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THE GROWTH OF COMPOSITE MEROMORPHIC FUNCTIONS WITH DEFICIENT FUNCTIONS

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Abstract. Let f be a transcendental meromorphic function of order ρ_f, g a transcendental entire function of lower order λ_g ($\lambda_g < +\infty$), and $a_i(z)$ ($i = 1, 2, \cdots, n; n \leq \infty$) be entire functions satisfying $T(r, a_i(z)) = \circ(T(r, g))$. If $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$, $\delta(a_i(z), g) > 0$ and $a_i(z) \not\equiv \infty$ for each i, then

$$\overline{\lim_{r\to\infty}} \log(T(r, f(g)))/T(r, g) = \pi\rho_f.$$

1. Introduction

In [8], Song and Huang proved the following result:

Theorem A. *Let f be a meromorphic function and let g be a transcendental entire function with $\sum_{a\neq\infty} \delta(a(z), g) = 1, (T(r, a(z)) = \circ(T(r, g))$. If f and g are of finite order, then*

$$\overline{\lim_{r\to\infty}} \log(T(r, f(g)))/T(r, g) \leq \pi\rho_f.$$

When f is entire, Theorem A is due to [10]. In this paper, we will prove that the above inequality holds as an equality.

Theorem 1. *Let f be a transcendental meromorphic function of order ρ_f, g a transcendental entire function of lower order λ_g ($\lambda_g < +\infty$), and $a_i(z)$ ($i = 1, 2, ..., n;\ n \leq \infty$) entire functions satisfying $T(r, a_i(z)) = \circ(T(r, g))$. If*

$$\sum_{i=1}^{n} \delta(a_i(z), g) = 1,$$

$\delta(a_i(z), g) > 0$ and $a_i(z) \not\equiv \infty$ for each i, then

$$\overline{\lim_{r\to\infty}} \log(T(r, f(g)))/T(r, g) = \pi\rho_f.$$

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2. Lemmas

Lemma 1 ([1]). *Let $f(z)$ be a transcendental meromorphic function, $g(z)$ a transcendental entire function, then for all $\sigma > 1$ and $r > r_0 = r_0(f,g)$ we have*

$$T(r, f(g)) \leq \ AT(BM(\sigma r, g), f) \log M(\sigma r, g),$$

where $A > 0, B > 0$ are constants.

Lemma 2 ([4]). *For any entire function f, if $0 \leq r < R < +\infty$ then we have*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

Lemma 3. *Let g be an entire function of order ρ_g and lower order λ_g ($\lambda_g < +\infty$), let $a_i(z)$ ($i = 1, 2, \cdots, n; n \leq \infty$) be entire functions satisfying $T(r, a_i(z)) = \circ(T(r, g))$. If $\sum\limits_{i=1}^{n} \delta(a_i(z), g) = 1$, $\delta(a_i(z), g) > 0$ and $a_i(z) \not\equiv \infty$ for each i, then*

1. ([3]) *$g(z)$ is of regular growth and $\rho_g = \lambda_g$ is a positive integer.*
2. ([7])
$$\lim_{r\to\infty} T(r, g)/\log M(r, g) = 1/\pi.$$
3. *For an arbitrary small $\varepsilon_1 > 0$, there exists $a_1(z), a_2(z), \cdots, a_k(z)$ such that*

$$\sum_{i=1}^{k} \delta(a_i(z), g) = 1 > 1 - \frac{\varepsilon_1}{2}. \tag{2.1}$$

Let $a_1(z), a_2(z), \cdots, a_h(z)$ ($h \leq k$) be maximal linearly independent group in $a_1(z), a_2(z), \cdots, a_k(z)$. Put

$$L(g) = \begin{vmatrix} g(z) & a_1(z) & a_2(z) & \cdots & a_h(z) \\ g^{(1)}(z) & a_1^{(1)}(z) & a_2^{(1)}(z) & \cdots & a_h^{(1)}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g^{(h)}(z) & a_1^{(h)}(z) & a_2^{(h)}(z) & \cdots & a_h^{(h)}(z) \end{vmatrix}.$$

