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ON THE ASYMPTOTIC BEHAVIOR OF GENERALIZED SOLUTION OF PARABOLIC SYSTEMS IN A NEIGHBORHOOD OF CONIC POINT

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Abstract. The purpose of this paper is to develop the well-known theory on the elliptic, hyperbolic and parabolic equations in nonsmooth domains that has been presented by such Russian mathematicians as V. A. Kondratiev, V. G. Mazya, B. A. Plamenevsky and S. A. Nazarov. We will obtain an asymptotic expansion of the generalized solutions of first initial boundary-value problems for strongly parabolic systems near the conic point on the boundary of the infinite cylinder.

1. Introduction

The boundary problems for elliptic equation in domains with smooth boundary have been well studied. Agmon, Duglis and Nirenberg [6] established the normal solvability of general boundary problem satisfying Sapiro-Lopatinsky condition, and they proved that if the right-hand parts, the coefficients and the boundary are infinitely differentiable so is the solution.

In the case where the boundary contains angle points (2-dimensional domain) or conic points (higher dimensional domain), the indicated methods can not be applied since it is impossible to straighten the boundary by a smooth transformation.

In this paper we consider the first boundary problem for strongly parabolic systems in domains, the boundary of which contains a finite number of conic points. We will obtain the asymptotic series for the solution, belonging to some Sobolev space, in a neighborhood of a conic point. According to this expression, the solution will expand to two terms. The first term has a polynomial form, and the second term has a sufficient smoothness.

2. Notations

Let Ω be a bounded domain in $\mathbb{R}^n$. The boundary $\partial\Omega$ of Ω is assumed to be an infinitely differentiable surface everywhere, except at the coordinate origin, in the neighborhood of which $\partial\Omega$ coincides with the cone

$$K = \Big\{ x \ \Big| \ \frac{x}{|x|} \in G \Big\},$$

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where G is a smooth domain on unit sphere. $\Omega_T = \Omega \times (0,T)$, $0 < T \leqslant \infty$, $S_T = \partial\Omega \times (0,T)$.

Let $H^{\ell,k}(\Omega_T)$ be the space consisting of all functions $u = (u_1, \dots, u_s)$ in $L_2(\Omega_T)$ which have the generalized derivatives up to order ℓ by x and up to order k by t belonging to $L_2(\Omega_T)$.

The norm in this space is defined as follows:

$$\|u\|_{H^{\ell,k}(\Omega_T)} = \Big[\int\limits_{\Omega_T} \Big(\sum_{|\alpha|=0}^{\ell} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t_j}|^2 dxdt \Big]^{1/2},$$

$\overset{\circ}{H}{}^{\ell,k}(\Omega_T)$ is the closure in $H^{\ell,k}(\Omega_T)$ of the set consisting of all infinitely differentiable in Ω_T functions which vanish near S_T.

$H^{\ell,k}(e^{-\gamma t}, \Omega_\infty)$ is the space consisting of all functions $u(x,t)$ satisfying

$$\|u\|_{H^{\ell,k}(e^{-\gamma t},\Omega_\infty)} = \Big[\int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{\ell} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{tj}|^2 \Big) e^{-2\gamma t} dxdt \Big]^{1/2} < \infty.$$

In the same way as above we define $\overset{\circ}{H}{}^{\ell,k}(e^{-\gamma t}, \Omega_\infty)$.

$H^{\ell,k}_\beta(e^{-\gamma t}, \Omega_\infty)$ is the space consisting of all functions $u(x,t)$ satisfying

$$\|u\|_{H^{\ell,k}_\beta(e^{-\gamma t},\Omega_\infty)} = \Big[\int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|+j=0}^{\ell} r^{2(\beta+|\alpha|-\ell)} |D^\alpha u|^2 + \sum_{j=0}^{k} |u_t j|^2 \Big) e^{-2\gamma t} dxdt \Big]^{1/2} < +\infty,$$

$H^{\ell}_\beta(e^{-\gamma t}, \Omega_\infty)$ is the space consisting of all functions $u(x,t)$ satisfying

$$\|u\|_{H^{\ell}_\beta(e^{-\gamma t},\Omega_\infty)} = \Big[\int\limits_{\Omega_\infty} \Big(\sum_{|\alpha|+j=0}^{\ell} r^{2(\beta+|\alpha|+j-\ell)} |D^\alpha u_{t^j}|^2 \Big) e^{-2\gamma t} dxdt \Big]^{1/2} < \infty.$$

We consider in Ω_∞ the first initial boundary value problem

$$Lu \equiv (-1)^m \Big[\sum_{|p|,|q|=1}^{m} D^p a_{pq}(x,t) D^q u + \sum_{|p|=1}^{m} a_p(x,t) D^p u + a(x,t)u \Big] - u_t = f(x,t), \tag{2.1}$$

$$\frac{\partial^j u}{\partial t^j}\Big|_{S_\infty} = 0, \quad j = \overline{0, m-1}, \tag{2.2}$$

$$u(x,0) = 0, \tag{2.3}$$

where a_{pq}, a_p, a are bounded measurable complex-value matrices $s \times s$, $a_{pq} = (-1)^{|p|+|q|}a^*_{qp}$. When $|p| = |q| = m$, suppose that a_{pq} are uniformly continuous with respect to t functions in $\overline{\Omega}_\infty$.

Assume that a_{pq}, a_p, a are infinitely differentiable functions in $\overline{\Omega}_\infty$.

