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ON THE ROBUSTNESS OF ASYMPTOTIC STABILITY FOR A CLASS OF SINGULARLY PERTURBED SYSTEMS WITH MULTIPLE DELAYS

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Abstract. This paper is concerned with the stability robustness of a class of singularly perturbed systems of linear functional differential equations. First, the stability radius for the reduced systems is proposed. Then, asymptotic behavior of the structured complex stability radius for the singularly perturbed systems is established as the small parameter tends to zero.

1. Introduction

In this paper we consider the singularly perturbed system (SPS) of functional differential equations (FDE-s)

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + L_{12}y_t \\ \varepsilon\dot{y}(t) &= L_{21}x_t + L_{22}y_t \end{aligned} \tag{1.1}$$

where $x \in \mathbb{C}^{n_1},\ y \in \mathbb{C}^{n_2},\ \varepsilon > 0$ is a small parameter;

$$L_{j1}x_t = \sum_{i=0}^{l} A_{j1}^i x(t-\tau_i) + \int_{-\tau_l}^{0} D_{j1}(\theta)x(t+\theta)d\theta \tag{1.2}$$

$$L_{j2}y_t = \sum_{k=0}^{m} A_{j2}^k y(t-\varepsilon\mu_k) + \int_{-\mu_m}^{0} D_{j2}(\theta)y(t+\varepsilon\theta)d\theta$$

$j = 1,2,\ A_{jk}^i$ are constant matrices of appropriate dimensions, $D_{jk}(.)$ are integrable matrix-valued functions, and $0 \le \tau_0 \le \tau_1 \le ... \le \tau_p,\ 0 \le \mu_0 \le \mu_1 \le ... \le \mu_m$.

A lot of problems arising in various fields of science and engineering can be modelled by SPS-s of differential equations with or without delay, e.g., see [8] and the references cited therein. The system (1.1) was analyzed by Dragan and Ionita in [2]. By extending the classical results of Klimusev and Krasovskii, e.g., see [14], the authors gave a parameter-independent sufficient condition ensuring the exponential-asymptotic stability of the zero solution of (1.1) for all sufficiently

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small ε. For characterizing the robustness of asymptotic stability for linear systems, appropriate measure is the so-called stability radii introduced by Hinrichsen and Pritchard [10, 11, 12]. A formula of the complex structured stability radius for linear systems was obtained in [11]. The result was extended to linear functional systems by Son and Ngoc [17]. The real stability radius for linear systems, which is a more difficult issue, was investigated in a remarkable paper of Qiu et.al. [15]. Recently, this result was extended to linear time-delay systems [13]. See also a fairly complete list of references on the topic in [1]. In this paper, we focus on the complex stability radius.

Let us assume that the system (1.1),(1.2) is asymptotically stable for all sufficiently small ε. We consider the system (1.1) with the coefficients subjected to structured perturbations as follows:

$$
\begin{aligned}
\tilde{L}_{j1}x_t =& \sum_{i=0}^{l}(A_{j1}^{i} + B_j\Delta_1^{i}C_1^{i})x(t-\tau_i) + \\
& \int_{-\tau_l}^{0}(D_{j1}(\theta) + B_j\delta_1(\theta)C_1^{l+1})x(t+\theta)d\theta \\
\tilde{L}_{j2}y_t =& \sum_{k=0}^{m}(A_{j2}^{k} + B_j\Delta_2^{k}C_2^{k})y(t-\varepsilon\mu_k) + \\
& \int_{-\mu_m}^{0}(D_{j2}(\theta) + B_j\delta_2(\theta)C_2^{m+1})y(t+\varepsilon\theta)d\theta,
\end{aligned}
\tag{1.3}
$$

where

$$
\begin{gathered}
\{\Delta_1^{i}\}_{i=0}^{l} \in \mathbb{C}^{p_1\times q_{1i}}, \{\Delta_2^{k}\}_{k=0}^{m} \in \mathbb{C}^{p_2\times q_{2k}}, \\
\delta_1(\theta) \in \mathbb{C}^{p_1\times q_{1(l+1)}}, \delta_2(\theta) \in \mathbb{C}^{p_2\times q_{2(m+1)}}
\end{gathered}
$$

are uncertain perturbations, $\delta_1(.), \delta_2(.)$ are integrable matrix-valued functions on the indicated intervals; $B_j \in \mathbb{C}^{n_j\times p_j}, j = 1,2;$ $C_1^{i} \in \mathbb{C}^{q_{1i}\times n_1}, i = 0,1,...,l+1; C_2^{k} \in \mathbb{C}^{q_{2k}\times n_2}, k = 0,1,...,m+1$ are sets of matrices determining the perturbation structure. For brevity, let us denote

$$
\begin{aligned}
\mathbf{A} &= \left\{\{A_{j1}^{i}\}_{i=0}^{l}, \{A_{j2}^{k}\}_{k=0}^{m}, \{D_{ij}(.)\}_{i,j=1}^{2}\right\}, \\
\mathbf{B} &= \{B_1, B_2\}, \\
\mathbf{C} &= \left\{\{C_1^{i}\}_{i=0}^{l}, \{C_2^{k}\}_{k=0}^{m}\right\}, \\
\Delta &= \left\{\{\Delta_1^{i}\}_{i=0}^{l}, \{\Delta_2^{k}\}_{k=0}^{m}, \delta_1(.), \delta_2(.)\right\}.
\end{aligned}
$$

Following the notion introduced in [17], measure of the robustness of asymptotic stability for (1.1),(1.2) can be defined as follows.