Then

(i) ([6]) *The order of $L(g)$ is equal to lower order of $L(g)$ and $L(g), g(z)$ have the same order.*

(ii) ([5])

$$K[L(g)] = \varlimsup_{r\to\infty} \frac{N(r, L(g)) + N(r, 1/L(g))}{T(r, L(g))} = 0.$$

Lemma 4 ([2]). *Let $f(z)$ be meromorphic function of lower order λ and order ρ, let P be an integer defined by $P \geq 1, P - \frac{1}{2} \leq \lambda < P + \frac{1}{2}, \rho < P + 1$. If for $A_0 > 0, 0 < \varepsilon \leq 1$, we have*

$$K(f) = \varlimsup_{r\to\infty} \frac{N(r, f) + N(r, 1/f)}{T(r, f)} < \frac{\varepsilon}{A_0(P+1)}.$$

Then, for $1 < \sigma \leq 36$ *and* $r > r_0$*, we have*

$$T(\sigma r, f) = \sigma^P T(r, f)(1 + \eta(r, \sigma)), \quad |\eta(r, \sigma)| < \varepsilon. \tag{2.2}$$

Lemma 5 ([6])**.** *Let* $f(z)$ *be meromorphic function,* $a_i(z)$ $(i = 1, 2, \cdots, k)$ *be distinct meromorphic functions satisfying* $T(r, a_i(z)) = \circ(T(r, f))$*. Let* $\{a_i(z)_{i=1}^h\}$*be maximal linearly independent group of* $\{a_i(z)_{i=1}^k\}$ $(h \leq k)$*. Put*

$$A_0 = \begin{vmatrix} a_1(z) & a_2(z) & \cdots & a_h(z) \\ a_1^{(1)}(z) & a_2^{(1)}(z) & \cdots & a_h^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(h)}(z) & a_2^{(h)}(z) & \cdots & a_h^{(h)}(z) \end{vmatrix} = A(a_1(z), a_2(z), \cdots, a_h(z)),$$

thus

$$L(f) = \frac{(-1)^h}{A_0} A(f, a_1, a_2, \cdots, a_h) = f^{(h)} + \frac{A_1}{A_0} f^{(h-1)} + \cdots + \frac{A_h}{A_0} f. \tag{2.3}$$

Then, the inequality

$$\sum_{i=1}^{h} m(r, \frac{1}{f - a_i(z)}) \leq m\big(r, \frac{1}{L(f)}\big) + \circ(T(r, f)) \tag{2.4}$$

holds outside a set E *of a finite linear measure except in positive real number axis* (mes$E < +\infty$).

Lemma 6. *Let* g *be an entire function of order* ρ *and lower order* λ $(\lambda < +\infty)$*, let* $a_i(z)$ $(i = 1, 2, \cdots, n; n \leq \infty)$ *be entire functions satisfying* $T(r, a_i(z)) = \circ(T(r, g))$*. If* $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$*,* $\delta(a_i(z), g) > 0$ *and* $a_i(z) \not\equiv \infty$ *for each* i*, then*

$$T(\sigma r, g) \sim \sigma^\rho T(r, g) \qquad (r \to \infty,\ 1 < \sigma \leq 36).$$

Proof. 1. Since $a_i(z) \not\equiv \infty$ and $\delta(a_i(z), g) > 0$, we may assume that $a_1(z) \not\equiv \infty$ and $\delta(a_1(z), g) > 0$. By (2.4) we have

$$\sum_{i=1}^{h} m\big(r, \frac{1}{g - a_i(z)}\big) \leq T(r, L(g)) + \circ(T(r, g)), \qquad (r \notin E).$$

So

$$\varliminf_{r\to\infty} \frac{T(r, L(g))}{T(r, g)} \geq \sum_{i=1}^{h} \varliminf_{r\to\infty} \frac{m\big(r, \frac{1}{g - a_i(z)}\big)}{T(r, g)} \geq \delta(a_1(z), g) > 0.$$

