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# A BOUNDED 2-HYPERCONVEX SPACE FAILING TO HAVE THE FIXED POINT PROPERTY FOR A STRICTLY NON-EXPANSIVE MAP

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ABSTRACT. It was shown in [3] that if $\lambda < 2$, then any bounded $\lambda$-hyperconvex space has the fixed point property for non-expansive maps. In this note we construct an example of a bounded 2-hyperconvex space with the fixed point free for any iteration of a strictly non-expansive map.

## 1. Introduction

Let $X$ be a metric space and let $\lambda \geqslant 1$. Following [3], a subset $A$ of $X$ is said to have the $\lambda$-*intersection property* if for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$ each of radius $r_\alpha$ centered at $x_\alpha \in A$ for $\alpha \in \Lambda$, the condition

$$d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta \quad \text{for every} \quad \alpha, \beta \in \Lambda,$$

implies

$$A \cap \bigcap_{\alpha\in\Lambda} B(x_\alpha, \lambda r_\alpha) \neq \emptyset.$$

We say that a subset $A$ in a metric space $X$ is *convex* if $A$ is an intersection of a family of closed balls. A metric space $X$ is said to be $\lambda$-*hyperconvex* if every non-empty convex set in $X$ has the $\lambda$-intersection property.

Following [1], a matric space $X$ is *hyperconvex* if the whole space $X$ itself has the 1-intersection property.

We recall that a map $f : X \to X$ is *non-expansive* if

$$d(f(x), f(y)) \leqslant d(x, y) \quad \text{for every } x, y \in X,$$

and $f$ is *strictly non-expansive* if

$$d(f(x), f(y)) < d(x, y) \quad \text{for every} \quad x, y \in X \text{ with } x \neq y.$$

It was shown in [2] that if $X$ is a bounded hyperconvex space, then any non-expansive map $f : X \to X$ has a fixed point. This result was extended to the case of $\lambda$-hyperconvexity in [3] as follows.

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**Theorem A** ([3]). *Let $X$ be a bounded $\lambda$-hyperconvex space. If $\lambda < 2$, then any non-expansive map $f : X \to X$ has a fixed point.*

From Theorem A it arises a question whether or not this result holds for $\lambda \geqslant 2$. In this note we show that Theorem $A$ fails for $\lambda = 2$. In fact, we are going to establish the following theorem which is the main result of this note.

**Theorem B.** *There exist a bounded 2-hyperconvex metric space $Q \subset l_\infty$ with* $\operatorname{diam} Q \leqslant 1$*, and a strictly non-expansive map $f : Q \to Q$ such that*

$$\|f^n(x) - x\| > 2^{-1} \quad \textit{for every } x \in Q \textit{ and } n \in \mathbb{N}.$$

Thus, Theorems A and B completely solve the problem on the fixed point property for non-expansive maps in $\lambda$-hyperconvex spaces.

In the next section we will describe the bounded 2-hyperconvex matric space stated in Theorem B. Our example is very elementary and self-contained. In fact, nothing than the definition of the $l_\infty$-space is used in our construction.

## 2. The example

Let $l_\infty$ denote the Banach space of all bounded sequences of real numbers equipped with the sup-norm, that is

$$\|x - y\| = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$$

for every $x = (x_n) \in l_\infty$ and $y = (y_n) \in l_\infty$.