We require that the considered system (2.1) - (2.3) is strongly parabolic, i.e, for each $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{C}^j \setminus \{0\}$

$$\sum_{|p|=|q|=0} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geqslant \mu_0|\xi|^{2m}|\eta|^2 \quad \forall (x,t) \in \overline{\Omega}_\infty,$$

where $\xi^p = \xi_1^{p_1}\cdots\xi_n^{p_n}$, μ_0 is a positive constant.

The function $u(x,t)$ is called a generalized solution of the first initial boundary value problem (2.1) - (2.3) in the space $\overset{\circ}{H}{}^{m,1}(e^{-\gamma t},\Omega_\infty)$ if $u(x,0) = 0$ and

$$\tag{2.4} \begin{aligned} \int\limits_{\Omega_T} \Big[- u_t\overline{\eta} + \sum_{|p|,|q|=1}^{m} (-1)^{m-1+|p|}a_{pq}D^q u\overline{D^p u} + {}\\ + \sum_{|p|=1}^{m}(-1)^{m-1}a_p D^q u\overline{\eta} + (-1)^{m-1}au\overline{\eta}\Big] dxdt = \int\limits_{\Omega_T} f\overline{\eta}dxdt \end{aligned}$$

for all $T > 0$ and all test functions $\eta \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\eta(x,T) = 0$.

3. Main results

Denote the main part of the operator L at the origin 0 by $L_0(0,t,D)$. First we consider in K the Dirichlet problem for the system

$$\tag{3.1} L_0(0,t,D)u = r^{-i\lambda_0(t)-2m}\sum_{s=0}^{m}\ln^s r f_s(\omega,t),$$

where ω is a local coordinate system on S^{n-1}.

Lemma 3.1. [3] *Let $f_s(\omega,t)$, $s = 0,\dots,M$, be infinitely differentiable functions of ω. Then there exists a solution of* (3.1) *having the form*

$$u(x,t) = r^{-i\lambda_0}\sum_{s=0}^{M+\mu}\ln^s r\tilde{f}_s(\omega,t),$$

where $\tilde{f}_s$, $s = 0,\dots,M+\mu$, are the infinitely differentiable functions of ω, $\mu = 1$ if λ_0 is a simple eigenvalue of the problem

$$\tag{3.2} Q(\omega,t,\lambda,D_\omega)v(\omega) = 0, \quad \omega \in G,$$

$$\tag{3.3} D^j_\omega v(\omega) = 0, \quad \omega \in \partial G,\ j = 0,\dots,m-1$$

if $L_0(0,t,D) = r^{-2m}Q(\omega,t,rD_r,D_\omega)$, where $D_r = \dfrac{i\partial}{\partial r}$, and $\mu = 0$ if λ_0 is not the eigenvalue of this problem.

Now we condider the Dirichlet problem for the system

$$(-1)^{m-1}L_0(0,t,D)u = F(x,t), \quad x \in K. \tag{3.4}$$

Lemma 3.2. *Let $u(x,t)$ be the generalized solution of (3.4) for almost everywhere $t \in [0,\infty)$ such that $u \equiv 0$ when $|x| > R = \text{const}$. Let*

$$u_{t^k} \in H_\beta^{2m+\ell,1}(e^{-\gamma_k t}, K_\infty), \quad F_{t^k} \in H_{\beta'}^{\ell,1}(e^{-\gamma_k t}, K_\infty)$$

for some γ, γ_k, $k \leqslant h$, $\beta' < \beta < m+\ell$. In addition we suppose that the straight lines

$$\text{Im}\lambda = -\beta + 2m + \ell - \frac{n}{2} \quad \textit{and} \quad \text{Im}\lambda = -\beta' + 2m + \ell - \frac{n}{2}$$

do not contain the points of the spectrum of the problem (3.2) - (3.3) for every $t \in [0,\infty)$ and in the strip

$$-\beta + 2m + \ell - \frac{n}{2} < \text{Im}\lambda < -\beta' + 2m + \ell - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$. Then the following representation holds

$$u(x,t) = c(t)r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t), \tag{3.5}$$

where ϕ is an infinitely differentiable function of (ω,t) and does not depend on the solution, $c_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$ and $(u_1)_{t^k} \in H_{\beta'}^{2m+\ell,1}(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h$.

Proof. From Theorem 3.2 in [2] it follows that

$$u(x,t) = c(t)r^{-i\lambda(t)}\phi(\omega,t) + u_1(x,t), \tag{3.6}$$

where $\phi(\omega,t)$ is the eigenfunction of the problem (3.2)-(3.3) which corresponds to the eigenvalue $\lambda(t)$, $u_1 \in H_{\beta'}^{2m+\ell,1}(e^{-\gamma t}, K_\infty)$, and

$$c(t) = i\int_K F(x,t)r^{-i\overline{\lambda(t)}+2m-n}\psi(x,t)dx,$$

where ψ is the eigenfunction of the problem conjugating to the problem (3.2)-(3.3) and which corresponds to the eigenvalue $\overline{\lambda(t)}$.

Since

$$\text{Im}\overline{\lambda(t)} > \beta' - 2m - \ell + \frac{n}{2} \quad \text{and} \quad F \in H_{\beta'}^{\ell,1}(e^{-\gamma t}, K_\infty),$$

$c(t) \in L_2(0,\infty)$ (see Theorem VIII.2.6 of [4]). Hence the assertion is proved for $h = 0$.