Thus, there exists $A \geq 1$ such that for $r > A$ we have

$$T(r, g) < \frac{1}{c_1} T(r, L(g)), \tag{2.5}$$

where $c_1 = \frac{1}{2}\delta(a_1(z), g)$. By (2.4) and (2.5) we get

$$\sum_{i=1}^{h} \varliminf_{r\to\infty} \frac{m\big(r, \dfrac{1}{g - a_i(z)}\big)}{T(r, L(g))} \leq \delta(0, L(g)).$$

Since

$$N(r, L(g)) \leq \sum_{i=1}^{h} N(r, \frac{A_i}{A_0}) + \sum_{i=1}^{h} N(r, g^{(i)}) + \circ(T(r, g)) = \circ(T(r, g)),$$

we have

$$\begin{aligned} T(r, L(g)) =& m(r, L(g)) + N(r, L(g)) \\ \leq& m(r, g) + m\big(r, \frac{L(g)}{g}\big) + \circ(T(r, g)) \\ =& T(r, g) + \circ(T(r, g)) = (1 + \circ(1))T(r, g). \end{aligned} \tag{2.6}$$

2. Using [9, (1.5.8)] we have

$$\begin{aligned} \sum_{i=1}^{h} m\big(r, \frac{1}{g - a_i(z)}\big) &\leq m\Big(r, \sum_{i=1}^{h} \frac{1}{g - a_i(z)}\Big) + \circ(T(r, g)) \\ &\leq T(r, L(g)) - N(r, \frac{1}{L(g)}) + \circ(T(r, g)). \end{aligned}$$

Hence, by (2.1) we obtain

$$\begin{aligned} 1 - \frac{\varepsilon_1}{2} < \sum_{i=1}^{n} \delta(a_i(z), g) &\leq \varliminf_{r\to\infty} \sum_{i=1}^{h} \frac{m(r, \frac{1}{g - a_i(z)})}{T(r, g)} \\ &\leq \varliminf_{r\to\infty} \Big(\frac{T(r, L(g))}{T(r, g)} - \frac{N(r, \frac{1}{L(g)})}{T(r, g)}\Big). \end{aligned}$$

Thus, for any $\varepsilon > 0$ $(\varepsilon > \frac{\varepsilon_1}{2})$, we have

$$\varliminf_{r\to\infty} \frac{T(r, L(g))}{T(r, g)} > 1 - \frac{\varepsilon_1}{2} > 1 - \varepsilon. \tag{2.7}$$

3. Since $g(z)$ is an entire function satisfying the condition $\sum\limits_{i=1}^{n} \delta(a_i(z), g) = 1$ and $\delta(a_i(z), g) > 0$ $(a_i(z) \not\equiv \infty)$, by the first assertion of Lemma 3 we see that $g(z)$ is of regular growth and $\rho = \lambda$ is a positive integer.

By the third of assertion of Lemma 3 we know that $L(g)$ is of regular growth, the order of $L(g)$ is equal to lower of $L(g)$ and $L(g), g(z)$ have the same order.

By Lemma 3 we see that $L(g)$ satisfies the conditions of Lemma 4 where $P = \rho$. Hence, for $0 < \varepsilon < 1,\ 1 < \sigma \leq 36$ and $r > r_0$, by the result in the subsection 2 of this proof we have

$$T(\sigma r, L(g)) = \sigma^{\rho} T(r, L(g))(1 + \eta(r, \sigma)), \quad |\eta(r, \sigma)| < \varepsilon. \tag{2.8}$$

It follows from (2.6)-(2.8) that

$$\begin{aligned}\overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)} &\le \overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(\sigma r,L(g))}\overline{\lim_{r\to\infty}}\frac{T(\sigma r,L(g))}{T(r,L(g))}\overline{\lim_{r\to\infty}}\frac{T(r,L(g))}{T(r,g)}\\ &\le \left(\underline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(\sigma r,L(g))}\right)^{-1}\overline{\lim_{r\to\infty}}\frac{T(\sigma r,L(g))}{T(r,L(g))}\overline{\lim_{r\to\infty}}\frac{(1+\circ(1))T(r,g)}{T(r,g)}\\ &\le \frac{1}{1-\varepsilon}\cdot\sigma^{\rho}(1+\varepsilon)=\frac{1+\varepsilon}{1-\varepsilon}\sigma^{\rho}.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\overline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)} \le \sigma^{\rho}. \tag{2.9}$$