Definition 1. Let the system (1.1),(1.2) be asymptotically stable. The complex structured stability radius for (1.1),(1.2) with respect to perturbation of the form (1.3) is defined by

$$
r_\varepsilon(\mathbf{A},\mathbf{B},\mathbf{C}) := \inf\{\|\Delta\|, \text{the system (1.1),(1.3) is not asym. stable}\},
\tag{1.4}
$$

where

$$\|\Delta\| := \sum_{i=0}^{l} \|\Delta_1^i\| + \sum_{k=0}^{m} \|\Delta_2^k\| + \int_{-\tau_l}^{0} \|\delta_1(\theta)\| d\theta + \int_{-\mu_m}^{0} \|\delta_2(\theta)\| d\theta,$$

and $\|.\|$ is a matrix norm induced by vector norms.

By multiplying both sides of the second equation in (1.1) with ε^{-1}, one obtains a regular explicit system of FDE-s. By applying [17, Theorem 3.3], a formula of the stability radius $r_\varepsilon(\mathbf{A}, \mathbf{B}, \mathbf{C})$ can easily be formulated. However, its practical computation uses to be very difficult because of the appearance of ε^{-1}. Therefore, we are interested in the asymptotic behavior of the stability radius as the parameter tends to zero. Such a robust stability analysis was done for the classical SPS of ordinary differential equations by Dragan in [3]. Recently, by using the implicit-system approach, Du and Linh have extended the result of [3] to a more general class of singularly perturbed differential equations [4] and to index-1 DAE-s containing a small parameter [5]. In this paper, a result similar to those in [3, 4] is obtained for the singularly perturbed system with multiple delays (1.1),(1.2). That is, the stability radius of the SPS is shown to converge to the minimum of the stability radii of the "reduced slow" system and of the "boundary layer fast" system as the parameter tends to zero.

The paper is organized as follows. In the next section, we recall the sufficient condition obtained in [2] for the exponential-asymptotic stability of system (1.1),(1.2). The main results come in Section 3 and 4. In Section 3, we formulate the complex stability radius for the reduced slow system, which is a semi-explicit index-1 system of functional differential-algebraic equations (FDAE-s). Then, in Section 4, the asymptotic behavior of the stability radius for the SPS is characterized as the parameter tends to zero. Finally, a conclusion will close the paper.

2. Parameter-independent stability condition

It is well-known that a linear system of functional differential equations is asymptotically stable if and only if all the roots of the associated characteristic equation are located in the open half plane $\mathbb{C}^-$, see [9]. However, in the case of the SPS (1.1),(1.2) it is not easy to check this condition. As we mentioned above, we should multiply both sides of the second equation with ε^{-1} in order to get a regular explicit system. Hence, the characteristic equation should contain ε^{-1}, too, which makes the computation of roots become difficult.

Taking $\varepsilon = 0$ in (1.1), we obtain

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + \bar{L}_{12}y(t) \\ 0 &= L_{21}x_t + \bar{L}_{22}y(t) \end{aligned} \tag{2.1}$$

where

$$\bar{L}_{j2} = \sum_{k=0}^{m} A_{j2}^{k} + \int_{-\mu_m}^{0} D_{j2}(\theta)d\theta, \quad j = 1,2. \tag{2.2}$$

That is, the second equation becomes an algebraic equation. Let us assume that $\bar{L}_{22}$ is invertible. The reduced slow system (2.1),(2.2) is called an index-1 FDAE of semi-explicit form, see [6]. By substituting $y(t) = \bar{L}_{22}^{-1} L_{21} x_t$ into the first equation, we obtain a linear functional differential equation

$$\dot{x}(t) = L_S x_t, \tag{2.3}$$

with

$$L_S x_t = \sum_{i=0}^{l} A_S^i x(t - \tau_i) + \int_{-\tau_l}^{0} D_S(\theta) x(t+\theta) d\theta, \tag{2.4}$$

where

$$A_S^i = A_{11}^i - \bar{L}_{12}\bar{L}_{22}^{-1} A_{21}^i, \ i = 0,1,...,l,$$
$$D_S(\theta) = D_{11}(\theta) - \bar{L}_{12}\bar{L}_{22}^{-1} D_{21}(\theta).$$

We also consider the fast boundary layer system

$$\dot{z}(\zeta) = L_F z_\zeta, \tag{2.5}$$

where

$$L_F z_\zeta = \sum_{k=0}^{m} A_{22}^k z(\zeta - \mu_k) + \int_{-\mu_m}^{0} D_{22}(\theta) y(\zeta + \theta) d\theta$$

and $\zeta = \varepsilon^{-1} t$ is the scaled time. We assume the following

Assumption A1. All the roots of the equation

$$\det\ (\lambda I_{n_2} - \sum_{k=0}^{m} A_{22}^i e^{-\lambda \mu_k} - \int_{-\mu_m}^{0} D_{22}(\theta) e^{\lambda\theta} d\theta) = 0$$

are located in the open left half plane $\mathbb{C}^-$ and

Assumption A2. Matrix $\bar{L}_{22}$ defined by (2.2) is nonsingular and all the roots of the equation

$$\det\ (\lambda I_{n_1} - \sum_{i=0}^{l} A_S^i e^{-\lambda \tau_i} - \int_{-\tau_l}^{0} D_S(\theta) e^{\lambda\theta} d\theta) = 0$$

are located in the open left half plane $\mathbb{C}^-$.

Note that these equations are the characteristic equations associated with the systems (2.5) and (2.3), respectively. Furthermore, they are independent of the small parameter ε.