Assume that assertion of the lemma is true for $h-1$. Denote u_{t^h} by v. From (3.4) we obtain

$$(-1)^{m-1}L_0(0,t,D) = F_{t^k} + (-1)^m\sum_{k=1}^{h}\binom{h}{k}L_{0t^k}(0,t,D)t_{t^{h-k}} \tag{3.7}$$

where

$$L_{0t^k} = \sum_{|p|=|q|=m} \frac{\partial^k a_{pq}(0,t)}{\partial t^k} D^p D^q.$$

Put $S_0(\omega,t) = r^{-i\lambda(t)}\phi(\omega,t)$.

Since $\phi(\omega,t) \in C^\infty(\omega,t)$ [1], from (3.7) it follows that

$$\sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D) = \sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)\big[(cS_0)_{t^{h-k}}\big] + \\ + \sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)(u_1)_{t^{h-k}}.$$

Using the induction hypothesis we obtain

$$\sum_{k=1}^{h} \binom{h}{k} L_{0t^k}(0,t,D)u_{t^{h-k}} = F_1 - \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}} L_0(0,t,D)(S_0)_{t^k}, \tag{3.8}$$

where $F_1 \in H^{\ell,1}_{\beta'}(e^{-\gamma t}, K_\infty)$. From (3.7)-(3.8) we see that

$$(-1)^{m-1}L_0(0,t,D)v = F_2 - (-1)^m \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}} L_0(0,t,D)(S_0)_{t^k}, \tag{3.9}$$

where $F_2 \in H^{\ell,1}_{\beta'}(e^{-\gamma t}, K_\infty)$. Hence by analogy to (3.6) we get

$$u_{t^h} = v = \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^k} + d(t)S_0 + u_2, \tag{3.10}$$

where $d(t)e^{-\gamma t} \in L_{2,loc}(0,\infty)$, $u_2 \in H^{2m+\ell}_{\beta'}(e^{-\gamma t}, K_\infty)$.

From this equality it follows that

$$\begin{aligned} S_{0,1} &= u_{t^h} - \sum_{k=2}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^h} - (h-1)c_{t^{h-1}}(S_0)_t \\ &= c_{t^{h-1}}(S_0)_t + dS_0 + u_2. \end{aligned} \tag{3.11}$$

Now differentiate the equality (3.6) $(h-1)$ times by t. As a result we obtain

$$u_{t^{h-1}} = \sum_{k=0}^{h-1} \binom{h-1}{k} (S_0)_{t^k} + (u_1)_{t^{h-1}}. \tag{3.12}$$

We rewrite (3.12) in the form

$$S_{0,2} = u_{t^{h-1}} - \sum_{k=1}^{h-1} \binom{h-1}{k} c_{t^{h-k-1}}(S_0)_{t^k} = c_{t^{h-1}}S_0 + u_{t^{h-1}}. \tag{3.13}$$

Then

$$(S_{0,2})_t = u_{t^h} - \sum_{k=1}^{h-1} \binom{h-1}{k} \left[c_{t^{h-k}}(S_0)_{t^k} + c_{t^{h-k-1}}(S_0)_{t^{k+1}}\right]$$
$$= u_{t^h} - \sum_{k=1}^{h} \binom{h}{k} c_{t^{h-k}}(S_0)_{t^k} + c_{t^{h-1}}(S_0)_t.$$

From this equality and (3.10) we obtain

$$(S_{0,2})_t = c_{t^{h-1}}(S_0)_t + dS_0 + u_2.$$

Put $S_1 = S_0^{-1}(u_1)_{t^{h-1}}$, $S_2 = S_0^{-1}u_2 - S_0^{-2}(S_0)_t(u_1)_{t^{h-1}}$. It is easy to check that

$$S_0^{-1}S_{0,2} = c_{t^{h-1}} + S_1, \quad (S_0^{-1}S_{0,2})_t = d + S_2,$$

It follows that

$$I(t) = c_{t^{h-1}}(t) - c_{t^{h-1}}(0) - \int_0^t d(\tau)d\tau = \int_0^t S_2(x,\tau)d\tau - S_1(x,t) + S_1(x,0).$$

Since $u_{t^{h-1}}$, $u_2 \in H^{2m+\ell,1}_{\beta'}(e^{-\gamma_{h-1}t}, K_\infty)$, S_1, $S_2 \in H^{0,1}_{-\frac{n}{2}}(e^{-\gamma_{h-1}t}, K_\infty)$. Therefore $I(t) \in H^0_{-\frac{n}{2}}(K)$, i.e., $I(t) \equiv 0$. Hence

$$c_{t^h}e^{-\gamma_h t} = de^{-\gamma_h t} \in L_{2,loc}(0,\infty),$$
$$(u_1)_{t^h} = u_2 \in H^{2m+\ell,1}_{\beta'}(e^{-\gamma_h t}, K_\infty).$$

The proof is complete. □

Theorem 3.1. *Let $u(x,t)$ be the generalized solution of the problem* (2.1)-(2.3) *such that $u \equiv 0$ with $|x| > R = \mathrm{const}$ and let $f_{t^k} \in L^\infty(0,\infty; L_2(K))$, $f_{t^k}(x,0) = 0$ with $k \leqslant h$. Assume that the straight lines*

$$\mathrm{Im}\lambda = m - \frac{n}{2} \quad \textit{and} \quad \mathrm{Im}\lambda = 2m - \frac{n}{2}$$

do not contain the points of the spectrum of (3.2)-(3.3) *for every $t \in [0,\infty)$, and in the strip*

$$m - \frac{n}{2} < \mathrm{Im} < 2m - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of the problem (3.2)-(3.3). *Then the following representation holds*

$$u(x,t) = \sum_{s=0}^{m-1} c_s(t) r^{-i\lambda(t)+s} P_{m-1,s}(\ln r) + u_1(x,t), \tag{3.14}$$

where $P_{m-1,s}$ is a polynomial with order less than m and its coefficients are infinitely differentiable functions of (ω,t), $(c_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H^{2m,1}_0(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h$.

