_p(h)$

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Abstract. The aim of this paper is to establish the strong comparison principle of Xing type ([Xi1], [Xi2]) for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$. As an application of the obtained results, we investigate the convergence in the capacity of the complex Monge-Ampère operator for the class $\mathcal{F}_p$ as well as solve the Dirichlet problem in the class $\mathcal{F}_p(h)$.

1. Introduction

After constructing the complex Monge-Ampère operator on the class of locally bounded plurisubharmonic functions Bedford and Taylor have proved the comparison principle for the class of bounded plurisubharmonic functions on a bounded domain Ω in $\mathbb{C}^n$ (see Theorem 4.1 in [Bed-Ta2]). Recently, after introducing and investigating many essential results for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$, Cegrell (see [Ce2]) established this principle for the class $\mathcal{F}_p$. However, in 1996 and 2000 Xing proved a stronger inequality than the comparison principle first for the class of bounded psh functions and next for psh functions in the class $\mathcal{B}$ (see [Xi1], [Xi 2]). In this paper we first prove the inequality of Xing type for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$. Next, we apply the obtained results to investigate the weak convergence in the capacity for the complex Monge-Ampère operator on the class $\mathcal{F}_p$ and to solve the Dirichlet problem in the class $\mathcal{F}_p(h)$.

2. Some notions

In this section we recall some definitions and results concerning the classes $\mathcal{E}_p$ and $\mathcal{F}_p$ introduced and investigated by Cegrell (see [Ce2], [Ce3]).

2.1. Let Ω be a hyperconvex domain in $\mathbb{C}^n$. By $\mathcal{E}_0 = \mathcal{E}_0(\Omega)$ we denote the class of negative and bounded psh functions φ on Ω such that $\lim\limits_{z\to\xi}\varphi(z) = 0\ \forall\xi \in \partial\Omega$ and $\int\limits_\Omega (dd^c\varphi)^n < \infty$.

For each $p \geq 1$, by $\mathcal{E}_p = \mathcal{E}_p(\Omega)$ we denote the class of psh functions φ on Ω such that there exists a sequence $\{\varphi_j\} \subset \mathcal{E}_0$ with $\varphi_j \downarrow \varphi,\ j \to \infty$, and

$$\sup_j \int\limits_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty.$$

Received October 22, 2004; in revised form July 1, 2005.

If also φ_j can be choosen so that $\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty$ then we say that $\varphi \in \mathcal{F}_p = \mathcal{F}_p(\Omega)$.

In [Ce2] Cegrell showed that $\mathcal{E}_0 \subset \mathcal{F}_p \subset \mathcal{E}_p$ and $\mathcal{F}_q \subset \mathcal{F}_p$ if $q > p$.

2.2. By Theorem 3.5 in [Ce2] it follows that the operator $(dd^c)^n$ is well-defined on the class $\mathcal{E}_p$. Moreover, Theorem 3.7 in [Ce2] says that if $\{u_j\} \subset \mathcal{E}_p$ and $u_j \uparrow u,\ j \to \infty$, then $u \in \mathcal{E}_p$ and $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$. Another result of Pesson showed that if $\{u_j\},\ u$ are in $\mathcal{E}_p$ and $u_j \downarrow u$ then $(dd^c u_j)^n \longrightarrow (dd^c u)^n$ weakly (see Corollary 3.8 in [Per]).

2.3. Next we deal with the comparison principle for the class $\mathcal{F}_p$. As in [Ce2] Cegrell proved that if $u, v \in \mathcal{F}_p$ and $u \leq v$ on Ω then

$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c v)^n.$$

Moreover, Lemma 4.4 in [Ce2] claims that if $u, v \in \mathcal{F}_p$ then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

From the above results it follows that the comparison principle is valid for the $\mathcal{F}_p$. Namely, if $u, v \in \mathcal{F}_p$ and $(dd^c u)^n \leq (dd^c v)^n$ then $u \geq v$ on Ω (see Theorem 4.5 [Ce2]).

2.4. Now we recall the notions about the convergence in C_n- capacity and the uniform absolute continuity in C_n- capacity and the uniform absolute continuity of a sequence of measures with respect to C_n in a domain Ω in $\mathbb{C}^n$.

Let C_n be the inner capacity given by Bedford-Taylor in [Be-Ta2], as defined by

$$C_n(E) = C_n(E, \Omega) = \sup \left\{ \int_E (dd^c u)^n : u \in PSH(\Omega), 0 < u < 1 \right\}$$

for any Borel subset E of Ω. A sequence of functions $\{u_j\}$ is said to converge to a function u in C_n- capacity on a set $E \subset \Omega$ if for each $\delta > 0$ we have

$$\lim_{j \to \infty} C_n(\{z \in E : |u_j(z) - u(z)| \geq \delta\}) = 0.$$

A sequence of positive Borel measures $\{\mu_j\}$ is said to be uniformly absolutely continuous with respect to C_n- capacity in Ω (briefly $\mu_j << C_n$ in Ω) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each Borel set $E \subset \Omega$ with $C_n(E) < \delta$ the inequality $\mu_j(E) < \varepsilon$ holds for all $j > 1$. For details concerning properties of C_n- capacity and the convergence in C_n- capacity as well as the uniform absolute continuity of a sequence of positive measures with respect to C_n- capacity we refer to the papers of Bedford-Taylor [Be-Ta2] and Xing [Xi2].

2.5. Now we deal with the classes $\mathcal{F}_p(h)$ and $\mathcal{E}_p(h)$ introduced and investigated in [Ce2]. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $h \in C(\partial\Omega)$. Put

$$U(0,h)(z) = \sup\left\{ v(z) : v \in PSH(\Omega) \cap L^{\infty}_{\text{loc}}(\Omega), \overline{\lim_{z\to\xi}} v(z) \leq h(\xi), \forall \xi \in \partial\Omega \right\}.$$

Then from [Ce-Ko] it follows that $U(0,h) \in PSH \cap L^{\infty}_{\text{loc}}(\Omega)$ and $\overline{\lim_{z\to\xi}} U(0,h)(z) \leq h(\xi),\ \forall \xi \in \partial\Omega$.

