ACTA MATHEMATICA VIETNAMICA
Volume 30, Number 1, 2005, pp. 1-13

STOCHASTIC PROCESSES INDEXED BY URBANIK CONVOLUTION ALGEBRAS

NGUYEN VAN THU AND CAO VAN NUOI

Abstract. The aim of the present paper is to study a class of second order stochastic processes indexed by Urbanik convolution algebras. We prove their spectral representation which stands for an analogue of that of processes indexed by hypergroups. Moreover, we show that they can be reduced to $*_\alpha$-correlated processes.

1. Notation and preliminaries

In [1, 2, 5, 6] Lasser, Hösel and Leitner introduced and studied stochastic processes indexed by hypergroups. In particular, they proved spectral representation and considered prediction problems for such processes. For processes indexed by orthogonal polynomial hypergroups their results stand for somewhat distinguishing from those for classical weakly stationary processes.

In what follows we will introduce and study similar stochastic processes which are indexed by Urbanik convolution algebras.

Recall some definitions of Urbanik convolutions. Let $\mathcal{P}$ denote the set of all probability measures (p.m's) on the positive half-line $R_+ = [0, \infty)$ endowed with the weak convergence. We write δ_x for the unit mass at point x and write T_x for the map given by

$$T_x\mu(B) = \mu(\{x^{-1}y : y \in B\})$$

for $x \geqslant 0$, $\mu \in \mathcal{P}$ and $B \in \mathcal{B}(R_+)$, the σ-field of Borel subsets of R_+. Note that if $x = 0$ then $T_0\mu = \delta_0$. Let C_b denote the Banach space of all real bounded continuous functions on R_+ with the supremum norm $\|.\|$.

A commutative and associative $\mathcal{P}$-valued binary operation $\circ$ on $\mathcal{P}$ with δ_0 as the unit element is called a Urbanik convolution (cf. Urbanik [7, 8]), if it is continuous in each variable separately and distributive with respect to convex combinations and maps T_x and if it satisfies the following law of large numbers (LLN): There exists a sequence of positive numbers c_n such that the sequence

Received September 24, 2001; in revised form December 23, 2003 and December 1, 2004.
1991 *Mathematics Subject Classification.* 60G10, 60G12, 60G99.
Key words and phrases. Urbanik convolution, $(\mathcal{P}, \circ)$-correlated process, spectral representation.
The paper is partially supported by the National basic research program in natural science, Vietnam.

$T_{c_n}\delta_1^{\circ n}$ converges to a limit other than δ_0. Here, for $\mu \in \mathcal{P}$, $\mu^{\circ n}$ denotes the n-th power of μ under the operation $\circ$.

The pair $(\mathcal{P}, \circ)$ is called a Urbanik convolution algebra. This notion was introduced by K. Urbanik in the standard paper [7] and studied by many researchers.

We assume throughout the paper that the algebra $(\mathcal{P}, \circ)$ is regular, i.e., it admits a characteristic function $\hat{\mu} \in C_b$ defined by the following properties: The correspondence $\mathcal{P} \ni \mu \longleftrightarrow \hat{\mu}$ is one-to-one, $\hat{\mu}$ is distributive with respect to convex combinations, $(\mu \circ \gamma)^\wedge = \hat{\mu}\hat{\gamma}$, $\widehat{T_x\mu}(t) = \hat{\mu}(tx)$ and the uniform convergence of $\hat{\mu}_n$ to $\hat{\mu}$ on every compact interval is equivalent to the weak convergence of μ_n to μ. The characteristic function is represented as

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\mu(dx), \tag{1.1}$$

where Ω is a continuous kernel which stands for the characteristic function of δ_1.

The limiting measure in (LLN) denoted by σ_κ is called the characteristic measure of $(\mathcal{P}, \circ)$ and (with c_n multiplied by a positive constant if necessary) has the following characteristic function

$$\hat{\sigma}_\kappa(t) = \exp(-t^\kappa), \tag{1.2}$$

where $t \geqslant 0$ and κ is a positive constant called the characteristic exponent of the Urbanik convolution $\circ$.

Now we quote some examples of regular Urbanik convolutions which will be given in terms of the kernel Ω and the characteristic measure σ_κ or its density g_κ.

