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SOME REMARKS ON THE NORMALITY OF A FAMILY OF HOLOMORPHIC MAPS INTO THE PROJECTIVE SPACE

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Abstract. We study the normality of a family of holomorphic maps from domain in $\mathbf{C}^n$ into the projective space. The main results of the paper is a generalization of a recent result of Z. H. Tu in [Tu].

1. Introduction

Let X and Y be complex manifolds. A family $\mathcal{F}$ of holomorphic maps from X into Y is said to be normal if every sequence in $\mathcal{F}$ contains a subsequence which is either convergent or compactly divergent. In [Al-Kr], there is a generalization of Zalcman criterion for the normality of a family of holormophic maps from a domain in $\mathbf{C}^n$ to a complex manifold. Using this criterion, Z. H. Tu proved the following theorem in [Tu].

Theorem A. *Let $\mathcal{F}$ be a family of holomorphic maps of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that for each $f \in \mathcal{F}$, there exist $q \geqslant 2N+1$ hyperplanes $H_1(f), \ldots, H_q(f)$ (which may depend on f) in $\mathbf{P}^N(\mathbf{C})$ such that f intersects $H_j(f)$ with multiplicity at least m_j $(j = 1, \ldots, q)$, where m_j $(j = 1, \ldots, q)$ are fixed positive integers which are independent of f and may be ∞, with*

$$\sum_{j=1}^{q} \Big(1 - \frac{N}{m_j}\Big) > N + 1$$

and

$$\inf\{D(H_1(f), \ldots, H_q(f)) : \ f \in \mathcal{F}\} > 0.$$

Then $\mathcal{F}$ is a normal family on D.

The first result of this paper (Theorem 3.1) is an analogue of the above theorem where the existence of hyperplanes are replaced by that of hypersurfaces. The proof proceeds along the lines of [Tu]. However we need a technical lemma which says roughly that the degrees of homogeneous polynomials which are defined by hypersurfaces $H_1(f), \ldots, H_{2N+1}(f)$, $f \in \mathcal{F}$, satisfying the second inequality in Theorem A are uniformly bounded.

The next result of the paper (see Proposition 3.3) is motivated by a theorem of Alexander which says that a family of holomorphic maps from the unit ball

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in $\mathbf{C}^n$ to $\mathbf{C}$ is normal if and only if the restriction of it to every complex line through 0 is normal. Namely we prove that a family $\mathcal{F}$ of holomorphic maps of unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ is uniformly Montel at $0 \in \mathbf{B}_n$ if and only if $\mathcal{F}|_l$ is uniformly Montel at $0 \in l \cap \mathbf{B}_n$ for all complex lines through the origin. We conclude the paper by giving a slight generalization of Theorem A on a sufficient condition for the normality of a family of holomorphic maps into $\mathbf{P}^N(\mathbf{C})$.

2. Preliminaries

In this section we first recall some concepts that will be used throughout the paper.

Definition 2.1. Let X and Y be complex manifolds. By $Hol(X,Y)$ we denote the space of holomorphic maps from X to Y equipped with the compact-open topology. A family $\mathcal{F} \subset Hol(X,Y)$ is called normal if every sequence in $\mathcal{F}$ contains a subsequence which either is convergent or compactly divergent in $Hol(X,Y)$.

We recall that a sequence $\{f_j\} \subset Hol(X,Y)$ is said to be compactly divergent if for every compact $K \subset X$ and $L \subset Y$ there exists j_0 such that $f_j(K) \cap L = \emptyset$ for $j > j_0$.

We need the following theorem which is a special case of a result due to Alardo-Krantz (see [Al-Kr]).