It is well-known that any ball in $l_\infty$ is hyperconvex (see [1]). Let

$$\begin{aligned} Q_1 &= [3/4, 1] \times \{0\} \times \{0\} \times \cdots \subset l_\infty, \\ Q_2 &= \{1\} \times [5/8, 3/4] \times \{0\} \times \{0\} \times \cdots \subset l_\infty. \end{aligned}$$

In general, we define $Q_n \subset l_\infty$ for $n \geqslant 3$ by setting

$$\begin{aligned} Q_n = \{1\} &\times \{3/4\} \times \cdots \times \{2^{-1} + 2^{-n+1}\} \\ &\times [2^{-1} + 2^{-n-1}, 2^{-1} + 2^{-n}] \times \{0\} \times \{0\} \times \cdots \end{aligned}$$

Observe that $Q_n$ is an interval in $l_\infty$, therefore it is hyperconvex. We define $Q \subset l_\infty$ by

$$Q = \bigcup_{n=1}^{\infty} Q_n \subset l_\infty.$$

The space $Q$ will be equipped with the metric induced from the norm of $l_\infty$. We are going to show that $Q$ satisfies the conditions of Theorem B. It is straightforward to check that

$$\|x - y\| \leqslant 1 \quad \text{for every } x, y \in Q.$$

Therefore $\operatorname{diam} Q \leqslant 1$.

Observe that for every $x, y \in Q$, we have $x \in Q_n$ and $y \in Q_m$ for some $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} x &= (1, 3/4, \dots, 2^{-1} + 2^{-n+1}, x_n, 0, 0, \dots), \\ &\quad \text{where } x_n \in [2^{-1} + 2^{-n-1}, 2^{-1} + 2^{-n}] \\ y &= (1, 3/4, \dots, 2^{-1} + 2^{-m+1}, y_m, 0, 0, \dots), \\ &\quad \text{where } y_m \in [2^{-1} + 2^{-m-1}, 2^{-1} + 2^{-m}]. \end{aligned}$$

Clearly, we may assume that $m \geqslant n$. Then the metric of $Q$ induced from the norm of $l_\infty$ is given by the formula

$$\|x - y\| = \begin{cases} |x_n - y_n| & \text{if } m = n, \\ y_{n+1} \in [2^{-1} + 2^{-n-2}, 2^{-1} + 2^{-n-1}] & \text{if } m = n+1, \\ 2^{-1} + 2^{-n-1} & \text{if } m \geqslant n+2. \end{cases} \tag{1}$$

Theorem B will be proved via the following two propositions.

**Proposition 1.** *$Q$ is 2-hyperconvex.*

The proof of Proposition 1 will be given in Section 3 and 4. We first prove the following proposition.

**Proposition 2.** *There exists a strictly non-expansive map $f : Q \to Q$ such that*

$$\|f^n(x) - x\| > 2^{-1} \quad \textit{for every } x \in Q \textit{ and } n \in \mathbb{N}.$$

*Proof.* We define a map $f : Q \to Q$ with the following properties

(i) $\|f(x) - f(y)\| < \|x - y\|$ for every $x, y \in Q$ with $x \neq y$;

(ii) $\|f^n(x) - x\| > 2^{-1}$ for every $x \in Q$ and $n \in \mathbb{N}$.

For every $x \in Q$, we have $x \in Q_n$ for some $n \in \mathbb{N}$. Then

$$x = (1, 3/4, \dots, 2^{-1} + 2^{-n+1}, x_n, 0, 0, \dots),$$

where $x_n \in [2^{-1} + 2^{-n-1}, 2^{-1} + 2^{-n}]$.

We define $f(x) \in Q_{n+1}$ by

$$f(x) = (1, 3/4, \dots, 2^{-1} + 2^{-n}, 2^{-1} + 2^{-n-2} + 2^{-1}(x_n - 2^{-1} - 2^{-n-1}), 0, 0, \dots).$$

Observe that

$$\|f(x) - x\| = 2^{-1} + 2^{-n-2} + 2^{-1}(x_n - 2^{-1} - 2^{-n-1}) > 2^{-1}.$$

Thus, condition (ii) holds for $n = 1$. Now assume that $n \geqslant 2$. By definition, if $x \in Q_m$ then $f^n(x) \in Q_{m+n}$. Since $n \geqslant 2$, from (1) we get

$$\|f^n(x) - x\| = 2^{-1} + 2^{-m-1} > 2^{-1} \quad \text{for every } x \in Q.$$

Consequently, condition (ii) holds.