Theorem 1 (Dragan and Ionita [2]). *Let Assumptions A1-2 be satisfied. There exists $\bar{\varepsilon}_0 > 0$ such that for arbitrary $\varepsilon \in (0, \bar{\varepsilon}_0)$, the zero solution of the system* (1.1),(1.2) *is exponential-asymptotically stable.*

We remark that Assumption A1 implies the nonsingularity of $\bar{L}_{22}$. Furthermore, it is possible to replace the open interval $(0, \bar{\varepsilon}_0)$ by a closed one $[0, \varepsilon_0]$ (the case of $\varepsilon = 0$ is discussed in details in the next section).

3. The stability radius for index-1 FDAE-s

Now let us consider the reduced slow system (2.1) again. This system of FDAE-s has index-1 if and only if $\bar{L}_{22}$ defined in (2.2) is nonsingular [6]. In this case, as we can see in the previous section, (2.1) can be reduced to a regular linear FDE by eliminating $y(t)$. Hence, we have

Proposition 1. *Suppose that $\bar{L}_{22}$ is nonsingular. There exists the unique solution of the initial value problem for the FDAE* (2.1)*, $t \geq 0$, with initial condition*

$$x(t) = \varphi(t), \quad t \in [-\tau_l, 0], \tag{3.1}$$

where $\varphi(.) \in C([-\tau_l, 0], \mathbb{C}^{n_1})$ is arbitrarily given.

Note that the initial condition should be assigned to the differential component $x(.)$, only. The algebraic component $y(.)$ can be determined uniquely and explicitly by $x(.)$.

Definition 2. Suppose that $\bar{L}_{22}$ is nonsingular. The zero solution of system (2.1) is said to be (exponential-) asymptotically stable if for any $\varphi(.) \in C([-\tau_l, 0], \mathbb{C}^{n_1})$, there exist positive constants c and α such that

$$\|(x(t)^T, y(t)^T)^T\| \leq c|\varphi|e^{-\alpha t}$$

holds $\forall t \geq 0$, where $(x(t)^T, y(t)^T)^T$ is the unique solution of (2.1),(3.1) and $|\varphi| = \sup\limits_{-\tau_l \leq t \leq 0} \|\varphi(t)\|$. Then, we also say that system (2.1) is asymptotically stable.

It is easy to check the following statement.

Proposition 2. *Suppose that $\bar{L}_{22}$ is nonsingular. The system* (2.1) *is asymptotically stable if and only if all the roots of the characteristic equation*

$$\det\left(\lambda\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sum\limits_{i=0}^{l} A_{11}^i e^{-\tau_i\lambda} + \int\limits_{-\tau_l}^{0} D_{11}(\theta)e^{\theta\lambda}d\theta & \bar{L}_{12} \\ \sum\limits_{i=0}^{l} A_{21}^i e^{-\tau_i\lambda} + \int\limits_{-\tau_l}^{0} D_{21}(\theta)e^{\theta\lambda}d\theta & \bar{L}_{22} \end{pmatrix}\right) = 0 \tag{3.2}$$

are located in $\mathbb{C}^-$.

Clearly, equation (3.2) is equivalent to that in Assumption A2. Accordingly to (1.3), we consider system (2.1) subjected to structured perturbations described as follows

(3.3)

$$\tilde{L}_{j1}x_t = \sum_{i=0}^{l}(A^i_{j1} + B_j\Delta^i_1 C^i_1)x(t-\tau_i) + \int_{-\tau_l}^{0}(D_{j1}(\theta) + B_j\delta_1(\theta)C_1^{l+1})x(t+\theta)d\theta,$$

$$\tilde{\bar{L}}_{j2}y(t) = \left(\sum_{k=0}^{m}(A^k_{j2} + B_j\Delta^k_2 C^k_2) + \int_{-\mu_m}^{0}(D_{j2}(\theta) + B_j\delta_2(\theta)C_2^{m+1})d\theta\right)y(t),$$

where $j = 1, 2$. The definition of the stability radius for system (2.1) is slightly modified as follows.

Definition 3. Let Assumption A2 be satisfied. The complex stability radius of (2.1) with respect to perturbation of the form (3.3) is defined by

$$\begin{aligned} &r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \\ &\inf\{\|\Delta\|, \text{the system (2.1),(3.3) is not asym. stable or } \tilde{\bar{L}}_{22} \text{ is singular}\}. \end{aligned} \tag{3.4}$$

First, we look for the so-called index-1 preserving radius of (2.1) defined by

$$r_{ind} := \inf\{\sum_{k=0}^{m}\|\Delta^k_2\| + \int_{-\mu_m}^{0}\|\delta_2(\theta)\|d\theta,\ \tilde{\bar{L}}_{22} \text{ is singular}\}.$$

Due to Definition 3, it is obvious that

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) \le r_{ind}.$$

The singularity of $\tilde{\bar{L}}_{22}$ means exactly that at least one eigenvalue of this matrix moves to zero under the effect of perturbation. The problem of finding r_{ind} is in fact a special "robust stability" problem, where the stable and unstable regions are set $\mathbb{C}_g = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}_b = \{0\}$, respectively. Using the same techniques in [11, 12, 17], it is easy to prove

Proposition 3. *Suppose that $\bar{L}_{22}$ is nonsingular. Then*

$$r_{ind} = \{\max_{k=0,1,\dots,m+1}\|C^k_2\bar{L}_{22}^{-1}B_2\|\}^{-1}.$$

Furthermore, there exists a minimal norm perturbation under which $\tilde{\bar{L}}_{22}$ is singular.