Now as in [Ce2] we consider functions $h \in C(\partial\Omega)$ such that $\lim_{z\to\xi} U(0,h)(z) = h(\xi),\ \forall \xi \in \partial\Omega$. For such functions we denote by $\mathcal{F}_p(h)$ (resp. $\mathcal{E}_p(h)$), $p \geq 1$, the class of plurisubharmonic functions u such that there exists $\varphi \in \mathcal{F}_p$ (resp. $\mathcal{E}_p$) with $U(0,h) \geq u \geq \varphi + U(0,h)$. By Theorem 7.2 in [Ce2] we know that $(dd^c.)^n$ is well-defined on $\mathcal{F}_p(h)$. A recent result of P.Ahag (see Theorem 4.11 in [Ah]) implies that $(dd^c.)^n$ is well-defined on $\mathcal{E}_p(h)$.

2.6. Finally we recall the class $\mathcal{E}$ introduced and investigated by Cegrell (see [Ce3]) recently. Let u be a negative psh function on a hyperconvex domain Ω. We say that $u \in \mathcal{E} = \mathcal{E}(\Omega)$ if for every $z_0 \in \Omega$ there exists a neighbourhood ω of z_0 in Ω and a decreasing sequence $h_j \in \mathcal{E}_0$ such that $h_j \downarrow u$ on ω and $\sup_j \int_\Omega (dd^c h_j)^n < \infty$.

In [Ce3] Cegrell showed that if $u \in \mathcal{E}$ then $(dd^c u)^n$ is well-defined and $\mathrm{PSH}^- \cap \mathrm{L}^{\infty}_{\text{loc}}(\Omega) \subset \mathcal{E}$ (see Definition 4.2 and the remark after Theorem 4.5 in [Ce3]).

3. The strong comparison principle for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$

As we say in the introduction of this paper, one of the main purposes of this paper is to establish the strong comparison principle for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$. First, the following result shows that the strong comparison principle holds for the class $\mathcal{E}_p$.

Theorem 3.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $u,\ v \in \mathcal{E}_p,\ p \geq 1$, with $\underline{\lim}_{z\to\xi} (u(z) - v(z)) \geq 0,\ \forall \xi \in \partial\Omega$. Then for all $r \geq 1$ and $w_j \in PSH(\Omega)$, $0 \leq w_j \leq 1,\ 1 \leq j \leq n$, the inequality*

$$\frac{1}{(n!)^2} \int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n \leq \int\limits_{\{u<v\}} (r-w_1)(dd^c u)^n$$

holds. Therefore, under the additional assumption $(dd^c v)^n \geq (dd^c u)^n$ in Ω we obtain that $u \geq v$ in Ω.

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{E}_p,\ p \geq 1$. Then $\lim_{c\to+\infty} c^n C_n\big(\{u < -c\}, \Omega\big) = 0$.*

Proof. Let $\mathcal{E}_0 \ni u_k \downarrow u$ be as in the definition of $\mathcal{E}_p$ satisfying the condition

$$\alpha = \sup_k \int_\Omega (-u_k)^p (dd^c u_k)^n < \infty.$$

Then for $c > 0$ we have

$$\{u_k < -c\} \uparrow \{u < -c\}$$

and

$$\{u < -c\} = \bigcup_{k \geq 1} \{u_k < -c\}.$$

Proposition 3.2 in [Be-Ta2] yields

$$C_n\big(\{u < -c\},\ \Omega\big) = \lim_{k \to \infty} C_n\big(\{u_k < -c\}, \Omega\big).$$

Let $w \in PSH(\Omega)$, $0 \leq w \leq 1$, be arbitrary. From Lemma 1 in [Xi1] we get the following estimations

$$\begin{aligned}
\int_{\{u_k<-c\}} (dd^c w)^n &\leq \int_{\{u_k<-c\}} (-1 - \frac{2u_k}{c})^n (dd^c w)^n \\
&\leq \frac{2^n}{c^n} \int_{\{u_k<-\frac{c}{2}\}} (-\frac{c}{2} - u_k)^n (dd^c w)^n \\
&\leq \frac{(n!)^2.2^n}{c^n} \int_{\{u_k<-\frac{c}{2}\}} (1 - w)(dd^c u_k)^n \\
&\leq \frac{(n!)^2.2^{n+p}}{c^{n+p}} \int_\Omega (-u_k)^p (dd^c u_k)^n \\
&\leq \frac{(n!)^2.2^{n+p}.\alpha}{c^{n+p}}.
\end{aligned}$$

Hence, for all $k \geq 1$,

$$C_n\big(\{u_k < -c\}\big) \leq \frac{(n!)^2.2^{n+p}.\alpha}{c^{n+p}}$$

and, consequently,

$$C_n\big(\{u < -c\}\big) = \lim_{k \to \infty} C_n\big(\{u_k < -c\}\big) \leq \frac{(n!)^2.2^{n+p}.\alpha}{c^{n+p}}$$

and the desired conclusion follows. $\square$

Lemma 3.2. *Let $u_j \in PSH(\Omega) \cap L^\infty(\Omega)$, $u_j \downarrow u$ on Ω, where $u \in \mathcal{E}$. Assume that $\lim_{s \to \infty} s^n C_n(\{u < -s\}) = 0$. Then $(dd^c u_j)^n$ is uniformly absolutely continuous with respect to C_n- capacity.*