Let us check (i). Let $x, y \in Q$ with $x \neq y$. Then $x \in Q_n$ and $y \in Q_m$ for some $m, n \in \mathbb{N}$. We may assume that $m \geqslant n$. Observe that if $m = n$ then from (1) we have

$$\|f(x) - f(y)\| = 2^{-1}|x_n - y_n| < |x_n - y_n| = \|x - y\|.$$

Now we assume that $m = n + 1$. Then from (1) we have

$$\|x - y\| = y_{n+1} \in [2^{-1} + 2^{-n-2}, 2^{-1} + 2^{-n-1}].$$

Since $y_{n+1} > 2^{-1}$, we get

$$\begin{aligned}\|f(x) - f(y)\| &= 2^{-1} + 2^{-n-3} + 2^{-1}(y_{n+1} - 2^{-1} - 2^{-n-2}) \\ &= 2^{-2} + 2^{-1}y_{n+1} < y_{n+1} = \|x - y\|.\end{aligned}$$

Finally, we assume that $m \geqslant n + 2$. Then from (1) we have

$$\|x - y\| = 2^{-1} + 2^{-n-1}.$$

It is easy to see that

$$\|f(x) - f(y)\| = 2^{-1} + 2^{-n-2}.$$

Therefore

$$\|f(x) - f(y)\| < \|x - y\|.$$

Consequently, $f$ is a strictly non-expansive and therefore Proposition 2 is proved. $\square$

## 3. Proof of Proposition 1: The first step

The proof of Proposition 1 is devided into several steps. In the first step we prove the following lemma.

**Lemma 1.** *Every convex set $A \subset Q$ is connected.*

We recall that a subset $A$ is convex if $A$ is an intersection of a family of closed balls in $Q$. First we observe that the constructed space $Q$ is a 1-dimensional piece-wise linear set containing no loops. Obviously $Q$ can be ordered by "$\leqslant$".

For $x, y \in Q$ we write

$$[x, y] = \{z \in Q : x \leqslant z \leqslant y\} \quad \text{and} \quad [x, \infty) = \{z \in Q : z \geqslant x\}. \tag{2}$$

We say that a subset $A$ in $Q$ is an *interval* if it is of the form (2). Observe that a closed set $A \subset Q$ is connected if and only if $A$ is an interval.

**Claim 1.** *if $x, y, z \in Q$ and $x \leqslant z \leqslant y$, then $\|x - z\| \leqslant \|x - y\|$.*

*Proof.* Let $x \in Q_n$ and $y \in Q_m$ for $m \geqslant n$. Let $z \in [x, y]$. Then $z \in Q_k$, $n \leqslant k \leqslant m$. Observe that

$$\begin{aligned} x &= (1, 3/4, \dots, 2^{-1} + 2^{-n+1}, x_n, 0, 0, \dots), \\ &\quad \text{where } x_n \in [2^{-1} + 2^{-n-1}, 2^{-1} + 2^{-n}] \\ y &= (1, 3/4, \dots, 2^{-1} + 2^{-m+1}, y_m, 0, 0, \dots), \\ &\quad \text{where } y_m \in [2^{-1} + 2^{-m-1}, 2^{-1} + 2^{-m}] \\ z &= (1, 3/4, \dots, 2^{-1} + 2^{-k+1}, z_k, 0, 0, \dots), \\ &\quad \text{where } z_k \in [2^{-1} + 2^{-k-1}, 2^{-1} + 2^{-k}]. \end{aligned}$$

