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THE SOLUTION OF ONE CLASS OF DUAL EQUATIONS INVOLVING HANKEL TRANSFORM

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ABSTRACT. The aim of the present work is to propose a method for investigating and solving one class of dual integral equations involving Hankel transfom.

1. INTRODUCTION

Let H_μ and $H'_\mu(\mu \geqslant -1/2)$ be the Zemanian spaces of test and generalized functions, respectively (see [8]). Denote by B_μ the Hankel integral transform defined on H'_μ. It is known that this operator is an automorphism on H'_μ with $B_\mu^{-1} = B_\mu$. For a suitable ordinary function $f(x)$ (for example, $f \in L_1(R_+)$, $R_+ = (0, \infty)$) the operator B_μ is defined by

$$\hat{f}(t) := B_\mu[f](t) := \int_0^\infty \sqrt{xt} J_\mu(xt) f(x) dx, t \in R_+,$$

where$J_\mu(x)$ is the Bessel function of the first kind.

Let $J = (a, b)$ be a certain bounded interval in R_+, $\bar{J} := [a, b]$ and m a non-negative integer number. Consider the following dual integral equation

$$B_\mu[t^{-2m}\hat{u}(t)](x) = f(x), x \in \bar{J}, \tag{1.1}$$

$$u(x) := B_\mu[\hat{u}](x) = 0, x \in R_+ \setminus \bar{J}, \tag{1.2}$$

where $f(x)$ is a given function, $\hat{u}(t)$ is an unknown regular generalized function in H'_μ. The function t^{-2m} is called the symbol of the dual equation (1.1)-(1.2).

We introduce the following definition.

Definition 1.1. Denote by H_μ^{-m} the class of functions $u(x)$ such that $u \in H'_\mu$, $\operatorname{supp} u \subset J$, $t^{-m} B_\mu[u](t) \in L_2(R_+)$.

It is clear that $H_\mu^0 \equiv L_2(J)$. The unknown function $u(x) = B_\mu[\hat{u}](x)$ shall be sought in the class H_μ^{-m}. Note that the case $m = 0$ is trivial. Indeed, substituting in (1) $\hat{u}$ by $B_\mu[u]$, where $u \in L_2(J)$, $\operatorname{supp} u \in J$ we obtain

$$u(x) = f(x), \quad a \leqslant x \leqslant b. \tag{1.3}$$

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In the sequel we shall consider the equation (1.1)-(1.2) only for $m \in N = \{1, 2, \dots\}$. Note that when $a > 0$, one can find examples of the equations (1.1)-(1.2) having an infinite number of solutions belonging to H_μ^{-m} if $m > \mu + 1$. Therefore, we shall make the following assumption

$$m \leqslant \mu + 1 \quad \text{if} \quad a > 0. \tag{1.4}$$

Dual equations of the form (1.1)-(1.2) were considered by many authors (see for example, [2, 3, 4, 6, 7]). Formal solutions of such equations have been given in [6] for $a = 0$ and in [2, 3] for $a > 0$. The validation of the case $a = 0$ may be found in [7]. The case for the symbol $t^{2m}A(t)(A(t) \not\equiv 1)$ was considered in [4].

The aim of the present work is to propose a method for investigating and solving dual equation (1.1)-(1.2). The method is based on the theory of generalized integral transformations [8] and fractional integrals of generalized functions [4].

2. Some auxiliary and integral operators

In the sequel we shall need the following differential operators [4]

$$M_\mu^m = x^{-\mu-1/2}\Big(\frac{d}{dx}\frac{1}{x}\Big)^m x^{m+\mu+1/2}\varphi(x),$$

$$N_\mu^m = x^{m+\mu+1/2}\Big(\frac{1}{x}\frac{d}{dx}\Big)^{-\mu-1/2},$$

where $m \in N$, $\mu \geqslant -1/2$.