Proof. First we will prove that if

$$m - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + m_0 - \frac{n}{2}\,, \quad 1 \leqslant m_0 \leqslant m,$$

then

$$u(x,t) = \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)+s} P_{m_0-1,s}(\ln r) + u_1(x,t), \tag{3.15}$$

where $P_{m_0-1,s}$ is a polynomial with order less than m_0 and its coefficients are infinitely differentiable functions of (ω,t), $(c_s)_{t^k} \in L_2(0,\infty)$ and

$$(u_1)_{t^k} \in H^{2m,1}_{m-m_0}(e^{-\gamma_k t}, K_\infty) \quad \text{for } k \leqslant h.$$

We introduce the notation

$$L_1 = (-1)^{m-1}\big[L_0(0,t,D) - L(x,t,D)\big].$$

From the system (2.1) we get

$$(-1)^{m-1} L_0(0,t,D)u = F, \tag{3.16}$$

where $F = u_t + f + L_1 u$.

From Lemma 3.1 of [5] it follows that $u \in H^{2m,1}_m(e^{-\gamma t}, K_\infty)$. On the other hand, $u_t \in L_2(K_\infty)$, $f \in L^\infty(0,\infty; L_2(K))$. Therefore $F \in H^{0,0}_{m-1}(e^{-\gamma t}, K_\infty)$.

Let $m - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + 1 - \frac{n}{2}$. From Lemma 3.2 it follows that

$$u(x,t) = c(t) r^{-i\lambda(t)} \phi(\omega,t) + u_1(x,t), \tag{3.17}$$

where ϕ is an infinitely differentiable function of (ω,t) which is independent of the solution $c_{t^h} e^{-\gamma_h t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H^{2m,1}_{m-1}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h$. So (3.15) is proved for $m_0 = 1$.

Assume that (3.15) holds for $m_0 \leqslant m-1$. We distinguish the following cases.

Case 1: $m - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + m_0 - \frac{n}{2}$.

Using the induction hypothesis we obtain (3.15). Put

$$S_{m_0} = (-1)^m \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)+s} P_{m_0-1,s}(\ln r). \tag{3.18}$$

Then

$$LS_{m_0} = F_1(x,t) + \sum_{j+s \leqslant m_0} \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)-2m+s+j} \widetilde{P}_{m_0-1.s,j}(\ln r), \tag{3.19}$$

where $(F_1)_{t^k} \in H^{0,0}_{m-m_0-1}(e^{-\gamma_k t}, K_\infty)$ when $0 \leqslant k \leqslant h$ and $\widetilde{P}_{m_0-1,s,j}$ is a polynomial with order less than m_0 and its coefficients are infinitely differentiable functions of (ω, t). From (3.15), (3.16) and (3.19) we obtain

(3.20)

$$(-1)^{m-1} L_0(0,t,D)u_1 = F_2(x,t) + \sum_{j+s\leqslant m_0} \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)-2m+s+j} P_{m_0-1,s,j}(\ln r),$$

where $F_2 = u_t + f + L_1 u_1 + F_1 \in H^{0,0}_{m-m_0-1}(e^{-\gamma t}.K_\infty)$.
By virtue of Lemma 3.1 there exists a function

$$\omega_1 = \sum_{j+s\leqslant m_0} \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)+s+j} P_{m_0,s,j}(\ln r) \tag{3.21}$$

such that

$$(-1)^{m-1} L_0(0,t,D)\omega_1 = \sum_{j+s\leqslant m_0} \sum_{s=0}^{m_0-1} c_s(t) r^{-i\lambda(t)-2m+s+j} P_{m_0-1,s,j}(\ln r). \tag{3.22}$$

Put $v_1 = u_1 - \omega_1$. From (3.20) and (3.22) it follows that

$$(-1)^{m-1} L_0(0,t,D)v_1 = F_2(x,t).$$

By means of Lemma 3.2 we obtain

$$v_1(x,t) = c(t) r^{-i\lambda(t)} \varphi(\omega,t) + u_2(x,t), \tag{3.23}$$

where φ is an infinitely differentiable function of (ω, t) which is the independent of the solution, $c_{t^k} e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_2)_{t^k} \in H^{2m,1}_{m-m_0-1}(e^{-\gamma_k t}, K_\infty)$ when $0 \leqslant k \leqslant h$.