Theorem 2.2. *Let $\mathcal{F}$ be a family of holomorphic maps of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Then the family $\mathcal{F}$ is not normal if only if there exist $\{f_j\} \subset \mathcal{F}$, $\{p_j\} \subset D$, $\{r_j\} \subset (0,\infty)$, $\{u_j\} \subset \mathbf{C}^n$ with $p_j \to p_0 \in D$, $r_j \to 0$, $||u_j|| = 1$, such that the sequence of holomorphic maps $g_j(z) = f_j(p_j + r_j u_j z) : \ \Delta_{\sigma_j} \to \mathbf{P}^N(\mathbf{C})$ converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic $g : \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ where $\sigma_j \to \infty$, $\ \Delta_\sigma = \{z \in \mathbf{C} : \ |z| < \sigma\}$.*

Next, we introduce some notations. If $Q_1, \ldots, Q_{N+1}$ are homogeneous polynomial on $\mathbf{C}^{N+1}$ then we set

$$D(Q_1, \ldots, Q_{N+1}) = \inf\{|Q_1(z)|^2 + ... + |Q_{N+1}(z)|^2 : \ ||z|| = 1\}$$

If $H_1, \ldots, H_q$ are hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ ($q \geqslant N+1$) then we let

$$D(H_1, \ldots, H_q) = \prod D(H_{i_1}, \ldots, H_{i_{N+1}}),$$

where the product is taken over all $\{i_1, \ldots, i_{N+1}\}$ with $1 \leqslant i_1 < \cdots < i_{N+1} \leqslant q$ and

$$D(H_{i_1}, \ldots, H_{i_{N+1}}) = \\ = \sup\{D(Q_1, \ldots, Q_{N+1}) : \ Q_j \text{ is defined by } H_{i_j} \text{ for } j = 1, \ldots, N+1\}.$$

Here we say that a polynomial

$$Q(z) = \sum_{|\alpha|=d} a_\alpha z^\alpha$$

is defined by a hypersurface H in $\mathbf{P}^N(\mathbf{C})$ if Q is a homogeneous polynomial of smallest degree d on $\mathbf{C}^{N+1}$ satisfying

$$H = \{(z_0 : \cdots : z_N) \in \mathbf{P}^N(\mathbf{C}) : \ Q(z_0, \dots, z_N) = 0\}$$

and

$$\sum_{|\alpha|=d} |a_\alpha|^2 = 1.$$

We have the following simple remark: If $H_1, \dots, H_q$ ($q \geqslant N+1$) are hypersurfaces in $\mathbf{P}^N(\mathbf{C})$, then $D(H_1, \dots, H_q) > 0$ if and only if $H_{i_1} \cap \cdots \cap H_{i_{N+1}} = \emptyset$ for all $1 \leqslant i_1 < \cdots < i_{N+1} \leqslant q$.

Finally we recall the concept of uniformly Montel in [Tu].

Definition 2.3. Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. $\mathcal{F}$ is said to be uniformly Montel on D if for any $f \in \mathcal{F}$, there exist $2N+1$ hypersurfaces $H_1(f), \dots, H_{2N+1}(f)$ in $\mathbf{P}^N(\mathbf{C})$ such that

$$f(D) \cap H_i(f) = \emptyset \ \forall \ i = 1, \dots, 2N+1$$

and

$$\inf\{D(H_1(f), \dots, H_{2N+1}(f) : \ f \in \mathcal{F}\} > 0.$$

$\mathcal{F}$ is said to be uniformly Montel at a point $z_0 \in D$ if $\mathcal{F}$ is uniformly Montel on some neighbourhood of z_0 in D.

3. Results

The main result of the note is a generalization of Theorem A. More precisely, we have

Theorem 3.1. *Let $\mathcal{F}$ be a family of holomorphic maps of a domain D in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Suppose that for each $f \in \mathcal{F}$, there exist $2N+1$ hypersurfaces $H_1(f), \dots, H_{2N+1}(f)$ in $\mathbf{P}^N(\mathbf{C})$ such that*

(a) *$f(D) \cap H_i(f) = \emptyset$ or $f(D) \subset H_i(f)$ $\forall \ i = 1, \dots, 2N+1$.*

(b) *$\inf\{D(H_1(f), \dots, H_{2N+1}(f)) : \ f \in \mathcal{F}\} > 0$.*

Then $\mathcal{F}$ is a normal family on D.