Proof. Without loss of generality we may assume that $u_j \leq 0, \forall j \geq 1$. By [Ce3] for each $j \geq 1$, $\exists u_j^k \in \mathrm{PSH} \cap C(\bar{\Omega})$, $u_j^k \downarrow u_j$ as $k \to \infty$ and $u_j^k \mid_{\partial\Omega} = 0$. As in [Ce Ko Ze] for $s > 0$ put

$$\Omega_{kj}(s) = \{u_j^k < -s\},\ \Omega_j(s) = \{u_j < -s\},\quad \Omega(s) = \{u < -s\},$$
$$a_{kj}(s) = C_n(\Omega_{kj}(s)),\ a_j(s) = C_n(\Omega_j(s)),\quad a(s) = C_n(\Omega(s)),$$
$$b_{kj}(s) = \int\limits_{\Omega_{kj}(s)} (dd^c u_j^k)^n,\ b_j(s) = \int\limits_{\Omega_j(s)} (dd^c u_j)^n,\quad b(s) = \int\limits_{\Omega(s)} (dd^c u)^n.$$

For $0 < s < t$ we have $\max(u_j^k, -t) = u_j^k$ on $\{u_j^k > -t\}$, an open neighbourhood of $\partial\Omega_{kj}(s)$. Then

$$a_{kj}(s) \geq t^{-n} \int\limits_{\Omega_{kj}(s)} (dd^c \max(u_j^k, -t))^n = t^{-n} \int\limits_{\Omega_{kj}(s)} (dd^c u_j^k)^n,$$

where the second equality follows from Lemma 4.1 in [Ce2]. Now if t tends to s, we get

$$s^n a_{kj}(s) \geq \int\limits_{\Omega_{kj}(s)} (dd^c u_j^k)^n,\ \forall\ k, j \geq 1,\ \forall s > 0. \tag{1}$$

Given $\varepsilon > 0$. By the hypothesis there exists $s_0 > 0$ such that

$$s_0^n a(s_0) < \varepsilon. \tag{2}$$

Let $E \subset \Omega$ be a Borel set with $C_n(E) < \dfrac{\varepsilon}{s_0^n}$. Take an open neighbourhood G of E such that $C_n(G) < \frac{\varepsilon}{s_0^n}$. It follows that

$$\begin{aligned}
\int\limits_E (dd^c u_j)^n &\leq \int\limits_G (dd^c u_j)^n \leq \varliminf_k \int\limits_G (dd^c u_j^k)^n \\
&\leq \varliminf_k \left[\int\limits_{\Omega_{kj}(s_0)} (dd^c u_j^k)^n + \int\limits_{G \setminus \Omega_{kj}(s_0)} (dd^c u_j^k)^n \right] \\
&\leq \varliminf_k \left[s_0^n a_{kj}(s_0) + s_0^n C_n(G) \right] \leq s_0^n a(s_0) + \varepsilon < 2\varepsilon \quad \forall j \geq 1.
\end{aligned}$$

Hence, $(dd^c u_j)^n$ is uniformly absolutely continuous in Ω. □

Lemma 3.3. *Let $u \in \mathcal{E}_p$ and $u_j \in \mathcal{E}_0$, $u_j \downarrow u$ as in the definition of the class $\mathcal{E}_p$. Then for every bounded psh function ω on Ω the sequence $\{\omega(dd^c u_j)^n\}$ converges weakly to $\omega(dd^c u)^n$.*

Proof. Without loss of generality we may assume that $-1 \leq \omega \leq 0$ on Ω. Given $\varphi \in C_0(\Omega)$. We can assume that $\sup\{|\varphi(z)| : z \in \Omega\} \leq 1$. Since ω is quasi-continuous (see [Bed-Ta2]), from Lemma 3.2 it follows that for each $\varepsilon > 0$ there exists an open subset $G \subset \Omega$ such that ω is continuous on $F = \Omega \setminus G$ and

$$\sup_j \int\limits_G (dd^c u_j)^n < \varepsilon. \tag{3}$$

Take a continuous function h on Ω such that $h = \omega$ on F. Since $\{(dd^c u_j)^n\}$ converges weakly to $(dd^c u)^n$ (see Theorem 3.5 in [Ce2]) it follows that there exists j_0 such that for $j > j_0$ we have

$$\Big| \int\limits_\Omega \varphi h (dd^c u_j)^n - \int\limits_\Omega \varphi h (dd^c u)^n \Big| < \varepsilon.$$

On the other hand, since G is open, by (3) we have

$$\Big| \int\limits_G \varphi \omega (dd^c u)^n \Big| \le \int\limits_G (dd^c u)^n \le \underline{\lim}_j \int\limits_G (dd^c u_j)^n < \varepsilon.$$

Similarly,

$$\Big| \int\limits_G \varphi h (dd^c u)^n \Big| \le M \int\limits_G (dd^c u)^n \le M \underline{\lim}_j \int\limits_G (dd^c u_j)^n < M\varepsilon$$

where $M = \sup\{|h(z)| : z \in \operatorname{supp} \varphi\}$.

Because $h = \omega$ on F then for $j > j_0$ we have

$$\begin{aligned}
&\Big| \int\limits_\Omega \varphi \omega (dd^c u_j)^n - \int\limits_\Omega \varphi \omega (dd^c u)^n \Big| \le \Big| \int\limits_\Omega \varphi h (dd^c u_j)^n - \int\limits_\Omega \varphi h (dd^c u)^n \Big| + \\
&+ \Big| \int\limits_G \varphi \omega (dd^c u_j)^n \Big| + \Big| \int\limits_G \varphi \omega (dd^c u)^n \Big| + \Big| \int\limits_G \varphi h (dd^c u_j)^n \Big| + \Big| \int\limits_G \varphi h (dd^c u)^n \Big| \\
&< (2M+3)\varepsilon.
\end{aligned}$$

The lemma is proved. □

The next lemma is an extension of Lemma 4.3 in [Ce2].