From (3.21) and (3.22) it follows that

$$u_1(x,t) = c(t) r^{-i\lambda(t)} \varphi(\omega,t) \sum_{j+s\leqslant m_0} \sum_{s=0}^{m_0-1} \Big(c_s(t) r^{-i\lambda(t)+s+j} P_{m_0,s,j}(\ln r) \Big) + u_2(x,t).$$

Hence from (3.15) we get

$$u(x,t) = \sum_{s=0}^{m_0} \widetilde{c}_s(t) r^{-i\lambda(t)+s} \widetilde{P}_{m_0,s}(\ln r) + u_2(x,t), \tag{3.24}$$

where $\widetilde{P}_{m_0,s}$ is a polynomial with order less than $m_0 + 1$ and its coefficients are infinitely differentiable functions of (ω, t), $(\widetilde{c}_s)_{t^k} e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_2)_{t^k} \in H^{2m,1}_{m-m_0-1}(e^{-\gamma_k t}, K_\infty)$ when $0 \leqslant k \leqslant h$.

Case 2: $m + m_0 - \dfrac{n}{2} < \mathrm{Im}\lambda < m + m_0 + 1 - \dfrac{n}{2}$.

Since in the strip $m - \dfrac{n}{2} \leqslant \mathrm{Im}\lambda, m + m_0 - \dfrac{n}{2}$ there does not exist any eigenvalue of problem (3.2)-(3.3), $u \in H^{2m,1} m - m_0(e^{-\gamma t}, K_\infty)$.

On the other hand,

$$m + m_0 - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + m_0 + 1 - \frac{n}{2}\cdot$$

Therefore, from Lemma 3.2 it follows that

$$u(x,t) = c(t)r^{-i\lambda(t)}\varphi(\omega,t) + u_1(x,t), \tag{3.25}$$

where φ is an infinitely differentiable function of (ω,t) and does not depend on the solution, $c_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H^{2m,1}_{m-m_0-1}(e^{-\gamma t}, K_\infty)$ when $k \leqslant h$.

Case 3: There exists t_0 such that $\mathrm{Im}\lambda(t_0) = m + m_0 - \dfrac{n}{2}$.

One may assume that

$$m + m_0 - \mu - \frac{n}{2} < \mathrm{Im}\lambda(t) < m + m_0 - \mu + 1 - \frac{n}{2}, \quad 0 < \mu < 1.$$

Repeating the arguments of the proof for Case 2 we obtain (3.25) where $(u_1)_{t^k} \in H^{2m,1}_{m-m_0-1+\mu}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h$. Hence we obtain (3.24). Using the above arguments we get (3.15). When $m_0 = m$, we obtain (3.14) from (3.15).

The proof of the theorem is complete. □

Theorem 3.2. *Let $u(x,t)$ be a generalized solution of problem* (2.1)-(2.2) *such that $u \equiv 0$ when $|x| > R = \mathrm{const}$ and let $f_{t^k} \in L^\infty(0,\infty,H^\ell(K))$, $f_{t^k}(x,0) = 0$ when $k \leqslant 2\ell + h$. We suppose that the straight lines*

$$\mathrm{Im}\lambda = m - \frac{n}{2} \quad \textit{and} \quad \mathrm{Im}\lambda = 2m + \ell - \frac{n}{2}$$

do not contain the points of spectrum of problem (3.2)-(3.3) *for every $t \in (0,\infty)$ and in the strip*

$$m - \frac{n}{2} < \mathrm{Im}\lambda < 2m + \ell - \frac{n}{2}$$

there exists only one simple eigenvalue $\lambda(t)$ of (3.2)-(3.3). *Then the next representation is true*

$$u(x,t) = \sum_{s=0}^{\ell+m-1} c_s(t) r^{-i\lambda(t)+s} P_{3\ell+m-1,s}(\ln r) + u_1(x,t), \tag{3.26}$$

where $P_{3\ell+m-1,s}$ is a polynomial with order less than $3\ell + m$ and its coefficients are infinitely differentiable functions of (ω,t), $(c_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H^{2m+\ell,1}_0(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h + \ell$.

Proof. We will use induction on ℓ. From Theorem 3.1 follows the assertion of the theorem for $\ell = 0$.

Assume that the theorem is true for $j \leqslant \ell - 2$.

Case 1: $m - \dfrac{n}{2} < \mathrm{Im}\lambda(t) < 2m + j - \dfrac{n}{2}$.

From the induction hypothesis we obtain

$$u(x,t) = \sum_{s=0}^{j+m-1} c_s(t) r^{-i\lambda(t)+s} P_{3j+m-1,s}(\ln r) + u_1(x,t), \tag{3.27}$$

where $P_{3j+m-1,s}$ is a polynomial with order less than $3j+m$ and its coefficients are infinitely differentiable functions of (ω,t), $(c_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H_0^{2m+j,1}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h+j$.

From (3.16) and (3.27) we find that

$$(-1)^{m-1}L_0(0,t,D)u_1 = F_3 + (-1)^m LS + S_t, \tag{3.28}$$

where $F_3 = (u_1)_t + f + L_1 u_1$ and $S = \sum\limits_{s=0}^{j+m-1} c_s(t) r^{-i\lambda(t)+s} P_{3j+m-1,s}(\ln r)$.

Since $f_{t^k} \in L^\infty(0,\infty; H^{j+1}(K_\infty))$ and $f_{t^k}(x,0) = 0$ with $k \leqslant 2(j+1)+h$ so $f_{t^k} \in L^\infty(0,\infty, H^j(K_\infty))$ and $f_{t^k}(x,0) = 0$ with $k \leqslant 2j+(h+2)$.

It follows that $(c_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$ and $(u_1)_{t^k} \in H_0^{2m+j,1}(e^{-\gamma_k t}, K_\infty)$ with $k \leqslant h+j+2$. Hence it follows that $(F_3)_{t^k} \in L^\infty(0,\infty, H^{j+1}(K))$ when $k \leqslant h+j+1$.