We introduce the following auxiliary functions

$$H_S(s) = s\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sum\limits_{i=0}^{l} A^i_{11}e^{-\tau_i s} + \int\limits_{-\tau_l}^{0} D_{11}(\theta)e^{\theta s}d\theta & \bar{L}_{12} \\ \sum\limits_{i=0}^{l} A^i_{21}e^{-\tau_i s} + \int\limits_{-\tau_l}^{0} D_{21}(\theta)e^{\theta s}d\theta & \bar{L}_{22} \end{pmatrix}$$

and

$$G^i_{S1}(s) = \begin{pmatrix} C^i_1 & 0 \end{pmatrix} H_S(s)^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad i = 0, 1, ..., l+1;$$

$$G^k_{S2}(s) = \begin{pmatrix} 0 & C^k_2 \end{pmatrix} H_S(s)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad k = 0, 1, ..., m + 1;$$

with $s \in \mathbb{C}$, $\Re s \geq 0$.

For computing the inverse matrix, we use a well-known factorization of block matrices, e.g., see [7]. By some matrix calculations, these functions can be reformulated as follows

(3.5)

$$\begin{aligned} G^i_{S1}(s) &= C^i_1 \left(sI - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}^{-1}_{22}\bar{L}_{21}(s)\right)^{-1} (B_1 - \bar{L}_{12}\bar{L}^{-1}_{22}B_2), \\ G^k_{S2}(s) &= -C^k_2\bar{L}^{-1}_{22}B_2 - C^k_2\bar{L}^{-1}_{22}\bar{L}_{21}(s)\left(sI - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}^{-1}_{22}\bar{L}_{21}(s)\right)^{-1} \\ &\qquad \times (B_1 - \bar{L}_{12}\bar{L}^{-1}_{22}B_2), \end{aligned}$$

where

$$\bar{L}_{11}(s) = \sum_{i=0}^{l} A^i_{11}e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{11}(\theta)e^{\theta s}d\theta,$$

$$\bar{L}_{21}(s) = \sum_{i=0}^{l} A^i_{21}e^{-\tau_i s} + \int_{-\tau_l}^{0} D_{21}(\theta)e^{\theta s}d\theta.$$

Let us denote

$$r_{stab} = \left(\max\{\max_{0\leq i\leq l+1} \sup_{s\in i\mathbb{R}} \|G^i_{S1}(s)\|, \max_{0\leq k\leq m+1} \sup_{s\in i\mathbb{R}} \|G^k_{S2}(s)\|\}\right)^{-1}.$$

Lemma 1. *Let Assumption A2 be satisfied. Then*

$$r_{ind} \geq r_{stab}.$$

Proof. Taking into consideration the boundedness of functions $\bar{L}_{11}(.)$, $\bar{L}_{21}(.)$ in $i\mathbb{R}$ and

$$\lim_{|s|\to+\infty} \| \left(sI_{n_1} - \bar{L}_{11}(s) + \bar{L}_{12}\bar{L}^{-1}_{22}\bar{L}_{21}(s)\right)^{-1} \| = 0,$$

one easily obtains the inequality

$$\max_{0\leq k\leq m+1} \sup_{s\in i\mathbb{R}} \|G^k_{S2}(s)\| \geq \max_{0\leq k\leq m+1} \lim_{|s|\to+\infty} \|G^k_{S2}(s)\| = \max_{k=0,1,...,m+1} \|C^k_2\bar{L}^{-1}_{22}B_2\|.$$

Due to the formulae of r_{ind} and r_{stab}, the proof is complete. □

Theorem 2. *Let Assumption A2 be satisfied, that is, $\bar{L}_{22}$ is nonsingular and the reduced slow system* (2.1) *is asymptotically stable. Then*

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \min\{r_{stab}, r_{ind}\} = r_{stab}.$$

Proof. For simplicity, we denote

$$\begin{aligned} &\mathcal{C}^i = \begin{pmatrix} C^i_1 & 0 \end{pmatrix}, \ i = 0, 1, ..., l + 1; \quad \mathcal{C}^{l+2+k} = \begin{pmatrix} 0 & C^k_2 \end{pmatrix}, \ k = 0, 1, ...m, \\ &\mathcal{G}^i_S(s) = G^i_{S1}(s), \ i = 0, 1, ..., l + 1; \quad \mathcal{G}^{l+2+k}_S(s) = G^k_{S2}(s), \ k = 0, 1, ...m, \end{aligned}$$

Due to Lemma 1, one of the following two cases holds.

Case A. $r_{stab} < r_{ind}$.

First, we prove that

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) \le r_{stab}. \tag{3.6}$$

To this end, we construct a destabilizing perturbation which has the norm arbitrarily close to r_{stab}. Suppose that $\epsilon > 0$ is an arbitrary, but sufficiently small number such that $r_{stab} + \epsilon < r_{ind}$. Then, there exist an index M, $0 \le M \le l + m + 2$, and $s_1 \in i\mathbb{R}$ such that

$$\|\mathcal{G}_S^M(s_1)\|^{-1} \le \left(\max_{0 \le i \le l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)^{-1} + \epsilon < r_{ind}.$$

Let the size of $\mathcal{G}_S^M(s_1)$ be $q \times p$. There exists a vector $u \in \mathbb{C}^p$, $\|u\| = 1$ such that

$$\|\mathcal{G}_S^M(s_1)u\| = \|\mathcal{G}_S^M(s_1)\|.$$

Invoking a corollary of the Hahn-Banach theorem, there exists a functional $v^* \in \mathbb{C}^q$, $\|v^*\| = 1$ such that

$$\|v^* \mathcal{G}_S^M(s_1)u\| = \|\mathcal{G}_S^M(s_1)u\|.$$

Let us define

$$\Delta_b := \|\mathcal{G}_S^M(s_1)\|^{-1} u v^* \in \mathbb{C}^{p \times q}.$$

It is easy to see that $\|\Delta_b\| = \|\mathcal{G}_S^M(s_1)\|^{-1}$. We construct a destabilizing perturbation Δ as follows:

- If $M \le l$, set $\Delta_1^M := \Delta_b e^{\tau_i s_1}$, and all the other perturbations are zero;
- If $M = l + 1$, set $\delta_1(\theta) := \tau_l^{-1} \Delta_b e^{-\theta s_1}$, and all the others are zero;
- If $l + 2 \le M \le l + m + 1$, set $\Delta_2^M := \Delta_b$, and all the others are zero;
- If $M = l + m + 2$, set $\delta_2(\theta) := \mu_m^{-1} \Delta_b$, and all the others are zero.