Lemma 3.4. *Let $\omega \in \mathcal{E}_p$ and $\mathcal{E}_0 \ni u_j \downarrow \omega$ as in the definition of $\mathcal{E}_p$. If $u,\ v \in PSH(\Omega)$ and $\varphi \in PSH(\Omega)$, $0 \le \varphi \le 1$ and $r \ge 1$, then*

$$\int\limits_{\{u<v\}} (r - \varphi)(dd^c \omega)^n \le \underline{\lim}_j \int\limits_{\{u<v\}} (r - \varphi)(dd^c u_j)^n. \tag{4}$$

Proof. Let $\varepsilon > 0$ be given. Because of the quasi- continuity of u and v, repeating the arguments of Lemma 3.3 shows that there exist an open subset $G \subset \Omega$ and two continuous functions $\tilde{u}$ and $\tilde{v}$ on Ω such that

$$\{u \neq \tilde{u}\} \cup \{v \neq \tilde{v}\} \subset G \text{ and } \sup_j \int\limits_G (dd^c u_j)^n < \frac{\varepsilon}{r}. \tag{5}$$

Then $\{u<v\} \subset \{\tilde{u}<\tilde{v}\} \cup G \subset \{u<v\} \cup G$. Hence, from Lemma 3.3 and (5) it follows that

$$\begin{aligned}\int\limits_{\{u<v\}} (r-\varphi)(dd^c\omega)^n &\le \int\limits_{\{\tilde{u}<\tilde{v}\}\cup G} (r-\varphi)(dd^c\omega)^n \\ &\le \underline{\lim}_j \int\limits_{\{\tilde{u}<\tilde{v}\}\cup G} (r-\varphi)(dd^c u_j)^n \\ &\le \underline{\lim}_j \int\limits_{\{u<v\}\cup G} (r-\varphi)(dd^c u_j)^n \\ &\le \underline{\lim}_j \int\limits_{\{u<v\}} (r-\varphi)(dd^c u_j)^n + \varepsilon.\end{aligned}$$

Now, if we let ε tend to zero and the desired conclusion follows. □

Proof of Theorem 3.1. Instead of u we consider $u+2\delta,\ \delta>0$, and notice that $\{u+2\delta<v\} \uparrow \{u<v\}$ as $\delta \downarrow 0$. Then we may assume that $\underline{\lim}_{z\to\partial\Omega}(u(z)-v(z)) \ge 2\delta$ on $\partial\Omega$. Thus $\{u<v+\delta\} \Subset \Omega$. Let $\mathcal{E}_0 \ni u_k \downarrow u$ and $\mathcal{E}_0 \ni v_j \downarrow v$ as in the definition of $\mathcal{E}_p$. Using Lemma 1 in [Xi1] we have

$$\begin{aligned}&\frac{1}{(n!)^2}\int\limits_{\{u_k<v_j\}} (v_j-u_k)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u_k<v_j\}} (r-w_1)(dd^c v_j)^n \\ &\le \int\limits_{\{u_k<v_j\}} (r-w_1)(dd^c u_k)^n.\end{aligned}$$

Since $\{u_k<v_j\}_{j\ge1}$ decreases to $\bigcap_{j=1}^{\infty}\{u_k<v_j\} \supset \{u_k<v\}$, by Fatou lemma and Lemma 3.4 it follows that

$$\begin{aligned}&\frac{1}{(n!)^2}\int\limits_{\{u_k<v\}} (v-u_k)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u_k<v\}} (r-w_1)(dd^c v)^n \\ &\le \underline{\lim}_j \Big[\frac{1}{(n!)^2}\int\limits_{\{u_k<v_j\}} (v_j-u_k)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u_k<v_j\}} (r-w_1)(dd^c v_j)^n\Big] \\ &\le \underline{\lim}_j \int\limits_{\{u_k<v_j\}} (r-w_1)(dd^c u_k)^n \\ (6)\quad &= \int\limits_{\{u_k\le v\}} (r-w_1)(dd^c u_k)^n\end{aligned}$$

for all $k \geq 1$. By applying the Lebesgue monotone convergence theorem to the two sides of (6) we obtain the inequality

$$\begin{aligned}
&\frac{1}{(n!)^2}\int\limits_{\{u<v\}}(v-u)^n dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^n\\
&\leq \overline{\lim_k}\int\limits_{\{u_k\leq v\}}(r-w_1)(dd^c u_k)^n \qquad (7)\\
&\leq \overline{\lim_k}\int\limits_{\{u\leq v\}}(r-w_1)(dd^c u_k)^n.
\end{aligned}$$

Now let $\varepsilon > 0$ be given. Take an open subset $G \subset \Omega$ with $\sup\limits_k \int\limits_G (dd^c u_k)^n < \varepsilon$ and $u,\ v$ continuous on $F = \Omega \setminus G$ as in Lemma 3.4. From the weak convergence of $\{(r - w_1)(dd^c u_k)^n\}$ to $(r - w_1)(dd^c u)^n$ and the compactness of $\{u \leq v\} \cap F$ it follows that

$$\begin{aligned}
&\frac{1}{(n!)^2}\int\limits_{\{u<v\}}(v-u)^n dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^n\\
&\leq \overline{\lim_k}\int\limits_{\{u\leq v\}\cap F}(r-w_1)(dd^c u_k)^n+r\varepsilon \qquad (8)\\
&\leq \int\limits_{\{u\leq v\}}(r-w_1)(dd^c u)^n+r\varepsilon.
\end{aligned}$$

Then the inequality

$$\begin{aligned}
&\frac{1}{(n!)^2}\int\limits_{\{u<v\}}(v-u)^n dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^n \qquad (9)\\
&\leq \int\limits_{\{u\leq v\}}(r-w_1)(dd^c u)^n
\end{aligned}$$

holds if in (8) ε tends to 0. Theorem 3.1 follows if we apply (9) to $\lambda v,\ \lambda > 1$ and notice that $\{u < \lambda v\} \uparrow \{u < v\}$ and $\{u \leq \lambda v\} \uparrow \{u < v\}$ as $\lambda \downarrow 1$. □

Similarly we get the following.