Note that the operators M_μ^1, N_μ^1 have been introduced in [8] and denoted there by M_μ, N_μ, respectively. By induction one gets the relations

$$M_\mu^m = \prod_{j=0}^{m-1} M_{\mu+j}, \qquad N_\mu^m = \prod_{j=0}^{m-1} N_{\mu+m-j-1}. \tag{2.1}$$

It is not difficult to show that

$$M_\mu^m[x^{-\mu-m+1/2}P_{m-1}(x^2)] = N_\mu^m[x^{\mu+1/2}P_{m-1}(x^2)], \tag{2.2}$$

where $P_{m-1}(x)$ is an arbitrary polynomial of degree $m-1$.

Using (2.1) and Lemma 5.3.3 in [8] one can prove that M_μ^m (respectively, N_μ^m) is a continuous mapping (an isomorphism) from $H_{\mu+m}$ into H_μ (from H_μ onto $H_{\mu+m}$). These operators may be extended to generalized functions by the equations

$$\langle M_\mu^m f, \varphi\rangle := \langle f, (-1)^m N_\mu^m \varphi\rangle, \quad \varphi \in H_\mu, f \in H'_{\mu+m}, \tag{2.3}$$

$$\langle N_\mu^m f, \varphi\rangle := \langle f, (-1)^m M_\mu^m \varphi\rangle, \quad \varphi \in H_{\mu+m}, f \in H'_\mu, \tag{2.4}$$

where $\langle f, \varphi\rangle$ denotes a value of a generalized function f on a test function φ [4].

Let $D'(J)$ be the space of distributions on the interval J [8] and let $C_0^\infty(J)$ denote the set of infinitely differentiable functions with a support contained in J. For $f \in D'(J)$ the operators M_μ^m and N_μ^m are defined by (2.3) and (2.4), respectively, where φ belongs the set $C_0^\infty(J)$. For the generalized operators M_μ^m, N_μ^m

the relations (2.2) are also valid. By means of these relations and Theorem 5.5.2 in [8] one can establish the following equalities

$$B_\mu M_\mu^m[f](x) = t^m B_{\mu+m}[f](x), \quad f \in H'_{\mu+m}, \tag{2.5}$$

$$N_\mu^m B_\mu[f](x) = B_{\mu+m}[(-t)^m f](x), \quad f \in H'_\mu. \tag{2.6}$$

Let $t^{-m-\mu+1/2}f(t) \in L_1(J)$, $J = (a,b)$. Denote by $N_{\mu,J}^{-m}[f](x)$ the following fractional integral

$$N_{\mu,J}^{-m}[f](x) := \frac{(-1)^m x^{\mu+1/2}}{2^{m-1}\Gamma(m)} \int_x^b f(t) t^{-m-\mu+1/2}(t^2-x^2)^{m-1}dt, \quad x \in J, \tag{2.7}$$

$$N_{\mu,J}^{-0}[f] = f.$$

where $m \in N$, $\Gamma(m)$ is the gamma-function. This operator has the properties:

$$N_\mu^m N_{\mu,J}^{-m}[f](x) = f(x), \tag{2.8}$$

$$N_{\mu,J}^{-m} N_\mu^m[f](x) = (-1)^m f(x) + x^{\mu+1/2} F_{m-1}[f](x^2), \tag{2.9}$$

where

(2.10)

$$F_{m-1}[f](x^2) = \sum_{k=1}^m (-1)^{k-1}\Big[\Big(\frac{1}{x}\frac{d}{dx}\Big)^{m-k} x^{-\mu-1/2} f(x)\Big]_{x=b} \frac{(b^2-x^2)^{m-k}}{2^{m-k}\Gamma(m-k+1)}.$$

We introduce the following function class.

Definition 2.1. Denote by $L_\mu^m(a,b)$ the class of functions $f(x)$ such that $N_\mu^k[f](x) \in C[a,b]$ $(k = 0, 1, \dots, m-1)$, $N_\mu^m[f](x) \in L_2(a,b)$.