It is clear that, in any case, $\|\Delta\| = \|\Delta_b\|$ holds. After some elementary calculations, one can easily verify that

$$\Delta_b \mathcal{G}_S^M(s_1)u = u \Rightarrow \Delta_b \mathbf{B} H_S(s_1)^{-1} \mathcal{C}^M u = u \Rightarrow \mathcal{C}^M \Delta_b \mathbf{B} w = H_S(s_1) w,$$

where $w := H_S(s_1)^{-1} \mathcal{C}^M u \ne 0$. With Δ defined as above, the characteristic equation associated with the perturbed system has the root $s_1 \in i\mathbb{R}$. On the other hand, by construction, $\|\Delta\| < r_{ind}$, hence the perturbed system remains index-1. By Proposition 2, the perturbed system is not asymptotically stable. Since ϵ is arbitrarily chosen, we obtain (3.6).

Now, we prove the inverse inequality of (3.6). Take an arbitrary perturbation set Δ such that $\|\Delta\| < r_{ind}$ and the perturbed system is not asymptotically stable. Since the perturbed system remains index-1, it follows that the associated characteristic equation has a root outside $\mathbb{C}^-$. Hence, there exist s_0, $\Re s_0 \ge 0$

and a nonzero vector $x_0 \in \mathbb{C}^n$ such that

$$
\begin{aligned}
H_S(s_0)x_0 = & \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \{ \sum_{i=0}^{l} \Delta_1^i e^{-\tau_i s_0} \begin{pmatrix} C_1^i & 0 \end{pmatrix} \\
& + \int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta \begin{pmatrix} C_1^{l+1} & 0 \end{pmatrix} + \sum_{k=0}^{m} \Delta_2^k \begin{pmatrix} 0 & C_2^k \end{pmatrix} \\
& + \int_{-\mu_m}^{0} \delta_2(\theta) d\theta \begin{pmatrix} 0 & C_2^{m+1} \end{pmatrix} \} x_0.
\end{aligned}
\tag{3.7}
$$

Multiplying both sides of (3.7) with $H_S(s_0)^{-1}$ from the left, we have

$$
\begin{aligned}
x_0 \ = \ & H_S(s_0)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \left\{ \sum_{i=0}^{l} \Delta_1^i e^{-\tau_i s_0} \begin{pmatrix} C_1^i & 0 \end{pmatrix} + \right. \\
& + \int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta \begin{pmatrix} C_1^{l+1} & 0 \end{pmatrix} + \sum_{k=0}^{m} \Delta_2^k \begin{pmatrix} 0 & C_2^k \end{pmatrix} + \\
& \left. + \int_{-\mu_m}^{0} \delta_2(\theta) d\theta \begin{pmatrix} 0 & C_2^{m+1} \end{pmatrix} \right\} x_0.
\end{aligned}
\tag{3.8}
$$

Let N be the index such that $0 \le N \le l + m + 2$ and

$$\|\mathcal{C}^N x_0\| = \max_{0 \le i \le l+m+2} \|\mathcal{C}^i x_0\|.$$

It is clear that $\mathcal{C}^N x_0 \neq 0$. Multiplying both sides of equality (3.8) with $\mathcal{C}^n$ from the left and taking norm, we obtain

$$\|\mathcal{C}^N x_0\| \le \|\mathcal{G}_S^N(s_0)\| \|\Delta\| \|\mathcal{C}^N x_0\|.$$

To verify this inequality, we use the estimates

$$\|\Delta_1^i e^{-\tau_i s_0}\| \le \|\Delta_1^i\| \text{ and } \| \int_{-\tau_l}^{0} \delta_1(\theta) e^{\theta s_0} d\theta \| \le \|\delta_1(.)\|$$

and the definition of $\|\Delta\|$. It follows that

$$\|\Delta\| \ge \|\mathcal{G}_S^N(s_0)\|^{-1} \ge \left(\max_{1 \le i \le l+m+2} \sup_{\Re s \ge 0} \|\mathcal{G}_S^i(s)\| \right)^{-1}.$$

Since each function $\mathcal{G}_S^i(s),\ i = 0, 1, ..., l + m + 2$, is analytic in $\mathbb{C} \setminus \mathbb{C}^-$, due to the the maximum principle, their least upper bound is attained in $i\mathbb{R}$ (at a finite point or at infinity). Hence,

$$r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) \ge \left(\max_{1 \le i \le l+m+2} \sup_{s \in i\mathbb{R}} \|\mathcal{G}_S^i(s)\| \right)^{-1} = r_{stab}. \tag{3.9}$$

Inequalities (3.6),(3.9) imply $r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{stab}$.

Case B. $r_{stab} = r_{ind}$.

Take an arbitrary perturbation set Δ such that $\|\Delta\| < r_{ind}$. It is clear that Δ cannot be a destabilizing perturbation. Otherwise, by repeating the above arguments, we would have $\|\Delta\| \geq r_{stab} = r_{ind}$ which yields a contradiction. It means that the perturbed system remains index-1 and asymptotically stable. By Definition 3, we have $r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}) = r_{ind} = r_{stab}$. □

Theorem 2 extends the result for index-1 DAE-s (without time-delay and perturbation structure) proposed in [16] to semi-explicit index-1 systems of FDAE-s. We also note that the index-1 preserving property is essential in the existence and the uniqueness of the solution for initial value problem (2.1),(3.1). For more details on delay DAE-s and their stability theory, e.g., see [6, 18] and the references therein.