Consider the following cases

**Cases 1:** $n = k = m$. In this cases we have $x_n \leqslant z_k \leqslant y_m$. Then the claim follows.

**Cases 2:** $n = k < k+1 \leqslant m$. In this case we have

$$x_n \leqslant z_k, \quad x_n, z_k \in [2^{-1} + 2^{-n-1}, 2^{-1} + 2^{-n}].$$

Then

$$\|x - z\| = |z_k - x_n| \leqslant 2^{-n-1} < y_m \leqslant \|x - y\|.$$

**Cases 3:** $n < n+1 = k = m$. Since $z_k \leqslant y_m$, we have

$$\|x - z\| = z_k \leqslant y_m = \|x - y\|.$$

**Cases 4:** $n < n+1 = k < m$. Then from (1) we have

$$\|x - z\| = z_{n+1} \leqslant 2^{-1} + 2^{-n-1} = \|x - y\|.$$

**Cases 5:** $n < n+1 < k \leqslant m$. Then from (1) we have

$$\|x - z\| = \|x - y\| = 2^{-1} + 2^{-n-1}.$$

The claim is proved. □

From Claim 1 we get

**Corollary 1.** *Let $A \subset Q$ be an interval, and let $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \Lambda}$ be a family of balls centered at $x_\alpha \in A$. If $\bigcap\limits_{\alpha \in \Lambda} B(x_\alpha, r_\alpha) \neq \emptyset$, then*

$$A \cap \bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha) \neq \emptyset.$$

*Proof.* Let $A \subset Q$ be an interval, and let $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \Lambda}$ be a family of balls centered at $x_\alpha \in A$ with

$$\bigcap_{\alpha \in \lambda} B(x_\alpha, r_\alpha) \neq \emptyset.$$

Let

$$z \in \bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha).$$

We may assume that $A = [x, y]$ and $z \geqslant y$ (the cases $A = [x, \infty)$ or $z \leqslant x$ are similar). Then

$$x_\alpha \leqslant y \leqslant z \quad \text{for every } \alpha \in \Lambda.$$

From Claim 1 we get

$$\|y - x_\alpha\| \leqslant \|z - x_\alpha\| \leqslant r_\alpha \quad \text{for every } \alpha \in \Lambda.$$

Consequently,

$$A \cap \bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha) \neq \emptyset.$$

The corollary is proved. □

*Proof of Lemma 1.* Let $A$ be a convex set in $Q$. Then

$$A = \bigcap_{i \in I} B(x_i, r_i) \quad \text{for some index set } I.$$

We will show that $A$ is an interval. It suffices to prove that if $x, y \in A$, then $[x, y] \subset A$. Let $x \in Q_n$ and $y \in Q_m$ for $m \geqslant n$. Let $z \in [x, y]$. Then $z \in Q_k$, $n \leqslant k \leqslant m$. Since $x, y \in A$ we have

$$\|x - x_i\| \leqslant r_i \quad \text{and} \quad \|y - x_i\| \leqslant r_i \quad \text{for every } i \in I.$$

We need to show that

$$\|z - x_i\| \leqslant r_i \quad \text{for every } i \in I.$$

Now fix $a \in Q_s \subset Q$ with

$$a = (1, 3/4, \dots, 2^{-1} + 2^{-s+1}, a_s, 0, 0, \dots),$$

where $a_s \in [2^{-1} + 2^{-s-1}, 2^{-1} + 2^{-s}]$, and $r > 0$. It suffices to show that

$$\|x - a\| \leqslant r \quad \text{and} \quad \|y - a\| \leqslant r \text{ implies } \|z - a\| \leqslant r. \tag{3}$$

Observe that

(a) If $s \leqslant n \leqslant k \leqslant m$ or $n \leqslant s \leqslant k \leqslant m$, then from Claim 1 we get

$$\|z - a\| \leqslant \|y - a\| \leqslant r.$$

(b) If $n \leqslant k \leqslant s \leqslant m$ or $n \leqslant k \leqslant m \leqslant s$, then from Claim 1 we get

$$\|z - a\| \leqslant \|x - a\| \leqslant r.$$

The proof of Lemma 1 is complete. □

## 4. Proof of Proposition 1: The second step

By Lemma 1, every convex set $A \subset Q$ is connected, therefore is an interval, i.e., $A$ is of the form (2). We are going to show that every interval $A \subset Q$ has the 2-intersection property.