We need the following lemma which is of independent interest

Lemma 3.2. *For each $\delta > 0$, there exists $M(\delta) > 0$ such that*

$$\max\{\deg Q_1, \dots, \deg Q_q\} < M(\delta)$$

for all homogeneous polynomials $Q_1, \dots, Q_q$ ($q \geqslant n$) on $\mathbf{C}^n$ with

$$D(Q_1, \dots, Q_q) > \delta.$$

Proof. Given an arbitrary normalized homogeneous polynomial on $\mathbf{C}^n$

$$Q(z) = \sum_{|\alpha|=d} a_\alpha z^\alpha$$

with

$$\sum_{|\alpha|=d} |a_\alpha|^2 = 1.$$

Then by writing $\alpha = (\alpha_1, \ldots, \alpha_n)$, $z = (z_1, \ldots, z_n)$, we have

$$|Q(z)|^2 \leqslant \Big(\sum_{|\alpha|=d} |a_\alpha|^2\Big)\Big(\sum_{|\alpha|=d} |z_1|^{2\alpha_1}\cdots|z_n|^{2\alpha_n}\Big) \leqslant \Big(\sum_{|\alpha|=d} |a_\alpha|^2\Big)||z||^{2d}. \tag{1}$$

Hence $|Q(z)| \leqslant 1$ for $||z|| = 1$. From (1) we also have

$$\begin{aligned}|Q(z)|^2 &\leqslant \Big(\sum_{|\alpha|=d} |a_\alpha|^2\Big)\Big(\sum_{|\alpha|=d} |z_1|^{2\alpha_1}\cdots|z_n|^{2\alpha_n}\Big)\\ &\leqslant \sum_{|\alpha|=d} |z_1|^{2\alpha_1}\cdots|z_n|^{2\alpha_n} < (d+1)^n r^{2d} \qquad (2)\end{aligned}$$

for $|z_k| < r,\ \ k = 1, \ldots, n$.

Let $Q_1, \ldots, Q_n$ be arbitrary normalized homogeneous polynomials on $\mathbf{C}^n$ such that $D(Q_1, \ldots, Q_n) > 0$. Then we can find $z^0 \in \mathbf{C}^n$, $||z^0|| = 1$, such that $Q_i(z^0) = 0$ for $i = 2, \ldots, n$. This yields that

$$D(Q_1, \ldots, Q_n) \leqslant |Q_1(z^0)|^2 \leqslant 1. \tag{3}$$

By (3), without loss of generality we may assume $q = n$ and $\deg Q_1 \leqslant \cdots \leqslant \deg Q_n$. Suppose that the lemma is not true. Then there exist homogeneous normalized polynomials $Q_1^j, \ldots, Q_n^j$ such that

$$\begin{aligned}&\inf\{D(Q_1^j, \ldots, Q_n^j):\ j = 1, 2, \ldots\} > \delta > 0,\\ &\deg Q_i^j = d_i \quad \forall\ i = 1, \ldots, k,\\ &\deg Q_i^j = d_i^j \to \infty \text{ as } j \to \infty\ \forall\ i = k+1, \ldots, n,\end{aligned}$$

where $0 \leqslant k \leqslant n-1$. Since $\deg Q_i^j = d_i$ for $i = 1, \ldots, k$ and $j \geq 1$, without loss of generality we may assume that $\{Q_i^j\}_{j\geqslant 1}$ converges uniformly on compact subsets of $\mathbf{C}^n$ to a normalized homogeneous polynomial Q_i for $i = 1, \ldots, k$. Let z^0 with $||z^0|| = 1$ such that $Q_i(z^0) = 0$ for $i = 1, \ldots, k$. For each $j \geqslant 1$, take $z^j = (z_1^j, \ldots, z_n^j)$ with $||z^j|| = 1$ such that $z^j \to z^0$ and $|z_s^j| < 1$ for $s = 1, \ldots, n$. Let $\varepsilon > 0$ be arbitrary. Since $Q_i^j \to Q_i$ uniformly on compact subsets of $\mathbf{C}^n$ and $z^j \to z^0$, there exists j_0 such that

$$\sum_{i=1}^k |Q_i^j(z^{j_0})| < \varepsilon \quad \forall\ j \geqslant j_0.$$

By (2), we have

$$\sum_{i=k+1}^n |Q_i^j(z^{j_0})|^2 < \sum_{i=k+1}^n (d_i^j + 1)^n r^{d_i^j} < \varepsilon \quad \forall\ j \geqslant j_1 \geqslant j_0$$

where $r = \sup\limits_{i=1,\ldots,n} |z_i|$. Hence $\lim\limits_{j\to\infty} D(Q_1^j, \ldots, Q_n^j) = 0$. This is impossible. □