In the sequel we shall need the following formula [1]

$$\int_0^\infty J_\mu(xy) J_\nu(ty) y^{\nu-\mu+1} dy = \frac{x^\mu t^{-\nu}(t^2-x^2)^{\nu-\mu-1}}{2^{\nu-\mu-1}\Gamma(\nu-\mu)} \vartheta(t-x), \tag{2.11}$$

where $\mathrm{Re}\nu > \mathrm{Re}\mu > -1$, $\vartheta(x)$ is the Heaviside function.

3. Solution of the dual equation

Suppose that $u(x) \in H_\mu^{-m}(a,b)$, $f(x) \in L_\mu^m(a,b)$. We find the function $u(x)$ in the form

$$u(x) = M_\mu^m v(x), v(x) \in L_2(R_+) \subset H'_{\mu+m}, \tag{3.1}$$

where M_μ^m in general is taken in the sense of generalized functions.

Taking the Hankel transformation B_μ in (3.1), by virtue of (2.5), we have

$$\hat{u}(t) = B_\mu[u](t) = t^m B_{\mu+m}[v](t). \tag{3.2}$$

Substituting for $u(x)$ and $\hat{u}(t)$ from (3.1) and (3.2) in (1.2) and (1.1) respectively, we get

$$B_\mu[t^{-m}B_{\mu+m}[v](t)](x) = f(x), \quad x \in [a,b], \tag{3.3}$$

$$M_\mu^m[v](x) = 0, \quad x \notin [a,b]. \tag{3.4}$$

Applying the operator N_μ^m to the equality (3.3), by virtue of (2.6) we have

$$v(x) = (-1)^m N_\mu^m[f](x), \quad a < x < b. \tag{3.5}$$

From (3.4) it follows

$$v(x) = \begin{cases} \sum\limits_{k=0}^{m-1} a_k x^{2k-m-\mu+1/2}, & 0 < x < a, \\ \sum\limits_{k=0}^{m-1} b_k x^{2k-m-\mu+1/2}, & b < x < \infty, \end{cases} \tag{3.6}$$

where a_k and b_k are arbitrary constants. If $a = 0$ then $a_k = 0$ $(k = 0, 1, \dots, m-1)$. When $a > 0$, according to the condition (1.4) in order $v(x) \in L_2(0,a)$ it is necessary and sufficient that $a_k = 0$ $(k = 0, 1, \dots, m-1)$. Denote by m_0, m_1 the integer numbers defined by

$$m_1 = \begin{cases} \min\Big\{m-1, \dfrac{m+\mu-1}{2} - 1\Big\}, & \text{if } \dfrac{m+\mu-1}{2} \text{ is integer}, \\ \min\Big\{m-1, \Big[\dfrac{m+\mu-1}{2}\Big]\Big\}, & \text{if } \dfrac{m+\mu-1}{2} \text{ is not integer}, \end{cases} \tag{3.7}$$

$$m_0 = \begin{cases} \min\{m_1, \mu-1\}, & \text{if } \mu \text{ is integer}, \\ \min\{m_1, [\mu]\}, & \text{if } \mu \text{ is not integer}. \end{cases} \tag{3.8}$$

In addition, we assume that the function $v(x)$ possesses the property: $v(x) \in L_2(b,\infty)$, $x^{m-\mu-3/2}v(x) \in L_1(b,\infty)$. The set of such functions $v(x)$ is denoted by $V_\mu^m(R_+)$. Thus, we have

$$v(x) = \begin{cases} 0, & 0 < x < a, \\ (-1)^m N_\mu^m[f](x), & a < x < b, \\ \sum\limits_{k=0}^{m_0} b_k x^{2k-m-\mu+1/2}, & b < x < \infty. \end{cases} \tag{3.9}$$