On the other hand,

$$(-1)^m LS + S_k = F_4 + \sum_{s=0}^{j+m} \widetilde{c}_s(t) r^{-i\lambda(t)-2m-1} \widetilde{P}_{3j+m+1,s}(\ln r),$$

where $\widetilde{P}_{3j+m+1,s}$ is a polynomial with order less than $3j+m+2$ and its coefficients are infinitely differentiable functions of (ω,t) with

$$(F_4)_{t^k} \in L^\infty(0,\infty; H^{j+1}(K)), \quad (\widetilde{c}_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$$

when $k \leqslant h_j + 1$. Therefore, from (3.28) we obtain

$$(-1)^{m-1}L_0(0,t,D)u_1 = F_5 + \sum_{s=0}^{j+m} \widetilde{c}_s(t) r^{-i\lambda(t)-2m+s} \widetilde{P}_{3j+m+1,s}(\ln r),$$

where $F_5 = F_3 + F_4 \in L^\infty(0,\infty; H^{j+1}(K)) \subseteq L^\infty(0,\infty; H^j(K))$.

By virtue of Lemma 3.2 and by analogy to the proof of Theorem 3.1 we can find that

$$u_1(x,t) = \sum_{s=0}^{j+m} \widetilde{c}_s(t) r^{-i\lambda(t)+s} \widetilde{P}_{3j+m+2,s}(\ln r) + u_2(x,t), \tag{3.29}$$

where $\widetilde{P}_{3j+m+2,s}$ is a polynomial with order less than $3j+m+3$ and its coefficients are infinitely differentiable functions of (ω,t), $(u_2)_{t^k} \in H_{-1}^{2m+j,1}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h+j+1$.

By virtue of Lemma 3.2 [5] we have $(u_2)_{t^k} \in H_0^{2m+j+1,1}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h+j+1$. Hence and from (3.27) it follows that

$$u(x,t) = \sum_{s=0}^{j+m} c_s(t) r^{-i\lambda(t)+s} P_{3j+m+2,s}(\ln r) + u_2(x,t), \tag{3.30}$$

where $\deg P_{3j+m+2} < 3j+m+3$, $(c_s)_{t^k}e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$ and $(u_2)_{t^k} \in H_0^{2m+j+1,1}(e^{-\gamma_k t}, K_\infty)$ for $0 \leqslant k \leqslant h+j+1$.

Case 2: $2m + j - \dfrac{n}{2} < \mathrm{Im}\lambda(t) < 2m + j + 1 - \dfrac{n}{2}$.

By virtue of Lemma 3.1 and Theorem 3.1, from [5] it follows that

$$u_{t^k} \in H_m^{2m,1}(e^{-\gamma_k t}, K_\infty).$$

On the other hand, in the strip $m - \dfrac{n}{2} \leqslant \mathrm{Im}\lambda \leqslant 2m - \dfrac{m}{2}$ there does not exist the eigenvalue of problem (3.2)-(3.3) for every $t \in (0,\infty)$. Hence, from theorem on smoothness of solution of elliptic problem in conic domain (see [3]) it follows that $u_{t^k} \in H_0^{2m,1}(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h + 2\ell$.

We will prove that if $f_{t^k} \in L^\infty(0,\infty; H^j(K))$ and $f_{t^k}(x,0) = 0$ for $k \leqslant 2j + h$ then

$$u_{t^k} \in H_0^{2m+j,1}(e^{-\gamma_k t}, K_\infty), \quad k \leqslant h + 2l - j.$$

This assertion was proved for $j = 0$. Assume that it is true for $j - 1$. Since $f_{t^k} \in L^\infty(0,\infty; H^{j-1}(K))$ and $f_{t^k}(x,0) = 0$ for $k \leqslant 2(j-1)+h+2$, from inductive hypothesis it follows that $u_{t^k} \in H_0^{2m+j-1,1}(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h + 2\ell - j + 3$. Therefore $u_{t^{k+2}} \in H_0^{2m+j-1,1}(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h + 2\ell - j$. From the fact that the strip

$$2m + j - 1 - \frac{n}{2} \leqslant \mathrm{Im}\lambda \leqslant 2m + j - \frac{n}{2}$$

does not contain the eigenvalues of (3.2)-(3.3) for every $t \in (0,\infty)$, we obtain

$$u_{t^k} \in H_{-1}^{2m+j-1,1}(e^{-\gamma_k t}, K_\infty), \quad k \leqslant h + 2\ell - j.$$

It follows that $u_{t^k} \in H_0^{2m+j,1}(e^{-\gamma_k t}, K_\infty)$ when $k \leqslant h + 2\ell - j$.

Due to Lemma 3.2, from the above arguments we obtain

$$u(x,t) = c(t) r^{-i\lambda(t)} \varphi(\omega,t) + u_1(x,t), \tag{3.31}$$

where φ is an infinitely differentiable function of (ω,t) and independent of the solution, $c_{t^k} e^{-\gamma_k t} \in L_{2,loc}(0,\infty)$, $(u_1)_{t^k} \in H_0^{2m+\ell,1}(e^{-\gamma_k t}, K_\infty)$ for $k \leqslant h + \ell$.

Case 3: There exists t_0 such that $\mathrm{Im}\lambda(t_0) = 2m + \ell - 1 - \frac{n}{2}\cdot$

Similarly to the proof of Theorem 3.1, this case can be easily managed.