Theorem 3.2. *Let $u \in \mathcal{E}_p, p \geq 1$ and $v \in PSH^-(\Omega)\cap L^\infty(\Omega)$ satisfying $\underline{\lim}_{z\to\partial\Omega}(u(z)-v(z)) \geq 0$. Then the inequality*

$$\frac{1}{(n!)^2}\int\limits_{\{u<v\}}(v-u)^n dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^n\leq \int\limits_{\{u<v\}}(r-w_1)(dd^c u)^n$$

holds for all $r \geq 1$ *and* $w_1,\ ..., w_n \in PSH(\Omega),\ 0 \leq w_j \leq 1,\ j = \overline{1, n}$.

Next we present the strong comparison principle for the class $\mathcal{F}_p,\ p \geq 1$. Note that in Theorems 3.1 and 3.2 the strong comparison principle holds for the class $\mathcal{E}_p,\ p \geq 1$, when u and v have to satisfy the condition $\varliminf_{z \to \partial\Omega} (u(z) - v(z)) \geq 0$. However, in contrast to the class $\mathcal{E}_p$ the above condition is superfluous for the class $\mathcal{F}_p$. Namely we prove the following result.

Theorem 3.3. *Let* Ω *be a bounded hyperconvex domain in* $\mathbb{C}^n$ *and* $u,\ v \in \mathcal{F}_p,\ p \geq 1$. *Then for all* $r \geq 1$ *and* $w_j \in PSH(\Omega),\ 0 \leq w_j \leq 1,\ 1 \leq j \leq n$, *the inequality*

$$\frac{1}{(n!)^2} \int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n \leq \int\limits_{\{u<v\}} (r-w_1)(dd^c u)^n$$

holds.

Proof. In the same notations as in the proof of Theorem 3.1 we get the inequality

$$\frac{1}{(n!)^2} \int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n$$

$$\leq \varlimsup_k \int\limits_{\{u \leq v\}} (r-w_1)(dd^c u_k)^n \tag{10}$$

and there exists an open subset $G \subset \Omega$ such that $\sup_k \int_G (dd^c u_k)^n < \varepsilon$ and u, v are continuous on $F = \Omega \backslash G$ where $\varepsilon > 0$ is given. Assume that g is any non-negative and continuous function which is bounded by 1 on Ω and there exists a domain $\Omega_0 \Subset \Omega$ such that $g = 1$ on $\Omega \setminus \overline{\Omega_0}$. Then we infer that

$$\begin{aligned}
&\varlimsup_k \int\limits_{\{u \leq v\}} (r-w_1)(dd^c u_k)^n \\
&= \varlimsup_k \Big(\int\limits_{\{u \leq v\} \cap F} (r-w_1)(dd^c u_k)^n + \int\limits_{\{u \leq v\} \cap G} (r-w_1)(dd^c u_k)^n \Big) \\
&\leq \varlimsup_k \int\limits_{\{u \leq v\} \cap F} (r-w_1)(dd^c u_k)^n + r\varepsilon \\
&\leq \varlimsup_k \Big(\int\limits_{\{u \leq v\} \cap F} (1-g)(r-w_1)(dd^c u_k)^n + \int\limits_{\{u \leq v\} \cap F} g(r-w_1)(dd^c u_k)^n \Big) + r\varepsilon \\
&\leq \varlimsup_k \int\limits_{\{u \leq v\} \cap F \cap \overline{\Omega_0}} (r-w_1)(dd^c u_k)^n
\end{aligned}$$

$$+\overline{\lim_k}\Big(r\int\limits_\Omega (g-1)(dd^cu_k)^n + r\int\limits_\Omega (dd^cu_k)^n\Big) + r\varepsilon. \tag{11}$$

However, since $u_k \geq u$ on Ω and $u_k,\ u \in \mathcal{F}_p$, from Lemma 4.2 in [Ce1] it follows that for all $k \geq 1$,

$$\int\limits_\Omega (dd^cu_k)^n \leq \int\limits_\Omega (dd^cu)^n. \tag{12}$$

Combining (12) with (11), from the compactness of $\{u \leq v\} \cap F \cap \overline{\Omega_0}$, Lemma 3.3 and $g-1 \in C_0(\Omega)$ it follows that the right-side of (10) does not exceed

$$\begin{aligned}
&\int\limits_{\{u\leq v\}} (r-w_1)(dd^cu)^n + r\int\limits_\Omega (g-1)(dd^cu)^n + r\int\limits_\Omega (dd^cu)^n + r\varepsilon\\
&= \int\limits_{\{u\leq v\}} (r-w_1)(dd^cu)^n + r\int\limits_\Omega g(dd^cu)^n + r\varepsilon.
\end{aligned} \tag{13}$$

From (13) and (10) we get the inequality

$$\begin{aligned}
&\frac{1}{(n!)^2}\int\limits_{\{u<v\}} (v-u)^n dd^cw_1\wedge ...\wedge dd^cw_n + \int\limits_{\{u<v\}} (r-w_1)(dd^cv)^n\\
&\leq \int\limits_{\{u\leq v\}} (r-w_1)(dd^cu)^n + r\int\limits_\Omega g(dd^cu)^n + r\varepsilon.
\end{aligned}$$

To complete the proof of the Theorem 3.3 we let g and ε tend to 0 and use the same argument as in the proof of Theorem 3.1. □

Repeating the proof of Theorem 3.3 we obtain the following result.

Theorem 3.4. *Let $u \in \mathcal{F}_p$ and $v \in PSH^-(\Omega) \cap L^\infty(\Omega)$. Then for all $r \geq 1$ and $w_1, ..., w_n \in PSH(\Omega), 0 \leq w_j \leq 1,\ j = \overline{1, n}$, the inequality*

$$\frac{1}{(n!)^2}\int\limits_{\{u<v\}} (v-u)^n dd^cw_1\wedge ...\wedge dd^cw_n + \int\limits_{\{u<v\}} (r-w_1)(dd^cv)^n \leq \int\limits_{\{u<v\}} (r-w_1)(dd^cu)^n$$

holds.