Taking into account (2.7) and (3.9) we can reduce the equation (3.3) to the form

$$N_{\mu,J}^{-m}[v](x) + \frac{x^{\mu+1/2}}{2^{m-1}\Gamma(m)} \sum_{k=0}^{m_0} b_k J_\mu^{m,k}(x^2) = f(x), \quad a \in [a,b], \tag{3.10}$$

where

$$J_\mu^{m,k}(x^2) = \frac{\Gamma(m)}{2b^{2m+2\mu-2k}} \sum_{j=1}^{m} \frac{(-1)^j (b^2-x^2)^{m-j} b^{2j}}{(-m-\mu+k+1)_j \Gamma(m-j+1)}, \tag{3.11}$$

$$(c)_j = c(c+1)...(c+j-1).$$

If $m_0 < 0$ then the sum on the left-hand side of (3.10) is replaced by zero. For determining b_k and conditions putted on the function $f(x)$, substitute for $v(x)$ from (3.5) in (3.10). By virtue of (2.8), (2.9), after some transformations we get

$$\sum_{k=0}^{m_0} b_k J_\mu^{m,k}(x^2) + (-1)^m 2^{m-1}\Gamma(m)F_{m-1}[f](x^2) = 0, \quad x \in [a,b], \tag{3.12}$$

where $F_{m-1}[f](x)$ and $J_\mu^{m,k}(x^2)$ are defined by (2.10) and (3.11), respectively. In the case $m_0 < 0$ it follows from (3.12) that

$$N_\mu[f]^k(b) = 0 \quad (k = 0, 1, \ldots, m-1). \tag{3.13}$$

If $m_0 \geq 0$ then from (3.12) it follows:

$$\sum_{k=0}^{m_0} b_k \frac{b^{2k}}{(-m-\mu+k+1)_j} = (-1)^m 2^j b^{m+\mu-j-1/2} N_\mu^{m-j}[f](b) \tag{3.14}$$

$(j = 1, 2, \ldots, m_0 + 1)$,

$$N_\mu^{m-j}[f](b) = 0 \quad (j = m_0+2, m_0+3, \ldots, m). \tag{3.15}$$

Using the problem 336 in [5] one can show that the constants b_k are one-valued determined from the system (3.13).

Thus, we have proved

Theorem 3.1. *Let $f(x) \in L_\mu^m(a,b)$ and conditions* (1.4), (3.15) *be fulfilled. Then the dual integral equation* (1.1)-(1.2) *has a unique solution $u(x) \in H_\mu^{-m}(a,b)$ defined by the formula* (3.1), *where the function $v(x)$ is given by* (3.9). *The constants b_k are determined by the system* (3.14).

To obtain the structure of the function $u(x)$ we need the following lemma.

Lemma 3.2. *Assume that the function $g(x) \in H_\lambda (x \in R_+)$ has ordinary "derivatives" $\{M_{\lambda-j}^j[g]\}(x)$ almost everywhere up to order k $(j = 0, 1, \ldots, k; \lambda - k > -1/2)$ inclusive, except possibly, for a point $x_0 > 0$. Denote by $\langle\{M_{\lambda-j}^j[g]\}\rangle_{x_0}$ the jump of $\{M_{\lambda-j}^j[g]\}(x)$ at the point x_0:*

$$\langle\{M_{\lambda-j}^j[g]\}\rangle_{x_0} = \{M_{\lambda-j}^j[g]\}(x_0+0) - \{M_{\lambda-j}^j[g]\}(x_0-0).$$

Then the following formula holds

(3.16)

$$M_{\lambda-k}^k[g](x) = \{M_{\lambda-k}^k[g]\}(x) + \sum_{j=0}^{k-1}\langle\{M_{\lambda-j}^j[g]\}\rangle_{x_0} M_{\lambda-k}^{k-j-1}\delta(x-x_0), \quad x \in R_+,$$

where $\delta(x-x_0)$ is the Dirac delta function, $M_{\lambda-j}^j$ is taken in the sense of generalized functions.