Proof of Theorem 3.1. Suppose that the family $\mathcal{F}$ is not normal. By [Al-Kr] there exist $\{f_j\} \subset \mathcal{F}$, $\{p_j\} \subset D$, $\{r_j\} \subset (0, \infty)$, $\{u_j\} \subset \mathbf{C}^n$ with $p_j \to p_0 \in D$, $r_j \to 0$, $||u_j|| = 1$ such that the sequence of holomorphic maps $g_j(z) = f_j(p_j + r_j u_j z) : \Delta_{\sigma_j} \to \mathbf{P}^N(\mathbf{C})$ converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic $g : \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ where $\sigma_j \to \infty$, $\Delta_\sigma = \{z \in \mathbf{C} : |z| < \sigma\}$. By condition (a) for each f_j, there exist 2N+1 hypersurfaces $H_1^j, \dots, H_{2N+1}^j$ in $\mathbf{P}^N(\mathbf{C})$ such that

$$f_j(D) \cap H_i^j = \emptyset \text{ or } f_j(D) \subset H_i^j \text{ for } i = 1, \dots, 2N+1,$$
$$\inf\{D(H_1^j, \dots, H_{2N+1}^j) : j = 1, 2, \dots\} > t > 0.$$

By passing to a subsequence we may assume that $f_j(D) \cap H_i^j = \emptyset$ for $i = 1, \dots, k$ and $f_j(D) \subset H_i^j$ for $i = k+1, \dots, 2N+1$ $(0 \leqslant k \leqslant 2N+1)$. Let $Q_1^j, \dots, Q_{2N+1}^j$ be $2N+1$ homogeneous polynomials on $\mathbf{C}^{N+1}$ which define $2N+1$ hypersurfaces $H_1^j, \dots, H_{2N+1}^j$ in $\mathbf{P}^N(\mathbf{C})$ such that

$$\inf\{D(Q_1^j, \dots, Q_{2N+1}^j) : j = 1, 2, \dots\} > t > 0.$$

By Lemma 3.2

$$\sup\{\deg Q_i^j : i = 1, \dots, 2N+1; \ j = 1, 2, \dots\} < +\infty.$$

Passing to a subsequence we may assume that $\{Q_i^j\}_{j \geqslant 1}$ converges uniformly on compact subsets of $\mathbf{C}^{N+1}$ to a normalized homogeneous polynomial Q_i for $i = 1, \dots, 2N+1$. Let $H_1, \dots, H_{2N+1}$ be $2N+1$ hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ which are defined by $2N+1$ homogeneous polynomials $Q_1, \dots, Q_{2N+1}$ on $\mathbf{C}^{N+1}$. We have

$$D(Q_1, \dots, Q_{2N+1}) = \lim_{j \to \infty} D(Q_1^j, \dots, Q_{2N+1}^j) > t > 0.$$

Thus $D(H_1, \dots, H_{2N+1}) > 0$, hence $H_1 \cap \dots \cap H_{2N+1} = \emptyset$.

Next, we claim that $g(\mathbf{C}) \subset H_i$ or $g(\mathbf{C}) \cap H_i = \emptyset$ for $1 \leq i \leqslant k$. Otherwise there exist $1 \leqslant i_0 \leqslant k$ and $a \in g^{-1}(H_{i_0})$. Take $\delta > 0$ such that $g_j = (g_0^j : \dots : g_N^j)$, $g = (g_0 : \dots : g_N)$ and $\{g_i^j\}$ converges uniformly on $\Delta(a, \delta) = \{z \in \mathbf{C} : |z - a| < \delta\}$ to g_i as $j \to \infty$ for $i = 0, \dots, N$. From the relation $g(\mathbf{C}) \not\subset H_i$, it follows that $g(\Delta(a, \delta)) \not\subset H_i$. By Hurwitz's theorem, we have $g_j(\Delta(a, \delta)) \cap H_i^j \neq \emptyset$, hence $f_j(D) \cap H_i^j \neq \emptyset$ for j sufficiently large. This proves the claim.