4. Asymptotic behavior of the stability radius for the SPS

Now, we turn to the main point of the paper, the asymptotic behavior of the stability radius for the SPS (1.1),(1.3).

First, we introduce the following auxiliary functions:

$$\begin{aligned}
\bar{L}_{12}(\varepsilon, s) =& \sum_{k=0}^{m} A_{12}^{k} e^{-\varepsilon\mu_k s} + \int_{-\mu_m}^{0} D_{12}(\theta) e^{\varepsilon\theta s} d\theta, \\
\bar{L}_{22}(\varepsilon, s) =& \sum_{k=0}^{m} A_{22}^{k} e^{-\varepsilon\mu_k s} + \int_{-\mu_m}^{0} D_{22}(\theta) e^{\varepsilon\theta s} d\theta, \\
H_\varepsilon(s) =& s \begin{pmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{pmatrix} - \begin{pmatrix} \bar{L}_{11}(s) & \bar{L}_{12}(\varepsilon, s) \\ \bar{L}_{21}(s) & \bar{L}_{22}(\varepsilon, s) \end{pmatrix}
\end{aligned}$$

with $s \in \mathbb{C}$, $\Re s \geq 0$, $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is provided by Theorem 1. The functions $\bar{L}_{11}(s)$, $\bar{L}_{21}(s)$ were previously introduced in (3.5). Furthermore,

$$G_{\varepsilon 1}^{i}(s) = \begin{pmatrix} C_1^i & 0 \end{pmatrix} H_\varepsilon(s)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad i = 0, 1, ..., l+1;$$

$$G_{\varepsilon 2}^{k}(s) = \begin{pmatrix} 0 & C_2^k \end{pmatrix} H_\varepsilon(s)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad k = 0, 1, ..., m+1;$$

Let us fix a closed interval $[0, \varepsilon_0]$ provided by Theorem 1.

Proposition 4. *Let Assumptions A1-A2 be satisfied. Then*

$$r_\varepsilon(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \left(\max\{ \max_{0 \leq i \leq l+1} \sup_{s \in i\mathbb{R}} \|G_{\varepsilon 1}^{i}(s)\|, \max_{0 \leq k \leq m+1} \sup_{s \in i\mathbb{R}} \|G_{\varepsilon 2}^{k}(s)\| \} \right)^{-1}.$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Proof. To prove this formula, the techniques used in the proof of [17, Theorem 3.3] can be applied without any difficulty. Thus, we omit the details. □

By some matrix calculations, similarly to (3.5), the latter functions can be reformulated as follows

$$
\begin{aligned}
(4.1)\qquad G^i_{\varepsilon 1}(s) =& C^i_1\left[sI_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}\bar{L}_{21}(s)\right]^{-1}\\
&\times (B_1 + \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}B_2),\\
G^k_{\varepsilon 2}(s) =& C^k_2(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}B_2 + C^k_2(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}\bar{L}_{21}(s)\\
&\times \left[sI_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}\bar{L}_{21}(s)\right]^{-1}\\
&\times (B_1 + \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22})^{-1}B_2).
\end{aligned}
$$

The following auxiliary result is easy to prove, too.

Lemma 2. *Let Assumptions A1-A2 be satisfied. Then the matrix functions*

$$\bar{L}_{j1}(.),\ \bar{L}_{j2}(\varepsilon, .),\ j = 1, 2,\ \text{and}\ (.\varepsilon I_{n_2} - \bar{L}_{22}(\varepsilon, .))^{-1}$$

are bounded in $i\mathbb{R}$ *and their bounds are independent of* $\varepsilon \in (0, \varepsilon_0]$.

Proof. The uniform boundedness of $\bar{L}_{j1}(s)$, $\bar{L}_{j2}(\varepsilon, s)$, $j = 1, 2$, is obvious. To verify the uniform boundedness of $(.\varepsilon I_{n_2} - \bar{L}_{22}(\varepsilon, .))^{-1}$, we observe that

$$\sup_{s\in i\mathbb{R}} \|(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}\| = \sup_{s\in i\mathbb{R}} \|(sI_{n_2} - \hat{L}_{22}(s))^{-1}\|,$$

where

$$\hat{L}_{22}(s) = \sum_{k=0}^{m} A^k_{22} e^{-\mu_k s} + \int_{-\mu_m}^{0} D_{22}(\theta)e^{\theta s}d\theta.$$

Furthermore,

$$\lim_{|s|\to+\infty} \|(sI_{n_2} - \hat{L}_{22}(s))^{-1}\| = 0.$$

Hence, the function in question is bounded in $i\mathbb{R}$ and its bound does not depend on ε. ☐

Let the fast boundary layer system (2.5) introduced in Section 2 be asymptotically stable. We associate to this system the auxiliary functions

$$(4.2)\qquad G^k_F(s) = C^k_2(sI_{n_2} - \hat{L}_{22}(s))^{-1}B_2, \quad k = 0, 1, ..., m + 1;\ \Re s \geq 0.$$

By applying [17, Theorem 3.3] to system (2.5) subjected to the corresponding structured perturbation, we have

$$(4.3)\qquad r(\mathbf{A_{22}}, B_2, \mathbf{C_2}) = \left(\max_{0\leq k\leq m+1} \sup_{s\in i\mathbb{R}} \|G^k_F(s)\|\right)^{-1},$$

where $\mathbf{A_{22}} = \left\{\{A^k_{22}\}^m_{k=0}, D_{22}(.)\right\}$, $\mathbf{C_2} = \{C^k_2\}^{m+1}_{k=0}$ and $r(\mathbf{A_{22}}, B_2, \mathbf{C_2})$ denotes the complex structured stability radius for (2.5).