Besides, we observe that

$$\begin{aligned}\underline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)} &\ge \underline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(\sigma r,L(g))}\underline{\lim_{r\to\infty}}\frac{T(\sigma r,L(g))}{T(r,L(g))}\underline{\lim_{r\to\infty}}\frac{T(r,L(g))}{T(r,g)}\\ &\ge \underline{\lim_{r\to\infty}}\frac{\frac{1}{1+\circ(1)}T(\sigma r,L(g))}{T(\sigma r,L(g))}\left(\overline{\lim_{r\to\infty}}\frac{T(r,L(g))}{T(\sigma r,L(g))}\right)^{-1}(1-\varepsilon)\\ &= \left(\overline{\lim_{r\to\infty}}\frac{1}{\sigma^{\rho}(1+\eta(r,\sigma))}\right)^{-1}(1-\varepsilon)\\ &\ge \left(\overline{\lim_{r\to\infty}}\frac{1}{1-|\eta(r,\sigma)|}\right)^{-1}(1-\varepsilon)\sigma^{\rho}\\ &\ge (1-\varepsilon)^2\sigma^{\rho}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\underline{\lim_{r\to\infty}}\frac{T(\sigma r,g)}{T(r,g)} \ge \sigma^{\rho}. \tag{2.10}$$

So, by (2.9) and (2.10) we have

$$T(\sigma r,g)\sim\sigma^{\rho}T(r,g), \qquad (r\to\infty,\ 1<\sigma\le 36).$$

□

Lemma 7 ([1]). *Let g be a transcendental entire function, $\phi(L),\psi(L)$ and $\lambda(L)$ be nondecreasing functions in $L \ge L_0, \phi(L)\to\infty$ and $\lambda(L)\to\infty$ as $L\to\infty$, $\psi(L)>1$. For sufficiently large r, let*

$$L=\log M(r,g), \quad R=r(1+1/\psi(L)).$$

If $\log M(R,g)\le\phi(L)$, $\log r\le\lambda(L)$ hold for all sufficiently large values of L, and

$$l=\overline{\lim_{r\to\infty}}\frac{\lambda(L)\psi(L)[4\log\psi(L)+6\log\phi(L)]}{L}<1,$$

then for any $\varepsilon > 0$ there exists a positive $K = K(g, \phi, \psi, \lambda, \varepsilon)$ such that the equation $g(z) = w$ has roots in circle

$$|z| < t\left(1 + \frac{1}{\psi(\frac{1}{1+\varepsilon}\log M(t,g))}\right),$$

when $|w| > K$. Here $t = t(|w|)$ is determined by $M(t, g) = |w|$.

Lemma 8. *Let g be a transcendental entire function of lower order λ_g ($\lambda_g < +\infty$), let $a_i(z)$ ($i = 1, 2, \cdots, n$; $n \leq \infty$) be entire functions satisfying $T(r, a_i(z)) = \circ(T(r, g))$, $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$ and $\delta(a_i(z), g) > 0$ ($a_i(z) \not\equiv \infty$). Then for any $\delta \in (0, 1)$ there exists a constant $K = K(g, \delta)$ such that the equation $g(z) = w$ has roots in the circle $|z| < t(|w|)(1 + \delta)$ when $|w| > K(g, \delta)$. Here $t = t(|w|)$ is determined by $M(t, g) = |w|$.*

Proof. Let $r = r(L)$ be the inverse of $L = \log M(r, g)$. Put

$$\lambda(L) = \log r(L), \quad \phi(L) = \log M(3r, g),$$

and $\psi(L) = 1/\delta > 1$. For $R = (1 + 1/\psi(L))r = (1 + \delta)r$ we have

$$\log M(R, g) \leq \phi(L),\ \log r = \lambda(L)$$

and $r \to +\infty$ when $L \to +\infty$. By Lemma 2 we obtain

$$T(r, g) \leq \log^+ M(r, g) \leq \log^+ M(3r, g) \leq 7T(4r, g).$$

From Lemma 3 and Lemma 6 we deduce that $\rho_g = \lambda_g$ and

$$T(\sigma r, g) \sim \sigma^{\rho_g} T(r, g) \qquad (r \to \infty, 1 < \sigma < 36). \tag{2.11}$$