Let $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$ be a family of balls centered at $x_\alpha \in A$ with

$$\|x_\alpha - x_\beta\| \leqslant r_\alpha + r_\beta \quad \text{for every } \alpha, \beta \in \Lambda. \tag{4}$$

We need to prove that

$$A \cap \bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \neq \emptyset.$$

By Corollary 1 it suffices to show that

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \neq \emptyset. \tag{5}$$

For every $n \in \mathbb{N}$, let

$$\Lambda(n) = \{\alpha \in \Lambda : x_\alpha \in Q_n\}.$$

Then we have

$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda(n).$$

**Lemma 2.** *Assume that*

(i) $\Lambda(n) \neq \emptyset$ *for infinitely many* $n \in \mathbb{N}$, *and*

(ii) $r_\alpha \geqslant 2^{-1}(2^{-1} + 2^{-n-1})$ *for every* $\alpha \in \Lambda(n)$ *and for every* $n \in \mathbb{N}$.

*Then there exists* $n_0 \in \mathbb{N}$ *such that* $r_\alpha \geqslant 2^{-1}(2^{-1} + 2^{-n_0-1})$ *for every* $\alpha \in \Lambda(n)$ *and for every* $n \geqslant n_0$.

*Proof.* Assume on the contrary that the lemma does not hold. Then there exist a sequence $\{n_k\} \subset \mathbb{N}$ and $\alpha(k) \in \Lambda(n_k)$ such that

$$r_{\alpha(k)} < 2^{-1}(2^{-1} + 2^{-n_k-1}) \quad \text{for every } k \in \mathbb{N}.$$

From (4) we get

$$\|x_{\alpha(1)} - x_{\alpha(k)}\| \leqslant r_{\alpha(1)} + r_{\alpha(k)} \quad \text{for every } k \in \mathbb{N}.$$

Therefore

$$\begin{aligned}\|x_{\alpha(1)} - x_{\alpha(k)}\| &\leqslant 2^{-1}(2^{-1} + 2^{-n_1-1}) + 2^{-1}(2^{-1} + 2^{-n_k-1})\\ &= 2^{-1} + 2^{-1}(2^{-n_1-1} + 2^{-n_k-1}),\end{aligned}$$

for every $k \in \mathbb{N}$. On the other hand, since $x_{\alpha(1)} \in Q_{n_1}$ and $x_{\alpha(k)} \in Q_{n_k}$, from (1) we get

$$\|x_{\alpha(1)} - x_{\alpha(k)}\| \geqslant 2^{-1} + 2^{-n_1-1} > 2^{-1} + 2^{-1}(2^{-n_1-1} + 2^{-n_k-1}),$$

for every $n_k \geqslant n_1 + 2$. This contradiction completes the proof of the lemma. $\square$

From Lemma 2 we get the following fact which proves Proposition 2 in a special case.

**Corollary 2.** *If $r_\alpha \geqslant 2^{-1}(2^{-1}+2^{-n-1})$ for every $\alpha \in \Lambda(n)$ and for every $n \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that*

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \supset \bigcup_{k=n_0+1}^{\infty} Q_k. \tag{6}$$

*In particular, $\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \neq \emptyset$.*

*Proof.* If $\Lambda(n) \neq \emptyset$ for infinitely many $n \in \mathbb{N}$, then by Lemma 2 there exists $n_0 \in \mathbb{N}$ such that $r_\alpha \geqslant 2^{-1}(2^{-1}+2^{-n_0-1})$ for every $\alpha \in \Lambda(n)$ and for every $n \geqslant n_0$.