4. The weak continuity of the complex Monge-Ampère operator in the class $\mathcal{F}_p$

The aim of this section is to apply the results of the above section to the investigation of the weak continuity of the complex Monge-Ampère operator in the class $\mathcal{F}_p$. Namely we prove the following.

Theorem 4.1. *Let* $\{u_j\},\ u$ *be in* $\mathcal{F}_p,\ p \geq 1$ *and* $u_j \longrightarrow u$ *in the* C_n*-capacity on every compact set of* Ω*. Assume that*

$$\lim_{j\to\infty} C_n\Big(\{z \in \Omega : \Big|u_j(z) - u(z)\Big| \geq \alpha\}\Big) = 0$$

for some $\alpha > 0$ *and* $(dd^c u_j)^n$ *is uniformly absolutely continuous with respect to the* C_n*- capacity in* Ω*. Then* $(dd^c u_j)^n$ *converges weakly to* $(dd^c u)^n$ *and* $(dd^c u)^n \ll C_n$ *in* Ω*.*

Proof. Given $\Phi \in C_0(\Omega)$, we may assume that

$$\|\Phi\| = \sup\{|\Phi(z)| : z \in \Omega\} \leq 1.$$

To see that $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$ we need to show that

$$A = \int_\Omega \Phi\big[(dd^c u_j)^n - (dd^c u)^n\big] \longrightarrow 0 \text{ as } \ j \to \infty.$$

Given $\varepsilon > 0$. By the hypothesis there exists $\delta > 0$ such that

$$\int\limits_E (dd^c u_j)^n < \frac{\varepsilon}{1 + 2^n (n!)^2} \tag{14}$$

for all $E \subset \Omega$ with $C_n(E) < \delta$ and $j \geq 1$.

For each $c > 0$ as in [Xi2] we write $A = A_1 + A_2 + A_3$ where

$$\begin{aligned}
A_1 &= \int_\Omega \Phi\big[(dd^c u_j)^n - (dd^c \max(u_j, -c))^n\big],\\
A_2 &= \int_\Omega \Phi\big[(dd^c \max(u_j, -c))^n - (dd^c \max(u, -c))^n\big],\\
A_3 &= \int_\Omega \Phi\big[(dd^c \max(u, -c))^n - (dd^c u)^n\big].
\end{aligned}$$

Since $\max(u, -c) \in \mathcal{F}_p$ and $\max(u, -c) \downarrow u$ as $c \to +\infty$, by Corollary 3.8 in [Per] we can find $c_0 > 0$ such that $|A_3| < \varepsilon$ for $c > c_0$.

Consider A_1. By Lemma 5.4 in [Ce2] we infer that

$$|A_1| \leq \int\limits_{\{u_j \leq -c\}} (dd^c u_j)^n + \int\limits_{\{u_j \leq -c\}} (dd^c \max(u_j, -c))^n.$$

Applying Theorem 3.4 we get

$$\begin{aligned}\int\limits_{\{u_j\le -c\}}(dd^c\max(u_j,-c))^n &\le \int\limits_{\{u_j\le -c\}}(-1-\frac{2u_j}{c})^n(dd^c\max(u_j,-c))^n\\ &\le 2^n\int\limits_{\{u_j<-\frac{c}{2}\}}(-\frac{c}{2}-u_j)^n(dd^c\max(\frac{u_j}{2},-1))^n\\ &\le 2^n(n!)^2\int\limits_{\{u_j<-\frac{c}{2}\}}(dd^cu_j)^n.\end{aligned}$$

Hence,

$$|A_1|<\big(1+2^n(n!)^2\big)\int\limits_{\{u_j<-\frac{c}{2}\}}(dd^cu_j)^n.$$

From Lemma 3.1 it follows that $\lim\limits_{c\to+\infty}C_n\big(\{u<-\frac{c}{4}\}\big)=0$, hence we may assume that for $c>c_0$,

$$C_n(\{u<-\frac{c}{4}\})<\frac{\delta}{2}.$$

Since $\lim\limits_{j\to\infty}C_n\big(\{|u_j-u|\ge\alpha\}\big)=0$, there exists j_0 such that for $j>j_0$ we have

$$C_n\big(\{|u_j-u|\ge\alpha\}\big)<\frac{\delta}{2}.$$

Take $c_1>4(c_0+\alpha)$. Then

$$\{|u_j-u|>\frac{c_1}{4}\}\subset\{|u_j-u|\ge\alpha\}$$

and, consequently, for $j>j_0$ we have

$$C_n\big(\{|u_j-u|>\frac{c_1}{4}\}\big)<\frac{\delta}{2}.$$

Hence, for $j>j_0$ we get

$$C_n(\{u_j<-\frac{c_1}{2}\})<\delta. \tag{15}$$

From the hypothesis on the uniformly absolute continuity of $(dd^cu_j)^n$ with respect to C_n-capacity and (14), (15) it follows that $|A_1|<\varepsilon$ for $j>j_0$. But since the inclusion

$$\Big\{\Big|\max(u_j,-c)-\max(u,-c)\Big|>\beta\Big\}\subset\{|u_j-u|>\beta\}$$

holds for all $\beta>0$, $\max(u_j,-c)\longrightarrow\max(u,-c)$ in the C_n-capacity on every compact set of Ω. Hence, by [Xi1] $|A_2|<\varepsilon$ for $j>j_1>j_0$ and, consequently, $|A|<3\varepsilon$ for $j>j_0$. It remains to show that $(dd^cu)^n\ll C_n$ in Ω.

Given $\varepsilon > 0$. By the hypothesis there exists $\delta > 0$ such that for all $E \subset \Omega$, $C_n(E) < \delta$ and all $j \geq 1$, $\int\limits_E (dd^c u_j)^n < \varepsilon$.