Since ρ_g is a positive integer by Lemma 3, we know that $T(r, g)/(\log r)^2 \to \infty$. Thus

$$\begin{aligned} &\lim_{r\to\infty} \frac{\log r \cdot \log(\log M(3r, g))}{\log M(r, g)} \leq \lim_{r\to\infty} \frac{\log r \cdot \log 7T(4r, g)}{T(r, g)} \\ &= \lim_{r\to\infty} \frac{\log r \cdot \ \log[7 \cdot 4^{\rho_g} T(r, g)(1 + \circ(1))]}{T(r, g)} = 0. \end{aligned}$$

Hence

$$\begin{aligned} &\overline{\lim_{L\to\infty}} \frac{\lambda(L)\psi(L)[4\log\psi(L) + 6\log\phi(L)]}{L} \\ &= \overline{\lim_{r\to\infty}}\ \log r \cdot \frac{1}{\delta}[4\log\frac{1}{\delta} + 6\log(\log M(3r, g))]/\log M(r, g) \\ &= 0. \end{aligned}$$

Thus, by Lemma 7 the equation $g(z) = w$ has roots in the circle

$$|z| \leq t\left(1 + \frac{1}{\psi(\frac{1}{1+\delta}\log M(t,g))}\right) = (1 + \delta)t(|w|),$$

for $|w| > K(g, \delta)$. $\square$

Lemma 9. *Let f be a transcendental meromorphic function, g be a transcendental entire function of lower order λ_g ($\lambda_g < +\infty$), let $a_i(z)$ ($i = 1, 2, \cdots, n$; $n \leq \infty$) be entire functions satisfying $T(r, a_i(z)) = \circ(T(r, g))$, $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$ and $\delta(a_i(z), g) > 0$ ($a_i(z) \not\equiv \infty$). Then, for any $\delta \in (0, 1)$ there exists a constant $r_0 > 0$ such that*

$$T(r, f(g)) \geq \frac{\delta}{8} \frac{T\big(M(\frac{r}{1+2\delta}, g), f\big)}{\log M\big(\frac{r}{1+2\delta}, g\big)} \qquad (r > r_0).$$

Proof. Thanks to Nevanlinna's theory we have

$$N(r, \frac{1}{f-a}) \sim T(r, f) \qquad (r \to \infty)$$

outside a set of value a with capacity 0. Without loss of generality, we may suppose that it holds for $a = 0$, that is

$$N(r, 1/f) \sim T(r, f) \qquad (r \to \infty). \tag{2.12}$$

Now, by Lemma 8, for any $\delta \in (0, 1)$ there exists $K = K(g, \delta)$ such that the equation $g(z) = w$ has roots in the circle $|z| < (1 + \delta)t(|w|)$, where t satisfies $M(t, g) = |w|$ with $|w| > K$.

If w_0 is a zero of $f(w)$ in the region $D = \{w : K < |w| \leq M(r, g)\}$, we have $|w_0| > K$. Hence there exists z_0, $|z_0| < (1+\delta)t(|w_0|)$, such that $g(z_0) = w_0$. This implies that z_0 is a zero of $f(g(z))$.