To obtain (6) it suffices to show that

$$B(x_\alpha, 2r_\alpha) \supset Q_k \quad \text{for every } k \geqslant n_0+1 \text{ and for every } \alpha \in \Lambda.$$

In fact, let $\alpha \in \Lambda(n)$. Then $x_\alpha \in Q_n$. For $y \in Q_k$, $k \geqslant n_0+1$, we need to show that

$$\|y - x_\alpha\| \leqslant 2r_\alpha.$$

Assume that

$$y = (1, 3/4, \dots, 2^{-1}+2^{-k+1}, y_k, 0, 0, \dots),$$
$$\text{where } y_k \in [2^{-1}+2^{-k-1}, 2^{-1}+2^{-k}].$$

Consider the following cases.

**Cases 1:** $n \geqslant k$. Then we have

$$\|y - x_\alpha\| \leqslant 2^{-1}+2^{-k-1} \leqslant 2^{-1}+2^{-n_0-2} < 2r_\alpha.$$

**Cases 2:** $n = k-1$. Then we have

$$\|y - x_\alpha\| = y_k \leqslant 2^{-1}+2^{-k} \leqslant 2^{-1}+2^{-n_0-1} \leqslant 2r_\alpha.$$

**Cases 3:** $n \leqslant k-2$. Then by the assumption we have

$$\|y - x_\alpha\| = 2^{-1}+2^{-n-1} \leqslant 2r_\alpha.$$

Consequently, (6) is valid.

If $\Lambda(n) \neq \emptyset$ for only finitely many $n \in \mathbb{N}$, say for $n = n_1, \dots, n_m$, then let

$$n_0 = \max\{n_1, \dots, n_m\}.$$

We are going to show that

$$B(x_\alpha, 2r_\alpha) \supset Q_k$$

for every $k \geqslant n_0+1$ and for every $\alpha \in \Lambda(n_i)$, $i = 1, 2, \dots, m$.

In fact, since $n_i \leqslant n_0 < n_0+1 \leqslant k$, we get for every $y \in Q_k$

$$\|y - x_\alpha\| \leqslant 2^{-1}+2^{-n_i-1} \leqslant 2r_\alpha.$$

Hence

$$y \in B(x_\alpha, 2r_\alpha).$$

Consequently,

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \supset \bigcup_{k\geqslant n_0+1} Q_\alpha.$$

□

Therefore in the remainder of this paper we will assume that

$$r_\alpha < 2^{-1}(2^{-1} + 2^{-n-1}) \quad \text{for at least } \alpha \in \Lambda(n) \text{ and an } n \in \mathbb{N}.$$

Let

$$\begin{aligned} n_0 &= \min\{n : \alpha \in \Lambda(n) \quad \text{and} \quad r_\alpha < 2^{-1}(2^{-1} + 2^{-n-1})\},\\ \Lambda^+(n_0) &= \{\alpha \in \Lambda(n_0) : r_\alpha \geqslant 2^{-1}(2^{-1} + 2^{-n_0-1})\},\\ \Lambda^-(n_0) &= \{\alpha \in \Lambda(n_0) : r_\alpha < 2^{-1}(2^{-1} + 2^{-n_0-1})\},\\ \Lambda^{--}(n_0) &= \{\alpha \in \Lambda^-(n_0) : r_\alpha < 2^{-1}(2^{-1} + 2^{-n_0-2})\}. \end{aligned}$$

**Lemma 3.** *If* $\Lambda^{--}(n_0) \neq \emptyset$ *then*

$$Q_{n_0} \subset \bigcap_{\alpha\in\Lambda\setminus\Lambda^-(n_0)} B(x_\alpha, 2r_\alpha).$$

*Proof.* It suffices to show that

$$Q_{n_0} \subset B(x_\alpha, 2r_\alpha) \quad \text{for every } \alpha \in \Lambda \setminus \Lambda^-(n_0).$$