Assume that E is a Borel subset of Ω with $C_n(E) < \delta$. Take an open set $G \subset \Omega$, $E \subset G$ with $C_n(G) < \delta$. Then

$$\int\limits_E (dd^c u)^n \leq \int\limits_G (dd^c u)^n \leq \varliminf_j \int\limits_G (dd^c u_j)^n < \varepsilon$$

and hence, $(dd^c u)^n \ll C_n$ in Ω. Theorem 4.1 is proved. □

5. The Dirichlet problem for the class $\mathcal{F}_p(h)$

In this section we are interested in the following Dirichlet problem in the class $\mathcal{F}_p(h)$. Suppose that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, $h \in C(\partial\Omega)$ and μ is a positive Borel measure on Ω. Find a psh function u on Ω such that

$$(*) \qquad \begin{cases} (dd^c u)^n = \mu \\ \varlimsup\limits_{z \to \xi} u(z) = h(\xi), \ \forall \xi \in \partial\Omega. \end{cases}$$

In the case Ω is a strictly pseudoconvex domain, Bedford and Taylor (see [Be-Ta1]) showed that if $\mu = f d\lambda$, $0 \leq f \in C(\bar{\Omega})$, $d\lambda$ is the Lebesgue measure in $\mathbb{C}^n$, then (*) has an unique solution $u \in PSH(\Omega) \cap C(\bar{\Omega})$. This was extended in [Ce1] as follows. If $\mu = f d\lambda$, $0 \leq f \in L^\infty(\Omega)$, then (*) has an unique solution $u \in PSH(\Omega) \cap L^\infty(\Omega)$. Next in [Ce-Sa] they have shown that if $\mu = f d\lambda$, $0 \leq f \in L^\infty_{\text{loc}}(\Omega)$ and there exists a function $w \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $f d\lambda \leq (dd^c w)^n$, then (*) has a solution $u \in PSH(\Omega) \cap L^\infty(\Omega)$. Here, by relying on some recent results concerning with the class $\mathcal{F}_p(h)$ in [Ce3] and [Ah] we solve (*) in the class $\mathcal{F}_p(h)$. More precisely we prove the following

Theorem 5.1. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, $f \in L^1(\Omega)$ and $h \in C(\partial\Omega)$ such that $\lim\limits_{z \to \xi} U(0,h)(z) = h(\xi)$ for all $\xi \in \partial\Omega$. Assume that $f d\lambda \leq (dd^c v)^n$ for some $v \in \mathcal{F}_p(h)$, $p \geq 1$. Then there exists $u \in \mathcal{F}_p(h)$ such that $(dd^c u)^n = f d\lambda$.*

Proof. Without loss of generality we may assume that $h \leq 0$. Take an increasing sequence of simple functions $f_k \uparrow f$. By [Ce1], for each $k \geq 1$ there exists $u_k \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $(dd^c u_k)^n = f_k d\lambda$ and $\lim\limits_{z \to \xi} u_k(z) = h(\xi)$ for all $\xi \in \partial\Omega$. By the comparison principle in [Be-Ta2] it follows that $u_k \geq u_{k+1}$ on Ω for $k \geq 1$. Set $u(z) = \lim\limits_{k \to \infty} u_k(z)$, $z \in \Omega$. First we show that $u \in \mathcal{F}_p(h)$. Since $v \in \mathcal{F}_p(h)$, it follows that there exists $\varphi \in \mathcal{F}_p$ such that $U(0,h) \geq v \geq \varphi + U(0,h)$.

On the other hand, since $\varphi \in \mathcal{F}_p$, there exists a sequence of continuous psh functions $\varphi_j \in \mathcal{E}_0$, $\varphi_j \downarrow \varphi$. Let $p(z) = \dfrac{||z||^2}{4}$, $z \in \mathbb{C}^n$. Then $(dd^c p)^n = n! d\lambda$. Choose $\varepsilon > 0$ and $\delta > 0$ such that $v_{\varepsilon\delta} < v$ on $\bar{\Omega}$, where $v_{\varepsilon\delta} = v + \varepsilon p - \delta$. Next, for $j \geq 1$ put

$$v_j = \max(v, \varphi_j + U(0,h)) + \varepsilon p - \delta \in \mathrm{PSH} \cap L^\infty(\Omega)$$

and $v_j \downarrow v_{\varepsilon\delta}$. We prove that $\lim\limits_{s\to\infty} s^n C_n(\{v_{\varepsilon\delta} < -s\}) = 0$. Indeed, let $M = \sup\limits_{z\in\bar{\Omega}} p(z)$. Then

$$\{v_{\varepsilon\delta} < -s\} = \{v + \varepsilon p < \delta - s\} \subset \{v < \delta - s - \varepsilon M\}.$$

Hence, it remains to show that

$$\lim_{s\to\infty} s^n C_n(\{v < -s\}) = 0.$$

Since $\varphi + U(0,h) \leq v$ we get

$$\{v < -s\} \subset \{\varphi + U(0,h) < -s\}.$$

Therefore,

$$s^n C_n(\{\varphi + U(0,h) < -s\}) \leq s^n C_n\{\varphi < -\frac{s}{2}\} + s^n C_n\{U(0,h) < -\frac{s}{2}\}.$$

Since $\varphi \in \mathcal{E}_p$ and Lemma 3.1 implies that

$$\lim_{s\to\infty} s^n C_n(\{\varphi < -\frac{s}{2}\}) = 0.$$

Notice that because $h \in C(\partial\Omega)$, $U(0,h) \in C(\bar{\Omega})$ by [Wa]. Hence for sufficiently large $s > 0$ the set $\{U(0,h) < -s\} = \emptyset$. Thus

$$\lim_{s\to\infty} s^n C_n(\{U(0,h) < -s\}) = 0.$$

Now by Lemma 3.2 we have $\left(dd^c v_j\right)^n << C_n$ in Ω uniformly for $j \geq 1$. Since $\varliminf\limits_{z\to\partial\Omega} (u_k(z) - v_{\varepsilon\delta}(z)) \geq 0$ (we choose ε and δ sufficiently small so that $\varepsilon M - \delta \leq 0$), using the arguments of the proof of the comparison principle (see Theorem 4.1 in [Be -Ta2]) we get

$$\begin{aligned}\int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c v)^n &\leq \int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c v)^n + \int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c(\varepsilon p - \delta))^n \\ &\leq \int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c v_{\varepsilon\delta})^n \leq \int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c u_k)^n \leq \int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c v).\end{aligned}$$