Denote by $n(r)$ (resp., $\overline{n}(r)$) the number of the zeros (resp., distinct zeros) of $f(w)$ in D. Then we have

$$\overline{n}\big((1+\delta)r, 1/f(g)\big) \geq \overline{n}(r) = \overline{n}\big(M(r, g), 1/f\big) - \overline{n}(K, 1/f) \tag{2.13}$$

and

$$n\big((1+\delta)r, 1/f(g)\big) \geq n(r) = n\big(M(r, g), 1/f\big) - n(K, 1/f). \tag{2.14}$$

Consequently, for $\rho > 1$ and $\delta > 0$,

$$N(\rho, \frac{1}{f}) - N(1, \frac{1}{f}) = \int_1^\rho \frac{n(t, 1/f)}{t} dt \leq n(\rho, 1/f) \log \rho \tag{2.15}$$

and

$$N\big((1+2\delta)r, \frac{1}{f(g)}\big) \geq \int_{(1+\delta)r}^{(1+2\delta)r} \frac{n(t, \frac{1}{f(g)})}{t} dt \geq n((1+\delta)r, \frac{1}{f(g)}) \log \frac{1+2\delta}{1+\delta}. \tag{2.16}$$

It follows from (2.14)-(2.16) that

$$
\begin{aligned}
&\frac{1}{\log\dfrac{1+2\delta}{1+\delta}} N\big((1+2\delta)r, \frac{1}{f(g)}\big) \geq n\big((1+\delta)r, \frac{1}{f(g)}\big) \\
&\geq n\big(M(r,g), 1/f\big) - n(K, 1/f) \qquad (2.17)\\
&\geq \frac{N\big(M(r,g),1/f\big)}{\log M(r,g)} - \frac{N(1,1/f)}{\log M(r,g)} - n(K,1/f).
\end{aligned}
$$

Since $\dfrac{x}{1+x} < \log(1+x) < x$ $(x>0)$, for any $\delta \in (0,1)$ we get

$$\frac{4}{3} < \frac{4}{\delta}\frac{\delta}{1+2\delta} < \frac{4}{\delta}\log\frac{1+2\delta}{1+\delta} < \frac{4}{\delta}\frac{\delta}{1+\delta} < 4.$$

By (2.12), for sufficiently large r,

$$N(M(r,g)1/f) > \frac{1}{\dfrac{4}{\delta}\log\dfrac{1+2\delta}{1+\delta}} T(M(r,g),f).$$

Thus

$$\log\frac{1+2\delta}{1+\delta}\frac{N(M(r,g),1/f)}{\log M(r,g)} > \frac{\delta}{4}\frac{T(Mr,g)f)}{\log M(r,g)}. \qquad (2.18)$$

From the first fundamental theorem we see that $T\big((1+2\delta)r, f(g)\big) - T\big((1+2\delta)r, 1/f(g)\big)$ is a bounded quantity.

Hence, for sufficiently large r we have

$$
\begin{aligned}
&\left|T\big((1+2\delta)r, \frac{1}{f(g)}\big) - T\big((1+2\delta)r, f(g)\big)\right| + \left(n(K,1/f) + \frac{N(1,1/f)}{\log M(r,g)}\right)\log\frac{1+2\delta}{1+\delta} \\
&< \frac{\delta}{8}\frac{T(M(r,g),f)}{\log M(r,g)}. \qquad (2.19)
\end{aligned}
$$

From (2.17)-(2.19) we obtain

$$
\begin{aligned}
T\big((1+2\delta)r, f(g)\big) \geq & N\big((1+2\delta)r, \frac{1}{f(g)}\big) + T\big((1+2\delta)r, f(g)\big) - T\big((1+2\delta)r, \frac{1}{f(g)}\big) \\
\geq & \frac{\delta}{8}\frac{T(M(r,g),f)}{\log M(r,g)}.
\end{aligned}
$$

Substituting r for $(1+2\delta)r$, we obtain the desired inequality. □

3. Proof of Theorem 1

For completeness we now give a simple proof of Theorem A. For any small $\varepsilon > 0$ there exists $r_0 > 0$ such that $T(r,f) < r^{\rho_f+\varepsilon}$ for $r > r_0$. By Lemma 1 we get

$$T(r, f(g)) \leq AT(BM(\sigma r, g), f)\log M(\sigma r, g) < A\big(BM(\sigma r,g)\big)^{\rho_f+\varepsilon}\log M(\sigma r, g),$$

hence

$$\overline{\lim_{r\to\infty}} \frac{\log T(r,f(g))}{\log M(\sigma r,g)} \le \rho_f. \tag{3.1}$$