Proof. First we prove (3.16) for the case $k = 1$. For every $\varphi(x) \in H_{\lambda-1}(\lambda > 1/2)$ we have

$$\langle M_{\lambda-1}[g], \varphi\rangle = -\langle g, N_{\lambda-1}[g]\rangle$$

$$= -\lim_{\varepsilon\to 0}\Big[\int_0^{x_0-\varepsilon} g(x)x^{\lambda-1/2}\Big(\frac{d}{dx}x^{-\lambda+1/2}\varphi(x)\Big)dx$$

$$+\int_{x_0+\varepsilon}^{\infty} g(x)x^{\lambda-1/2}\Big(\frac{d}{dx}x^{-\lambda+1/2}\varphi(x)\Big)dx\Big]. \tag{3.17}$$

Integrating by parts, taking into account that $\varphi(x) = 0(x^{\lambda-1/2})$ $(x \to +0)$, $\varphi(x) = 0(x^{-\infty})$ $(x \to \infty)$, passing to the limit $(\varepsilon \to +0)$ in (3.17), we have

$$\langle M_{\lambda-1}[g], \varphi\rangle = \langle\{M_{\lambda-1}[g]\}, \varphi\rangle + \langle g\rangle_{x_0}\delta(x - x_0).$$

From here it follows:

$$M_{\lambda-1}[g](x) = \{M_{\lambda-1}[g]\}(x) + \langle g\rangle_{x_0}\delta(x - x_0). \tag{3.18}$$

Now we apply $M_{\lambda-2}$ to (3.18) and use this formula again. Taking (2.1) into account we obtain the formula (3.16) for $k = 2$. Continuing this process, step by step for $M_{\lambda-3}, \dots, M_{\lambda-k}$ we arrive finally to the formula (3.16). The proof of Lemma 3.2 is complete. □

The following theorem gives the structure of the solution of the dual equation (1.1)-(1.2).

Theorem 3.3. *Let the function $v(x)$ defined by* (3.9) *have ordinary "derivatives" $\{M^j_{\mu+m-j}[v]\}(x)$ almost everywhere on R_+ up to order m inclusive, except possibly for the points $x = a$ and $x = b$, and let $\mu > -1/2$. Then the function $u(x)$ defined by* (3.1) *may be represented in the form*

$$u(x) = F(x) + \sum_{j=0}^{m-1}[\alpha_j M_\mu^{m-j-1}\delta(x-a) + \beta_j M_\mu^{m-j-1}\delta(x-b)], \tag{3.19}$$

where

$$F(x) = \begin{cases}(-1)^m\{M_\mu^m N_\mu^m[f]\}(x), & x \in (a, b),\\ 0, & x \in R_+\backslash[a, b],\end{cases} \tag{3.20}$$

$$\alpha_j = (-1)^m\{M^j_{\mu+m-j}N^m_\mu[f]\}(a + 0), \tag{3.21}$$

(3.22)

$$\beta_j = \sum_{k=0}^{m_0} b_k\frac{2^j\Gamma(k+1)}{\Gamma(k+1-j)}b^{-\mu-m-j+2k+1/2} - (-1)^m\{M^j_{\mu+m-j}N^m_\mu[f]\}(b - 0).$$

Proof. Replacing in (3.16) $k = m$, $\lambda = \mu + m$, $g(x) = v(x)$, where the function $v(x)$ is defined by the formula (3.9) and taking into account that

$$M^j_{\mu+m-j}[x^{-\mu-m+2k+1/2}] = \frac{2^j\Gamma(k+1)}{\Gamma(k+1-j)}x^{-\mu-m-j+2k+1/2}$$

we obtain (3.19), where $F(x)$, α_j and β_j are defined by (3.20), (3.21) and (3.22), respectively. The proof of Theorem 3.3 is complete. □

Remark 3.4. If $a = 0$ then in (3.19) it is necessary to put $\alpha_j = 0$ $(j = 0, 1, \dots, m-1)$. If in (3.22) $j \geqslant k+1$, there are absent members corresponding to the set (j, k) by virtue of $\Gamma(-n) = \infty$ $(n = 0, 1, \dots)$.