Example 1.1. α-convolution $*_\alpha$ $(0 < \alpha < \infty)$:

$$\Omega(t) = \exp(-t^\alpha), \kappa = \alpha, \sigma_\kappa = \delta_1.$$

For $\alpha = 1$ we get the ordinary convolution i.e. $*_1 = *$

Example 1.2. Symmetric convolution $*_{1,1}$

$$\Omega(t) = \cos t, \quad \kappa = 2, \quad g_\kappa(x) = \frac{1}{\sqrt{\pi}} \exp\Big(-\frac{x^2}{4}\Big).$$

Example 1.3. Kingman convolution $*_{1,\beta}$ $(\beta = 2(s+1) > 1)$. We have

$$\kappa = 2 \quad \text{and} \quad \Omega(t) = \Lambda_s(t) = \Gamma(s+1)J_s(t)/\Big(\frac{1}{2}t\Big)^s,$$

where J_s is the Bessel function and

$$g_\kappa(x) = 2^{-2s-1}x^{2s+1}\exp(-4^{-1}x^2)/\Gamma(s+1)$$

The limiting case $s = -\frac{1}{2}$ reduces to the symmetric convolution.

Let $\{X_t\}$, $t \in R_+$, be a second-order stochastic process defined on a probability space $(\mathcal{X}, \mathcal{F}, P)$. Let $L^2(P)$ denote the real Hilbert space of all r.v.'s with finite second moments. In the sequel, the elements in $L^2(P)$ will be regarded as equivalence classes such that each class is consisted of r.v.'s which are equal with probability one. Thus, the variables of such a second-order process $\{X_t\}$ can be thought of as elements in $L^2(P)$. Let us denote $m_t = EX_t, t \in R_+$. As usual, the correlation function $R(t,s)$, $t,s \in R_+$, of $\{X_t\}$ is defined as $R(t,s) = E(X_t - m_t)(X_s - m_s)$, $t,s \in R_+$. Further, let $R_0(t) = R(0,t)$, $t \in R_+$.

Given a regular Urbanik convolution $\circ$ we say that $\{X_t\}$ is $(\mathcal{P}, \circ)$-*correlated* if $R_0(t) \in C_b$ and for any $t,s \in R_+$

$$R(t,s) = \int_0^\infty R_0(u)\delta_t \circ \delta_s(du). \tag{1.3}$$

It should be noted that the definition (1.3) of $(\mathcal{P}, \circ)$-correlatedness is analogous to that of K-stationary processes, K being a hypergroup (cf. [1, 2]). Moreover, our concept of $(\mathcal{P}, \circ)$-correlated processes is a slight generalization of the concept of additively correlated r.v.'s which was introduced and studied in [3, 4].

2. Spectral representation of $(\mathcal{P}, \circ)$-correlated processes

Given a Urbanik convolution algebra $(\mathcal{P}, \circ)$ the generalized translation operators (g.t.o) τ^x, $x \in R_+$, are defined on $f \in C_b$ and $x,y \in R_+$ by the formula

$$\tau^x f(y) = \int_0^\infty f(u)\delta_x \circ \delta_y(du).$$

Definition. A real function φ on R_+ is said to be $\circ$-positive definite, if for any $x_1, x_2, \ldots, x_n \in R_+$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in R$,

$$\sum_{i,j=1}^n \lambda_i \lambda_j \tau^{x_i}\varphi(x_j) \geqslant 0.$$

Since every regular Urbanik convolution is a strong regular stochastic convolution in the sense of Vol'kovich [9] it follows from Theorem 1 in [9] the following theorem.

Theorem 2.1. *Let $\circ$ be a regular Urbanik convolution with the kernel Ω. Then, a continous function f, $f(0) = 1$, is $\circ$-positive definite if and only if*

$$f(t) = \hat{\mu}(t), \quad t \in R_+$$

where $\mu \in \mathcal{P}$.

As a simple consequence of Theorem 2.1 we have

Theorem 2.2. *Suppose that f is a continuous function on R_+. Then it is a correlation function corresponding to a $(\mathcal{P}, \circ)$-correlated process $\{X_t\}$ if and only if f is $\circ$-positive definite. Consequently, there exists a unique finite measure ν on*

R_+ such that

$$f(t) = \int_0^\infty \Omega(tx)\nu(dx), \quad t \in R_+. \tag{2.2}$$

In the sequel, ν in (2.2) is named as a spectral measure of $\{X_t\}$. Next, for the simplicity of notation, we will assume in what follows that the mean function m_t, $t \in R_+$, is identically zero.