From Lemma 3 we observe that

$$\lim_{r\to\infty} \log M(\sigma r,g)/T(\sigma r,g) = \pi. \tag{3.2}$$

Now for any $\sigma \in (1,36)$, by Lemma 6 we obtain

$$\lim_{r\to\infty} T(\sigma r,g)/T(r,g) = \sigma^{\rho_g}. \tag{3.3}$$

Combining (3.1)-(3.3) we get

$$\overline{\lim_{r\to\infty}} \frac{\log T(r,f(g))}{T(r,g)} = \overline{\lim_{r\to\infty}} \frac{\log T(r,f(g))}{\log M(\sigma r,g)} \cdot \frac{\log M(\sigma r,g)}{T(\sigma r,g)} \cdot \frac{T(\sigma r,g)}{T(r,g)} \le \rho_f \cdot \pi \cdot \sigma^{\rho_g}.$$

Letting $\sigma \to 1$ gives

$$\overline{\lim_{r\to\infty}} \log T(r,f(g))/T(r,g) \le \pi\rho_f. \tag{3.4}$$

Next we prove the converse inequality. For $\varepsilon > 0$ and any $\delta \in (0,1)$, there exists a sequence $\{r_n\}$, $r_n \to \infty$ $(n \to \infty)$, such that

$$T(R_n,f) > R_n^{\rho_f-\varepsilon}, \qquad R_n = M(\frac{r_n}{1+2\delta},g).$$

Now, by Lemma 9,

$$\begin{aligned} T(r_n,f(g)) &\ge \frac{\delta}{8} T(M(\frac{r_n}{1+2\delta},g),f)/\log M(\frac{r_n}{1+2\delta},g) \\ &\ge \frac{\delta}{8}\left(M(\frac{r_n}{1+2\delta},g)\right)^{\rho_f-\varepsilon} / \log M(\frac{r_n}{1+2\delta},g). \end{aligned}$$

Hence

$$\overline{\lim_{r\to\infty}} \frac{\log T(r,f(g))}{\log M(\frac{r}{1+2\delta},g)} \ge \overline{\lim_{n\to\infty}} \frac{\log T(r_n,f(g))}{\log M(\frac{r_n}{1+2\delta},g)} \ge \rho_f. \tag{3.5}$$

By Lemma 3,

$$\overline{\lim_{r\to\infty}} \frac{\log M(\frac{r}{1+2\delta},g)}{T(\frac{r}{1+2\delta},g)} = \pi. \tag{3.6}$$

Since $1 < 1+2\delta < 36$ by Lemma 6, we get

$$
\begin{aligned}
\frac{T(\frac{r}{1+2\delta}, g)}{T(r,g)} &= \frac{T(\frac{r}{1+2\delta}, g)}{T\big((1+2\delta)\frac{r}{1+2\delta}, g\big)} \\
&= \frac{T(\frac{r}{1+2\delta}, g)}{\big(1+2\delta\big)^{\rho_g} T\big(\frac{r}{1+2\delta}, g\big)(1+\circ(1))} \to \frac{1}{\big(1+2\delta\big)^{\rho_g}} \qquad (r\to\infty).
\end{aligned} \tag{3.7}
$$

It follows from (3.5)-(3.7) that

$$
\begin{aligned}
\overline{\lim_{r\to\infty}} \frac{\log T(r, f(g))}{T(r,g)} &= \overline{\lim_{r\to\infty}} \frac{\log T(r, f(g))}{\log M(\frac{r}{1+2\delta}, g)} \frac{\log M(\frac{r}{1+2\delta}, g)}{T(\frac{r}{1+2\delta}, g)} \frac{T(\frac{r}{1+2\delta}, g)}{T(r,g)} \\
&\geq \rho_f \cdot \pi \cdot 1/\big(1+2\delta\big)^{\rho_g}.
\end{aligned}
$$

Letting $\delta \to 0$ we obtain

$$
\overline{\lim_{r\to\infty}} \frac{\log T(r, f(g))}{T(r,g)} \geq \rho_f \pi. \tag{3.8}
$$

The desired conclusion follows from (3.8) and (3.4). □

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