Obviously, we have $g(\mathbf{C}) \subset H_i$ for $i = k+1, \dots, 2N+1$. Hence there exists a subset $I \subset \{1, \dots, 2N+1\}$ such that

$$g(\mathbf{C}) \subset \Big(\bigcap_{i \in I} H_i\Big) \backslash \Big(\bigcup_{i \notin I} H_i\Big).$$

By [No-Wi, Corollary 1.4(ii)], $\Big(\bigcap_{i \in I} H_i\Big) \backslash \Big(\bigcup_{i \notin I} H_i\Big)$ is hyperbolic. Hence g is constant. This is a contradiction. □

Proposition 3.3. *Let $\mathcal{F}$ be a family of holomorphic maps of the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$. Then the following are equivalent*

(i) $\mathcal{F}|_\ell$ *is uniformly Montel at* $0 \in \mathbf{B}_n$ *for every complex line* ℓ *through the origin.*

(ii) $\mathcal{F}|_\ell$ *is normal at* $0 \in \mathbf{B}_n$ *for every complex line* ℓ *through the origin.*

(iii) $\mathcal{F}$ *is normal at* $0 \in \mathbf{B}_n$.

(iv) $\mathcal{F}$ *is uniformly Montel at* $0 \in \mathbf{B}_n$.

Proof. (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii) follow from Theorem A.

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) follow from [Tu, Theorem 4].

(ii) $\Rightarrow$ (iii) Conversely, assume that $\mathcal{F}|_\ell$ is normal at point $0 \in \mathbf{B}_n$ for every complex line ℓ through the origin. Given a sequence $\{f_j\} \subset \mathcal{F}$. We prove that there exists a subsequence of $\{f_j\}$ such that it converges uniformly on some neighbourhood of $0 \in \mathbf{B}_n$. Without loss of generality, we may assume that $f_j(0) \to w^0 = (w_0^0 : \cdots : w_N^0)$ as $j \to \infty$ with $w_0^0 \neq 0$. Write $f_j = (h_0^j : \cdots : h_N^j)$ where $h_0^j, \ldots, h_N^j$ are holomorphic functions on $\mathbf{B}_n$ with $\{z \in \mathbf{B}_n : \ h_0^j(z) = ... = h_N^j(z) = 0\} = \emptyset$ for $j \geqslant 1$. Put

$$W = \{w = (w_0 : \cdots : w_N) \in \mathbf{P}^N(\mathbf{C}) : \ z_0 \neq 0\}.$$

Then W is a neighbourhood of w^0 which is biholomorphic to $\mathbf{C}^N$. We show that exist $r > 0$, and j_0 such that $f_j(\mathbf{B}(0, r)) \subset W$ for $j \geqslant j_0$ where

$$\mathbf{B}(0, r) = \{z \in \mathbf{C}^n : \ ||z|| < r\}.$$

Indeed, in the converse case, by passing to a subsequence we can find a sequence $\{z_j\}$ such that $\{z_j\} \to 0$, and $f_j(z_j) \notin W$ $(h_0^j(z_j) = 0)$ for $j \geqslant 1$. For each $j \geqslant 1$, put $V_j = \{z \in \mathbf{B}_n : \ h_0^j(z) = 0\}$. Then $0 \notin V_j$ for j sufficiently large because $f_j(0) \to w^0 \in W$. On the other hand, since $z_j \in V_j$ then $\varliminf_{j\to\infty} \operatorname{dist}(V_j, 0) = 0$.