Given a $(\mathcal{P}, \circ)$-correlated process $\{X_t\}$, $t \in R_+$, let $\mathcal{H}$ denote a closed subspace of $L^2(P)$ spanned by r.v.'s $X_t,\ t \in R_+$. Let ν be the spectral measure associated with $\{X_t\}$. Then we have the following.

Lemma 2.1. *Let $\mathcal{K}$ denote the closed subspace of $L^2(R_+, \nu)$ spanned by functions $\Omega_t(x) = \Omega(tx),\ t, x \in R_+$. Then, for any $\mu, \gamma \in P$ we have $\hat{\mu}, \hat{\gamma} \in \mathcal{K}$ and $\hat{\mu}\hat{\gamma}(\cdot) \in \mathcal{K}$. Therefore, $\hat{\mu}^n(\cdot) \in \mathcal{K}$ for every $n = 1, 2, \ldots$*

Proof. Since for each $\mu \in P$ there exits a sequence $\{\mu_n\} \subset P$ such that each μ_n is supported by a finite number of points in R_+ and μ_n weakly converges to μ, we infer, by (1.1), that

$$\hat{\mu}(t) = \lim_{n\to\infty} \int_0^\infty \Omega(tx)\mu_n(dx), \tag{2.3}$$

where the limit is uniform on every bounded interval.

Observe that each function of the right-hand side of (2.3) belongs to $\mathcal{K}$ and for every $\varepsilon > 0$ there exists $A > 0$ such that $\nu([A, \infty)) < \varepsilon$. Hence and by (2.3), it follows that for $n = 1, 2, \ldots$

$$\begin{aligned}\int_0^\infty |\hat{\mu}(t) - \hat{\mu}_n(t)|^2\nu(dt) &= \int_0^A |\hat{\mu}(t) - \hat{\mu}_n(t)|^2\nu(dt) + \int_A^\infty |\hat{\mu}(t) - \hat{\mu}_n(t)|^2\nu(dt) \\ &\leqslant \int_0^A |\hat{\mu}(t) - \hat{\mu}_n(t)|^2\nu(dt) + 4\varepsilon.\end{aligned}$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \int_0^\infty |\hat{\mu}(t) - \hat{\mu}_n(t)|^2\nu(dt) \leqslant 4\varepsilon,$$

Since ε is arbitrary the last inequality implies that $\hat{\mu}_n(\cdot)$ converges to $\hat{\mu} \in \mathcal{K}$ and, consequently, $\hat{\mu} \in \mathcal{K}$.

The last statement is obvious. Thus the lemma is proved. □

The following lemma seems well-known in literature but for an easy reference we quote its proof here.

Lemma 2.2. *For each finite measure η on R_+ the set of funtions $\gamma_x(t) = \exp(-t^\kappa x^\kappa)$ where $t, x \in R_+$ is linearly dense in $L^2(R_+, \eta)$.*

Proof. It suffices to prove that if $f \in L^2(R_+, \eta)$ and f is orthogonal to all functions $\gamma_x(\cdot)$, $x \in R_+$, then $f(t) = 0 \quad \eta$-everywhere. Indeed, let us denote

$$\tau(dt) = f(t)\eta(dt).$$

Then, for each $x \in R_+$,

$$\int_0^\infty \exp(-x^\kappa t^\kappa)\tau(dt) = 0$$

which, by changing variables $x^\kappa \mapsto u$ and by the uniqueness of Laplace transform for signed measures, implies that $\tau = 0$ and, consequently, $f(t) = 0 \quad \eta$-everywhere. □

Lemma 2.3. $\mathcal{K} = L^2(R_+, \nu)$.

Proof. By virtue of LLN it follows that there exists positive numbers c_n such that for every $x \geq 0$ the sequence $T_{xc_n}\delta_1^{on}$ weakly converges to $T_x\sigma_\kappa$.