The theorem is proved. □

Theorem 3.3. *Let $u(x,t)$ be a generalized solution of problem* (2.1)-(2.3) *such that $u \equiv 0$ when $|x| > R = \mathrm{const}$ and let $f_{t^k} \in L^\infty(0,\infty, H^\ell(K))$, $f_{t^k}(x,0) = 0$ for $k \leqslant 2\ell + h$. Assume that the straight lines*

$$\mathrm{Im}\lambda = m - \frac{n}{2} \quad \textit{and} \quad \mathrm{Im}\lambda = 2m + \ell - \frac{n}{2}$$

do not contain the points of spectrum of problem (3.2)-(3.3) *for every $t \in [0,\infty)$ and in the strip*

$$m - \frac{n}{2} < \mathrm{Im}\lambda < 2m + \ell - \frac{n}{2}$$

exist only simple eigenvalues $\lambda_1(t), \lambda_2(t), \ldots, \lambda_{N_0}(t)$ *of problem* (3.2)-(3.3) *such that*

$$\mathrm{Im}\lambda_1(t) < \cdots < \mathrm{Im}\lambda_{N_0}(t),$$
$$\mathrm{Im}\lambda_j(t) \neq \mathrm{Im}\lambda_k(t) + N, \quad j \neq k; N \in \mathbb{Z}^+; j,k = 1,\ldots,N_0.$$

Furthermore, assume that there exist $T > 0$ *and* $\mu_j^* = \mathrm{const} \geqslant 0$ *such that*

$$m - \frac{n}{2} < \mathrm{Im}\lambda_1(t) < m + \mu_1^* - \frac{n}{2} < \mathrm{Im}\lambda_2(t) < \ldots$$
$$< m + \mu_{N_0-1}^* - \frac{n}{2} < \mathrm{Im}\lambda_{N_0}(t) < 2m + \ell - \frac{n}{2} \quad \forall\, t \in [T, \infty).$$

Then the following representation is true

$$u(x,t) = \sum_{j=1}^{N_0} \sum_{s=0}^{\ell+m-i} c_{s,j}(t) r^{-i\lambda_j(t)+s} P_{3\ell+m-1,s,j}(\ln r) + u_1(x,t), \tag{3.32}$$

where $P_{3\ell+m-1,s,j}$ *is a polynomial with order less than* $3\ell + m$ *and its coefficients are infinitely differentiable functions of* (ω, t), $(c_{s,j})_{t^k} e^{-\gamma_k t} \in L_2(0,\infty)$, $(u_1)_{t^k} \in H_0^{2m+\ell,1}(e^{-\gamma_k t}, K_\infty)$ *when* $k \leqslant h + \ell$.

Proof. For every $t_0 \in [0,\infty)$ there exists $\varepsilon > 0$ such that

$$m + \mu_{j-1} - \frac{n}{2} < \mathrm{Im}\lambda_j(t) < m + \mu_j - \frac{n}{2}\,,$$

for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\mu_j = \mathrm{const} \geqslant 0$, $j = 0, 1, \ldots, N_0$. By the compactness of the interval $[0,T]$, there exist the numbers $T_0 = 0$, $T_1, \ldots, T_{M_1}, T_M = T$ such that

$$m + \mu_{j-1,s} - \frac{n}{2} < \mathrm{Im}\lambda_j(t) < m + \mu_{j,s} - \frac{n}{2}$$

$t \in [T_{s-1}, T_s]$, $\mu_{j,s} = \mathrm{const}$, $j = 1, \ldots, N_0$, $s = 0, \ldots, M$.

Without loss of generality we may assume that

$$m - \frac{n}{2} < \mathrm{Im}\lambda_1(t) < m + \mu_1 - \frac{n}{2} < \mathrm{Im}\lambda_2(t) < \ldots$$
$$< m + \mu_{N_0-1} - \frac{n}{2} < \mathrm{Im}\lambda_{N_0}(t) < 2m + \ell - \frac{n}{2}, \quad t \in [0,T].$$

To prove the theorem we will use induction on N_0. For $N_0 = 1$ the statement of theorem follows from Theorem 3.2. Let this statement be true for $N_0 - 1$. For simplicity we will consider that $\mu_{N_0-1} = \ell_0 < \ell$, $\mu_{N_0-1}^* = \ell_0^* < \ell$.

From the induction hypothesis we obtain that when $t < T$

$$u(x,t) = \sum_{j=1}^{N_0-1} \sum_{s=0}^{\ell_0+m-1} c_{s,j} r^{-i\lambda_j(t)+s} P_{3\ell_0+m-1,s,j}(\ln r) + u_1(x,t), \tag{3.33}$$

where $P_{3\ell_0+m-1,s,j}$ is a polynomial with order less than $3\ell_0 + m$ and its coefficients are infinitely differentiable functions of (ω, t), $(c_{s,j})_{t^k} \in L_2[0,T]$, $(u_1)_{t^k} \in H_0^{2m+\ell_0,1}(e^{-\gamma_k t}, K_T)$ when $k \leqslant h + \ell_0$ and (3.33) is also true when $t > T$ with

$$(u_1)_{t^k} \in H_0^{2m+\ell_0^*,1}(e^{-\gamma_k t}, K_\infty), \quad k \leqslant h + \ell_0^*.$$