Assume that $\alpha \in \Lambda(k)$. Then we have

$$\begin{aligned} x_\alpha = (1, 3/4, \dots, 2^{-1} + 2^{-k+1}, x_k, 0, 0, \dots),\\ \text{where } x_k \in [2^{-1} + 2^{-k-1}, 2^{-1} + 2^{-k}]. \end{aligned}$$

For $y \in Q_{n_0}$, we have

$$\begin{aligned} y = (1, 3/4, \dots, 2^{-1} + 2^{-n_0+1}, y_{n_0}, 0, 0, \dots),\\ \text{where } y_{n_0} \in [2^{-1} + 2^{-n_0-1}, 2^{-1} + 2^{-n_0}]. \end{aligned}$$

Consider the following cases.

**Case 1:** $k \leqslant n_0 - 1$. Then by the definition of $n_0$ we have

$$\|y - x_\alpha\| \leqslant 2^{-1} + 2^{-k-1} \leqslant 2r_\alpha.$$

This means $y \in B(x_\alpha, 2r_\alpha)$.

**Case 2:** $k = n_0$. Since

$$\alpha \in \Lambda \setminus \Lambda^-(n_0) = \Lambda^+(n_0), \quad r_\alpha \geqslant 2^{-1}(2^{-1} + 2^{-n_0-1}),$$

we have

$$\|y - x_\alpha\| = \|x_{n_0} - y_{n_0}\| \leqslant 2^{-n_0-1} < 2^{-1} + 2^{-n_0-1} \leqslant 2r_\alpha.$$

So $y \in B(x_\alpha, 2r_\alpha)$.

**Case 3:** $k = n_0 + 1$. Since $\Lambda^{--}(n_0) \neq \emptyset$, there exists $\beta_0 \in \Lambda^{--}(n_0)$ such that
$$r_{\beta_0} < 2^{-1}(2^{-1} + 2^{-n_0-2}).$$
Observe that
$$\|x_{\beta_0} - x_\alpha\| = x_{n_0+1} \leqslant r_{\beta_0} + r_\alpha.$$
Therefore
$$2r_\alpha \geqslant 2x_{n_0+1} - 2r_{\beta_0} > 2x_{n_0+1} - (2^{-1} + 2^{-n_0-2}) \geqslant x_{n_0+1}.$$
Consequently,
$$\|y - x_\alpha\| = x_{n_0+1} \leqslant 2r_\alpha.$$
This means
$$y \in B(x_\alpha, 2r_\alpha).$$

**Case 4:** $k \geqslant n_0 + 2$. For $\beta \in \Lambda^{-}(n_0)$ we have
$$\|x_\beta - x_\alpha\| = 2^{-1} + 2^{-n_0-1} \leqslant r_\alpha + r_\beta.$$
Therefore
$$\begin{aligned} 2r_\alpha &\geqslant 2(2^{-1} + 2^{-n_0-1}) - 2r_\beta \\ &> 2(2^{-1} + 2^{-n_0-1}) - (2^{-1} + 2^{-n_0-1}) \\ &= 2^{-1} + 2^{-n_0-1}. \end{aligned}$$
Consequently,
$$\|y - x_\alpha\| = 2^{-1} + 2^{-n_0-1} \leqslant 2r_\alpha.$$
This means
$$y \in B(x_\alpha, 2r_\alpha).$$

Thus, Lemma 3 is proved. □

**Corollary 3.** *If* $\Lambda^{--}(n_0) \neq \emptyset$ *then*
$$\bigcap_{\alpha \in \Lambda} B(x_\alpha, 2r_\alpha) \neq \emptyset.$$

*Proof.* Since $Q_{n_0}$ is hyperconvex, we have
$$\bigcap_{\alpha \in \Lambda^{-}(n_0)} B(x_\alpha, 2r_\alpha) \neq \emptyset.$$
Then by Corollary 1 we get
$$Q_{n_0} \cap \bigcap_{\alpha \in \Lambda^{-}(n_0)} B(x_\alpha, 2r_\alpha) \neq \emptyset.$$
Let
$$a \in Q_{n_0} \cap \bigcap_{\alpha \in \Lambda^{-}(n_0)} B(x_\alpha, 2r_\alpha).$$