Our main result is the following

Theorem 3. *Let Assumptions A1-A2 be satisfied. Then,*

$$\lim_{\varepsilon\to+0} r_\varepsilon(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \min\{r_0(\mathbf{A}, \mathbf{B}, \mathbf{C}), r(\mathbf{A_{22}}, B_2, \mathbf{C_2})\}.$$

Proof. The key point of the proof is the uniform convergence

$$\left\| \left[sI_{n_1} - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon, s)(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1} \bar{L}_{21}(s) \right]^{-1} \right\| \rightrightarrows 0 \tag{4.4}$$

as $|s| \to +\infty$ with respect to $\varepsilon \in [0, \varepsilon_0]$. We recall that, throughout the proof, the variable s is varying strictly in the line $i\mathbb{R}$, only. Due to Lemma 2, (4.4) is evident.

By using the formula in Theorem 2, Proposition 4, and (4.3), it is sufficient to prove first,

$$\lim_{\varepsilon \to +0} \sup_{s \in i\mathbb{R}} \|G^i_{\varepsilon 1}(s)\| = \sup_{s \in i\mathbb{R}} \|G^i_{S1}(s)\|, \quad i = 0, 1, ..., l+1, \tag{4.5}$$

and secondly,

$$\lim_{\varepsilon \to +0} \sup_{s \in i\mathbb{R}} \|G^k_{\varepsilon 2}(s)\| = \max\{\sup_{s \in i\mathbb{R}} \|G^k_{S2}(s)\|, \sup_{s \in i\mathbb{R}} \|G^k_F(s)\|\}, \quad k = 0, 1, ..., m+1. \tag{4.6}$$

Fix an arbitrary index i, $0 \le i \le l+1$ and an arbitrarily small number $\rho > 0$. From (4.4), it is easy to see that $\|G^i_{\varepsilon 1}(s)\|$ converges uniformly to zero as $|s|$ tends to infinity. Therefore, there exists a bound T_1 independent of ε such that

$$\|G^i_{\varepsilon 1}(s)\| \le \rho, \quad \forall\ |s| \ge T_1.$$

On the other hand, in the compact domain $\{(s, \varepsilon),\ |s| \le T_1,\ 0 \le \varepsilon \le \varepsilon_0\}$, $\left\|G^i_{\varepsilon 1}(s)\right\|$ is continuous as a two-variable function, hence uniformly continuous, too. Therefore, there exists a sufficiently small $\varepsilon_1 = \varepsilon_1(\rho)$ such that for $\varepsilon \le \varepsilon_1$, we have

$$\sup_{|s| \le T_1} \left\|G^i_{\varepsilon 1}(s)\right\| \le \sup_{|s| \le T_1} \left\|G^i_{S1}(s)\right\| + \rho \le \sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| + \rho.$$

Thus, for $\varepsilon \le \varepsilon_1$, we obtain

$$\sup_{s \in i\mathbb{R}} \left\|G^i_{\varepsilon 1}(s)\right\| \le \sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| + \rho.$$

Since $\sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\|$ is finite, there exists a number $s_1 = s_1(\rho) \in i\mathbb{R}$ such that

$$\left\|G^i_{S1}(s_1)\right\| \ge \sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| - \rho.$$

Furthermore, because of the continuity of $\left\|G^i_{\varepsilon 1}(s_1)\right\|$ as a function of ε, there exists a sufficiently small $\varepsilon_2 = \varepsilon_2(\rho)$ such that for $\varepsilon \le \varepsilon_2$, we obtain

$$\sup_{s \in i\mathbb{R}} \left\|G^i_{\varepsilon 1}(s)\right\| \ge \left\|G^i_{\varepsilon 1}(s_1)\right\| \ge \left\|G^i_{S1}(s_1)\right\| - \rho \ge \sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| - 2\rho.$$

Therefore, for $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$, the estimate

$$\sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| - 2\rho \le \sup_{s \in i\mathbb{R}} \left\|G^i_{\varepsilon 1}(s)\right\| \le \sup_{s \in i\mathbb{R}} \left\|G^i_{S1}(s)\right\| + \rho$$

holds. This proves (4.5).

To prove (4.6), we proceed as in [4] and [5]. Analogously to above, fix an index k, $0 \le k \le m+1$ and an arbitrarily small $\varrho > 0$. We show that the inequalities

$$\begin{aligned}\max\left\{\sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\|, \sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\|\right\} - 2\varrho \le \sup_{s\in i\mathbb{R}}\left\|G^k_{\varepsilon 2}(s)\right\| \\ \le \max\left\{\sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\|, \sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\|\right\} + \varrho\end{aligned} \tag{4.7}$$

hold for all sufficiently small ε.