We now consider some examples.

Example 1. Consider the dual equation

(3.23) $$B_0[t^{-2}\hat{u}(t)](x) = \sqrt{x}(b^2 - x^2) \quad (0 < a \leqslant x \leqslant b),$$

(3.24) $$u(x) := B_0[\hat{u}(t)](x) = 0 \quad (0 < x < a, b < x < \infty).$$

In this case we have $\mu = 0, m = 1$. Obviously $m_0 = -1$ and conditions (1.4), (3.15) are fulfilled. According to formula (3.19) we have

(3.25) $$v(x) = x^{3/2}[\theta(x-a) - \theta(x-b)], \quad 0 < x < \infty,$$

where $\theta(x)$ is the Heaviside step function. According to (3.19)-(3.22) and (3.25) the function $u(x)$ has the form

(3.26)
$$u(x) = 4x^{1/2}[\theta(x-a) - \theta(x-b)] + 2a^{3/2}\delta(x-a) - 2b^{3/2}\delta(x-b), \quad 0 < x < \infty.$$

Using the formulas

$$B_0[\delta(x-c)](t) = \sqrt{ct}J_0(ct)(c > 0) \qquad \text{(see [7])}$$
$$2nz^{-1}J_n(z) - J_{n-1}(z) = J_{n+1}(z) \qquad \text{(see [1])}$$

we can show that

(3.27) $$\hat{u}(t) = B_0[u](t) = 2\sqrt{t}[b^2J_2(bt) - a^2J_2(at)].$$

Using the following propreties of Bessel functions:

$$J\mu(x) = 0(x^\mu)\ (x \to +0), \quad J_\mu(x) = 0(x^{1/2})\ (x \to \infty)$$

one can show that the function $u(x)$ defined by (3.26) belongs to the class $H_0^{-1}(a, b)$. Besides, by means of (2.11) it is not difficult to get the following formula

$$B_0[t^{-2}\hat{u}(t)](x) = \sqrt{x}[(b^2 - x^2)\theta(b-x) - (a^2 - x^2)\theta(a-x)],$$

where $\hat{u}(t)$ is defined by (3.27). This means that equations (3.23), (3.24) are fulfilled.

Example 2. Consider the homogeneous dual equation

(3.28) $$B_0[t^{-4}\hat{u}(t)](x) = 0\,, \quad (0 < a \leqslant x \leqslant b),$$

$$u(x) := B_0[\hat{u}(t)](x) = 0\,, \quad (0 < x < a\,,\ b < x < \infty). \tag{3.29}$$

In this case the condition (1.4) is not fulfilled and we have $m_0 = -1$ (see (3.7), (3.8)). Putting

$$u(x) = M_0^2[v](x), \quad \hat{u}(t) = t^2 B_2[v](t),$$

where $v(x) \in L_2(R_+), x^{1/2}v(x) \in L_1(b, \infty)$ into (3.28), (3.29) we get

$$v(x) = \begin{cases} \alpha x^{1/2}, & 0 \leqslant x \leqslant a, \\ 0, & x > a. \end{cases}$$

Here α is an arbitrary constant. Therefore we have

$$u(x) = -\frac{5\alpha}{2\sqrt{a}}\delta(x-a) - \alpha\sqrt{a}\delta'(x-a),$$

$$B_0[t^{-4}\hat{u}(t)](x) = \alpha\big(\frac{a^2 - x^2}{2} - x^2 \ln\frac{a}{x}\big)\theta(a-x), \quad x > 0.$$

Thus, the homogeneous dual equation (3.28)-(3.29) has an infinite number of solutions in the class $H_0^{-2}(a, b)$.

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