Then by Lemma 3 we have

$$a \in \bigcap_{\alpha \in \Lambda \setminus \Lambda^-(n_0)} B(x_\alpha, 2r_\alpha).$$

Therefore $\alpha \in \bigcap_{\alpha \in \Lambda} B(x_\alpha, 2r_\alpha)$ and Corollary 3 is proved. □

To complete the proof of Proposition 1, it remains to consider the case

$$\Lambda^{--}(n_0) = \emptyset. \tag{7}$$

**Lemma 4.** *Let* $\Lambda^{--}(n_0) = \emptyset$*, If we take*

$$b = (1, 3/4, \dots, 2^{-1} + 2^{-n_0}, 2^{-1} + 2^{-n_0-2}, 0, 0, \dots) \in Q_{n_0+1},$$

*then*

$$b \in \bigcap_{\alpha \in \Lambda} B(x_\alpha, 2r_\alpha).$$

*Proof.* We will show that $b \in B(x_\alpha, 2r_\alpha)$ for every $\alpha \in \Lambda$. Let $\alpha \in \Lambda(k)$. Consider the following cases

**Case 1:** $k \leqslant n_0 - 1$. Then by the definition of $n_0$ we have

$$\|b - x_\alpha\| \leqslant 2^{-1} + 2^{-k-1} \leqslant 2r_\alpha.$$

This means $b \in B(x_\alpha, 2r_\alpha)$.

**Case 2:** $k = n_0$. Since $r_\alpha \geqslant 2^{-1}(2^{-1} + 2^{-n_0-2})$ for every $\alpha \in \Lambda(n_0)$,

$$\|b - x_\alpha\| = 2^{-1} + 2^{-n_0-2} \leqslant 2r_\alpha.$$

This means $b \in B(x_\alpha, 2r_\alpha)$.

**Case 3:** $k = n_0 + 1$. For $\beta \in \Lambda^-(n_0)$ we have

$$\|x_\beta - x_\alpha\| = x_k = x_{n_0+1} \leqslant r_\alpha + r_\beta.$$

Therefore

$$2r_\alpha \geqslant 2x_{n_0+1} - 2r_\beta > 2x_{n_0+1} - (2^{-1} + 2^{-n_0-1}) \geqslant 2^{-1}.$$

Then we have

$$\|b - x_\alpha\| = x_{n_0+1} - 2^{-1} - 2^{-n_0-2} \leqslant 2^{-n_0-2} < 2^{-1} \leqslant 2r_\alpha.$$

This means $b \in B(x_\alpha, 2r_\alpha)$.

**Case 4:** $k \geqslant n_0 + 2$. For $\beta \in \Lambda^-(n_0)$ we have

$$\|x_\beta - x_\alpha\| = 2^{-1} + 2^{-n_0-1} \leqslant r_\alpha + r_\beta.$$

Therefore

$$\begin{aligned} 2r_\alpha &\geqslant 2(2^{-1} + 2^{-n_0-1}) - 2r_\beta \\ &> 2(2^{-1} + 2^{-n_0-1}) - (2^{-1} + 2^{-n_0-1}) \\ &= 2^{-1} + 2^{-n_0-1}. \end{aligned}$$

Then we have

$$\|b - x_\alpha\| \leqslant 2^{-1} + 2^{-n_0-2} < 2^{-1} + 2^{-n_0-1} \leqslant 2r_\alpha.$$

This means $b \in B(x_\alpha, 2r_\alpha)$. Thus, the assertion that $\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \neq \emptyset$ is also true in the case (7).

The proof of Proposition 1 is complete. □

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