Hence $\int\limits_{\{u_k<v_{\varepsilon\delta}\}} (dd^c p)^n = 0$. This shows that $v_{\varepsilon\delta} \leq u_k$ for all $k \geq 1$. Letting $k \to +\infty$ and $\varepsilon,\ \delta \downarrow 0$ we obtain that $\varphi + U(0,h) \leq u \leq U(0,h)$. Thus $u \in \mathcal{F}_p(h)$. Since $\mathcal{F}_p(h) \subset \mathcal{E}_p(h)$ and $h \leq 0$, Lemma 4.9 in [Ah] implies that $u \in \mathcal{E}$. On the other hand, $0 \geq u_k \in \mathrm{PSH} \cap \mathrm{L}^\infty(\Omega)$, $u_k \downarrow u$, $u \in \mathcal{E}$. Hence Theorem 4.5 in [Ce3] implies that $(dd^c u_k)^n$ converges weakly to $(dd^c u)^n$. Hence $(dd^c u)^n = f d\lambda$. Theorem 5.1 is completely proved. $\square$

Remark 5.1. There exists $f \in L^1(\Omega)$ such that $fd\lambda$ is not a complex Monge-Ampère measure $(dd^c u)^n$ for any $u \in \mathcal{E}_1$. Indeed, take a sequence $\{z_j\}$ of distinguished points in Ω converging to $\xi \in \partial\Omega$. Then we can find $r_j \downarrow 0$ such that $\mathbb{B}(z_j, r_j)$ are pairwise disjoint and $j^{2(n+1)} C_n(\mathbb{B}(z_j, r_j)) \longrightarrow 0$ as $j \to \infty$. Consider the intergrable function f on Ω given by

$$f = \sum_{j=1}^{\infty} \frac{1}{d_n r_j^{2n} j^2} \chi_{\mathbb{B}(z_j, r_j)},$$

where d_n is the volume of the unit ball in $\mathbb{C}^n$. Assume that there exists $u \in \mathcal{E}_1$ such that $fd\lambda = (dd^c u)^n$. Take a sequence $\mathcal{E}_0 \ni u_k \downarrow u$ as the definition of $\mathcal{E}_1$. By Lemma 3.3, $\{-\varphi(dd^c u_k)^n\} \longrightarrow (-\varphi)(dd^c u)^n$ weakly for $\varphi \in \mathcal{E}_0(\Omega)$. Theorem 4.2 in [Ce2] implies that

$$\int (-\varphi)(dd^c u)^n \le \varliminf_k \int (-\varphi)(dd^c u_k)^n \le A\Big(\int (-\varphi)(dd^c \varphi)^n\Big)^{\frac{1}{n+1}}, \tag{16}$$

where

$$A = D_{0,1} \sup_k \Big(\int (-u_k)(dd^c u_k)^n\Big)^{\frac{1}{n+1}} < \infty.$$

Applying (16) to $\varphi = h_{\overline{\mathbb{B}}(z_j,r_j)}$, where $h_{\overline{\mathbb{B}}(z_j,r_j)}$ is the relatively extremal function with respect to $\overline{\mathbb{B}}(z_j, r_j)$, we get the following inequalities

$$\begin{aligned}
\frac{1}{j^2} = \int\limits_{\overline{\mathbb{B}}(z_j,r_j)} fd\lambda &= \int\limits_{\overline{\mathbb{B}}(z_j,r_j)} -h_{\overline{\mathbb{B}}(z_j,r_j)} fd\lambda \le \int\limits_{\Omega} -h_{\overline{\mathbb{B}}(z_j,r_j)} fd\lambda \\
&\le A\Big(\int\limits_{\Omega} -h_{\overline{\mathbb{B}}(z_j,r_j)} (dd^c h_{\overline{\mathbb{B}}(z_j,r_j)})^n\Big)^{\frac{1}{n+1}} \\
&\le A\Big(\int\limits_{\overline{\mathbb{B}}(z_j,r_j)} (dd^c h_{\overline{\mathbb{B}}(z_j,r_j)})^n\Big)^{\frac{1}{n+1}} = AC_n(\overline{\mathbb{B}}(z_j, r_j)\Big)^{\frac{1}{n+1}}.
\end{aligned}$$

Hence

$$\varliminf_j j^2 C_n(\overline{\mathbb{B}}(z_j, r_j))^{\frac{1}{n+1}} \ge \frac{1}{A} > 0.$$

We reach a contradiction because $j^{2(n+1)} C_n(\overline{\mathbb{B}}(z_j, r_j)) \to 0$.

Remark 5.2. In [Ce2, Theorem 7.7], under the assumption that Ω is a smoothly bounded, strictly pseudoconvex domain in $\mathbb{C}^n$, $n \ge 2$, $p \ge 1$, μ is a positive measure on Ω with finite mass and $h \in C^\infty(\partial\Omega)$, Cegrell has shown that $\mu = (dd^c u)^n$ for some $u \in \mathcal{F}_p(h)$ if and only if there is a constant A such that

$$\int\limits_{\Omega} (-\varphi)^p d\mu \le A\Big(\int\limits_{\Omega} (-\varphi)^p (dd^c \varphi)^n\Big)^{\frac{p}{n+p}}, \quad \forall \ \ \varphi \in \mathcal{E}_0.$$

In the proof of the above result of Cegrell the hypothesis $h \in C^\infty(\partial\Omega)$ is an essential condition because under this hypothesis the function $U(0, -h) + U(0, h) \in \mathcal{E}_0$

and the arguments in the proof of the author is suitable. However, in Theorem 5.1 above we give a weaker hypothesis that $h \in \mathrm{C}(\partial\Omega)$ and hence we obtain a weaker result than Theorem 7.7 in [Ce2].

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