First, we prove the second inequality in (4.7). By a similar argument as when proving (4.5), there exists a sufficiently large number $T_2 = T_2(\varrho)$, T_2 is independent of ε, such that

$$\begin{aligned}&\Big\| C^k_2\big(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s)\big)^{-1}\bar{L}_{21}(s).\\ &\times\Big[sI - \bar{L}_{11}(s) - \bar{L}_{12}(\varepsilon, s)\big(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s)\big)^{-1}\bar{L}_{21}(s)\Big]^{-1}\\ &\times\Big(B_1 + \bar{L}_{12}(\varepsilon, s)\big(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s)\big)^{-1}B_2\Big)\Big\| \le \varrho, \quad |s| \ge T_2.\end{aligned}$$

Therefore, for s with $|s| \ge T_2$, we have

$$\left\|G^k_{\varepsilon 2}(s)\right\| \le \left\|C^k_2(\varepsilon s I_{n_2} - \bar{L}_{22}(\varepsilon, s))^{-1}B_2\right\| + \varrho.$$

Hence, we obtain

$$\begin{aligned}\sup_{|s|\ge T_2}\left\|G^k_{\varepsilon 2}(s)\right\| &\le \sup_{|s|\ge T_2}\left\|C^k_2(\varepsilon s I_{n_2} - \hat{L}_{22}(\varepsilon s))^{-1}B_2\right\| + \varrho = \\ &= \sup_{|s|\ge \varepsilon T_2}\left\|G^k_F(s)\right\| + \varrho \le \sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\| + \varrho.\end{aligned} \tag{4.8}$$

On the other hand, in the compact domain $\{(s,\varepsilon),\ |s| \le T_2,\ 0 \le \varepsilon \le \varepsilon_0\}$, $\left\|G^k_{\varepsilon 2}(s)\right\|$ is continuous as a two-variable function, hence uniformly continuous, too. Therefore, there exists a sufficiently small $\varepsilon_3 = \varepsilon_3(\varrho)$ such that for $\varepsilon \le \varepsilon_3$, we have

$$\sup_{|s|\le T_2}\left\|G^k_{\varepsilon 2}(s)\right\| \le \sup_{|s|\le T_2}\left\|G^k_{S2}(s)\right\| + \varrho \le \sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\| + \varrho.$$

Thus, for $\varepsilon \le \varepsilon_3$, we obtain

$$\sup_{s\in i\mathbb{R}}\left\|G^k_{\varepsilon 2}(s)\right\| \le \max\left\{\sup_{s\in i\mathbb{R}}\left\|G^k_{S2}(s)\right\|, \sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\|\right\} + \varrho.$$

Now, we prove the first inequality in (4.7). Analogously to (4.8), we have

$$\sup_{|s|\ge T_2}\left\|G^k_{\varepsilon 2}(s)\right\| \ge \sup_{|s|\ge \varepsilon T_2}\left\|G^k_F(s)\right\| - \varrho.$$

Since $\left\|G^k_F(s)\right\|$ is continuous, $s \in i\mathbb{R}$, there exists a sufficiently small $\varepsilon_4 = \varepsilon_4(\varrho)$ such that for $\varepsilon \le \varepsilon_4$ and the inequality

$$\sup_{|s|\ge \varepsilon T_2}\left\|G^k_F(s)\right\| \ge \sup_{s\in i\mathbb{R}}\left\|G^k_F(s)\right\| - \varrho$$

holds. Hence, we obtain

$$\sup_{|s|\geq T_2} \left\|G^k_{\varepsilon 2}(s)\right\| \geq \sup_{s\in i\mathbb{R}} \left\|G^k_F(s)\right\| - 2\varrho.$$

On the other hand, since $\sup_{s\in i\mathbb{R}} \left\|G^k_{S2}(s)\right\|$ is finite, there exists a number $s_2 = s_2(\varrho) \in i\mathbb{R}$ such that

$$\left\|G^k_{S2}(s_2)\right\| \geq \sup_{s\in i\mathbb{R}} \left\|G^k_{S2}(s)\right\| - \varrho.$$

Furthermore, because of the continuity of $\left\|G^k_{\varepsilon 2}(s_2)\right\|$ as a function of ε, there exists a sufficiently small $\varepsilon_5 = \varepsilon_5(\varrho)$ such that for $\varepsilon \leq \varepsilon_5$ we obtain

$$\sup_{s\in i\mathbb{R}} \left\|G^k_{\varepsilon 2}(s)\right\| \geq \left\|G^k_{\varepsilon 2}(s_2)\right\| \geq \left\|G^k_{S2}(s_2)\right\| - \varrho \geq \sup_{s\in i\mathbb{R}} \left\|G^k_{S2}(s)\right\| - 2\varrho.$$

Therefore, for $\varepsilon \leq \min\{\varepsilon_4, \varepsilon_5\}$ the inequality

$$\sup_{s\in i\mathbb{R}} \left\|G^k_{\varepsilon 2}(s)\right\| \geq \max\left\{\sup_{s\in i\mathbb{R}} \left\|G^k_{S2}(s)\right\|, \sup_{s\in i\mathbb{R}} \left\|G^k_F(s)\right\|\right\} - 2\varrho$$

holds.

Then, for $\varepsilon \leq \min\{\varepsilon_3, \varepsilon_4, \varepsilon_5\}$, the inequalities in (4.7) hold. The proof of (4.6) is complete.

Since (4.5),(4.6) hold for all $i = 0, 1, ..., l + 1$ and $k = 0, 1, ..., m + 1$, the proof of Theorem 3 is complete. □

5. Conclusion

In this paper, a class of SPS-s of differential equations with multiple delays has been considered. Motivated by and considered as a continuation of the stability analysis given in [2], the stability robustness of the SPS-s has been launched. The notion of the structured stability radius is extended to the reduced systems which are index-1 FDAE-s. By using the implicit-system approach, asymptotic behavior of the stability radius for the SPS-s is characterized as the parameter tends to zero. It is known that the complex stability radius for explicit linear systems depends continuously on data [12]. Here, we have shown that this property does not hold for the SPS-s, namely, the stability radius may be discontinuous in parameter. The SPS analyzed here includes that investigated in [3] as a special case. An extension of the results to more general systems of FDAE-s containing a small parameter would be of interest.

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