Repeating the arguments that are analoguos to the proof of (3.29) we have
(3.34)
$$(-1)^{m-1}L_0(0,t,D)u_1 = \widetilde{F} + \sum_{j=1}^{N_0-1}\sum_{s=0}^{\ell_0+m} \widetilde{c}_{s,j}(t)r^{-i\lambda_j(t)-2m+s}\widetilde{P}_{3\ell_0+m+1,s,j}(\ln r),$$
when $t \leqslant T$, where $\widetilde{F} \in H_0^{\ell_0+1,1}(K_T)$, $\deg \widetilde{P}_{3\ell_0+m+1,s,j} < 3\ell_0+2$, $(\widetilde{c}_{s,j})_{t^k} \in L_2[0,T]$ and when $t > T$ this representation is also true if we substitute ℓ_0 by ℓ_0^*, $\widetilde{F} \in H_0^{\ell_0^*+1,1}(e^{-\gamma t}, K_\infty)$, $(\widetilde{c}_{s,j}) \in L_2[T,\infty)$. Hence it follows that if
$$2m+\ell_1-\frac{n}{2} < \mathrm{Im}\lambda_{N_0}(t) < 2m+\ell_1+1-\frac{n}{2}$$
with $\ell_1 \geqslant \max(\ell_0,\ell_0^*)$ then
$$u_1(x,t) = \sum_{j=1}^{N_0}\sum_{s=0}^{\ell_1+m} \widetilde{c}_{s,j}(t)r^{-i\lambda_j(t)+s}\widetilde{P}_{3\ell_1+m+2,s}(\ln r) + u_2(x,t), \tag{3.35}$$
where $\deg \widetilde{P}_{3\ell_1+m+2,s} < 3\ell_1+m+3$ and its coefficient are infinitely differentiable functions of (ω,t),
$$(u_2)_{t^k} \in H_0^{2m+\ell_1+1,1}(e^{-\gamma_k t}, K_\infty), \quad k \leqslant h+\ell_1.$$

Since in the strip
$$2m+\ell_1+1-\frac{n}{2} \leqslant \mathrm{Im}\,\lambda \leqslant 2m+\ell-\frac{n}{2}$$
there do not exist eigenvalues of problem (3.2)-(3.3), from (3.33), (3.34) and (3.35) we obtain (3.32).

If there exists t_0 such that
$$\mathrm{Im}\,\lambda(t_0) = 2m+\ell_1-\frac{n}{2},$$
then from Lemma 3.2 and by an arguments analogous to the proof of Case 3 of Theorem 3.1 we obtain (3.32). The theorem is proved. □

Now we will formulate the theorem on the asymptotic behavior of generalized solution of the first boundary value problem for the strongly parabolic systems in a bounded domain with a conic point on the boundary.

Using arguments as in the proof of Theorem 3.4 [5], from Theorem 3.3 we obtain the following theorem.

Theorem 3.4. *Let $u(x,t)$ be a generalized solution of problem* (2.1)-(2.3) *and let $f_{t^k} \in L^\infty(0,\infty,H^\ell(\Omega))$, $k \leqslant 2\ell+1$. Assume that the straight lines*
$$\mathrm{Im}\lambda = m-\frac{n}{2} \quad \textit{and} \quad \mathrm{Im}\lambda = 2m+\ell-\frac{n}{2}$$
do not contain the points of the spectrum of problem (3.2)-(3.3) *for every $t \in [0,\infty)$, and in the strip*
$$m-\frac{m}{2} < \mathrm{Im}\lambda < 2m+\ell-\frac{n}{2}$$

do only the simple eigenvalues $\lambda_1(t), \dots, \lambda_{N_0}(t)$ *of problem* (3.2)-(3.3) *exist such that*

$$\begin{aligned}&\mathrm{Im}\lambda_1(t) < \mathrm{Im}\lambda_{N_0}(t),\\ &\mathrm{Im}\lambda_j(t) \neq \mathrm{Im}\lambda_k(t) + N, \quad j \neq k; N \in \mathbb{N}; j,k = 1,\dots,N_0; t \in [0,T].\end{aligned}$$

Furthermore, assume that there exist $T > 0$ *and* $\mu_j^* = \mathrm{const} \geqslant 0$ *such that*

$$\begin{aligned}&m - \frac{n}{2} < \mathrm{Im}\lambda_1(t) < m + \mu_1^* - \frac{n}{2} < \mathrm{Im}\lambda_2(t) < \dots\\ &< m + \mu_{N_0-1}^* - \frac{n}{2} < \mathrm{Im}\lambda_{N_0}(t) < 2m + \ell - \frac{n}{2} \quad \forall\, t \in [T,\infty).\end{aligned}$$

Then the following representation is true in a neighborhood of a conic point

$$u(x,t) = \sum_{j=1}^{N_0} \sum_{s=0}^{\ell+m-1} c_{s,j}(t) r^{-i\lambda_j(t)+s} P_{3\ell+m-1,s,j}(\ln r) + u_1(x,t),$$

where $P_{3\ell+m-1,s,j}$ *is a polynomial with order less than* $3\ell + m$ *and its coefficients are infinitely differentiable functions of* (ω, t), $(c_{s,j})_{t^k} e^{\gamma_k t} \in L_2(0,\infty)$ *and*

$$(u_1)_{t^k} \in H_0^{2m+\ell,1}(e^{-\gamma_k t}, \Omega_\infty) \quad \text{when} \quad k \leqslant h + \ell.$$

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