Proposition 6.1(b) in [Al] implies that there exists a complex line ℓ through $0 \in B_n$ such that $\varliminf_{j\to\infty} \operatorname{dist}(V_j \cap \ell, 0) = 0$. By passing to a subsequence we can find $\{z_j'\}$ such that $z_j' \to 0$ and $z_j' \in V_j \cap \ell$. It follows that $f_j(z_j') \notin W$ for $j \geqslant 1$. This is impossible because by passing to a sequence we may assume that $f_j|_\ell$ is convergent on a neighbourhood of $0 \in B_n \cap \ell$. Hence, $f_j(\mathbf{B}(0, r)) \subset W$ for j sufficiently large. By Proposition 6.2(b) in [Al] we complete the proof of the theorem. □

By a result of Nochka [No] and the heuristic principle, Tu proved a criterion on the normality of a family of holomorphic maps into $\mathbf{P}^N(\mathbf{C})$ (Theorem A). However we note that this criterion can be formulated in a more general form as follows.

Proposition 3.4. *Let* $\mathcal{F}$ *be a family of holomorphic maps of a domain* D *in* $\mathbf{C}^n$ *into* $\mathbf{P}^N(\mathbf{C})$. *Suppose that for each* $f \in \mathcal{F}$, *there exist* $q(f) \geq 2N+1$ *hyperplanes* $H_1(f), \ldots, H_{q(f)}(f)$ *in* $\mathbf{P}^N(\mathbf{C})$ *and* $m_1(f), \ldots, m_{q(f)}(f)$ *such that* f *intersects*

$H_j(f)$ with multiplicity at least $m_j(f)$ $(j = 1, \ldots, q(f))$ satisfying

$$\sum_{j=1}^{q(f)} \Big(1 - \frac{N}{m_j(f)}\Big) > N + 1$$

and

$$\inf\{D(H_1(f), \ldots, H_{q(f)}(f)) : \ f \in \mathcal{F}\} > 0.$$

Then $\mathcal{F}$ is a normal family on D.

Proof. Without loss of generality we may assume that $m_1(f) \geqslant \cdots \geqslant m_{q(f)}(f)$. Put $q_0 = (N+1)^2 + 1$. First we note that, if $q(f) \geqslant q_0$, then

$$\sum_{j=1}^{q_0} \Big(1 - \frac{N}{m_j(f)}\Big) > N + 1.$$

Indeed, put

$$I(f) = \{1 \leqslant j \leqslant q(f) \ : \ m_j(f) \geqslant N + 1\}.$$

Then

$$\sum_{j=1}^{q_0} \Big(1 - \frac{N}{m_j(f)}\Big) \geqslant \sum_{j=1}^{q_0} \Big(1 - \frac{N}{N+1}\Big) = \frac{q_0}{N+1} > N + 1$$

if card $I(f) \geqslant q_0$ and

$$\sum_{j=1}^{q_0} \Big(1 - \frac{N}{m_j(f)}\Big) \geqslant \sum_{j=1}^{q(f)} \Big(1 - \frac{N}{m_j(f)}\Big) > N + 1$$

if card $I(f) < q_0$. Thus, since

$$D(H_1(f), \ldots, H_{q_0}(f)) \geqslant D(H_1(f), \ldots, H_{q(f)}(f)),$$

we may always assume that $2N + 1 \leqslant q \leqslant q_0$. Suppose that the family $\mathcal{F}$ is not normal. Then we can find $\{f_j\} \subset \mathcal{F}$, $\{p_j\} \subset D$, $\{r_j\} \subset (0, \infty)$, $\{u_j\} \subset \mathbf{C}^n$ with $p_j \to p_0 \in D$, $r_j \to 0$, $||u_j|| = 1$ such that the sequence of holomorphic maps g_j as in Theorem 2.2 converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic $g : \ \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$. By passing to a subsequence we may assume that $q(f_j) = q_1$ with $2N + 1 \leqslant q_1 \leqslant q_0$ for $j \geqslant 1$ and $\{m_k^j = m_k(f_j)\}_{j \geqslant 1}$ are increasing sequences for $k = 1, \ldots, q_1$. Obviously, we may assume that $m_k^j = m_k$ for every $k = 1, \ldots, q_1$. Then, by the proof of Theorem 3.1, g is constant. This is a contradiction. □

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