By Lemma 2.1, for every $n = 1, 2, \ldots (T_{xc_n}\delta_1^{on})(t)$ belongs to $\mathcal{K}$ which, by virtue of (1.2) and by similar arguments in the proof of Lemma 2.1, implies that, in $L^2(R_+, \nu)$,

$$\lim_{n\to\infty}(T_{xc_n}\delta_1^{on})(t) = \exp(-x^\kappa t^\kappa).$$

Hence all functions $\exp(-x^\kappa t^\kappa)$, $t, x \in R_+$ belong to $\mathcal{K}$, which together with Lemma 2.2 implies that the set $\mathcal{K}$ is linearly dense in $L^2(R_+, \nu)$. But $\mathcal{K}$ is a closed subspace of $L^2(R_+, \nu)$, which shows that $\mathcal{K} = L^2(R_+, \nu)$. □

Lemma 2.4. *The Hilbert spaces $\mathcal{H}$ and $L^2(R_+, \nu)$ are isometrically isomorphic and the isomorphism, denoted by J, is determined by the relation*

$$\mathcal{H} \ni X_u \overset{J}{\longmapsto} \Omega_u(\cdot) \in L^2(R_+, \nu), \quad u \in R_+. \tag{2.4}$$

Proof. We first show that the map

$$J: \quad X_u \longmapsto \Omega_u(\cdot)$$

$u \in R_+$, can be extended to a continuous map on the whole space $\mathcal{H}$.

Accordingly, if $u_1, \ldots, u_k \in R_+$ and $\lambda_1, \ldots, \lambda_k \in R$ we put

$$J\Big(\sum_{j=1}^k \lambda_j X_{u_j}\Big) = \sum_{j=1}^k \lambda_j \Omega_{u_j}(\cdot).$$

By (1.3), (2.2) and by Fubini's Theorem we get

$$\begin{aligned} E\Big(\sum_{j=1}^{k}\lambda_j X_{u_j}\Big)^2 &= \sum_{i,j=1}^{k}\lambda_i\lambda_j R(u_i,u_j) \\ &= \sum_{i,j=1}^{k}\lambda_i\lambda_j \int_0^\infty R_o(t)\delta_{u_i}\circ\delta_{u_j}(dt) \\ &= \sum_{i,j=1}^{k}\lambda_i\lambda_j \int_0^\infty\int_0^\infty \Omega(tx)\nu(dx)\delta_{u_i}\circ\delta_{u_j}(dt) \\ &= \int_0^\infty \sum_{i,j=1}^{k}\lambda_i\lambda_j\Omega(u_i x)\Omega(u_j x)\nu(dx) \\ &= \int_0^\infty \Big|\sum_{j=1}^{k}\lambda_j\Omega(u_j x)\Big|^2\nu(dx) \end{aligned}$$

which together with Lemma 2.3 implies that the map J is well-defined on the set $\mathcal{H}^1$ of linear combinations of $X_u, u \in R_+$. Let $\mathcal{K}^1$ denote a linear subspace of $L^2(R_+,\nu)$ spanned by functions $\Omega_t(x) = \Omega(tx)$, $t, x \in R_+$. Then, by the above equalities, the map J stands for an isomorphism between linear spaces $\mathcal{H}^1$ and $\mathcal{K}^1$. Moreover, J can be extended to an isometric isomorphism between $\mathcal{H}$ and $L^2(R_+,\nu)$. Indeed, suppose that V is an element of $\mathcal{H}$ such that $V = \lim_{n\to\infty} V_n$, where $V_n \in \mathcal{H}^1$. By virtue of the last equalities it follows that

$$\lim_{n,m\to\infty}(V_n - V_m) = \lim_{n,m\to\infty}(JV_n - JV_m) = 0,$$

where the limits are taken in $\mathcal{H}$ and in $L^2(R_+,\nu)$, respectively. Thus, the sequence JV_n, $n = 1, 2, \ldots,$ is a Cauchy sequence. Consequently, there is a function $\xi \in Ł^2(R_+,\nu)$ such that

$$\lim_{n\to\infty} JV_n = \xi.$$

Now, putting $JV = \xi$ and taking into an account the fact that if W_n is another sequence in $\mathcal{H}^1$ converging to V then, by the foregoing arguments, it follows that the sequence JW_n converges in $L^2(R_+,\nu)$. But, since $V = \lim_{n\to\infty} V_n = \lim_{n\to\infty} W_n$ and the map J is an isomorphism between $\mathcal{H}^1$ and $\mathcal{K}^1$, the sequences $\{JV_n\}$ and $\{JW_n\}$ must converge to the same limit. Therefore, the map J is well defined. Hence and by the fact that $\mathcal{H}^1$ and $\mathcal{K}^1$ are dense in $\mathcal{H}$ and $\mathcal{K}$ (see Lemma 2.3), respectively the proof is complete. □

Let $\mathcal{G}$ denote a real Hilbert space with the inner product $< \cdot, \cdot >$ and the norm $\|\cdot\|$. A set function

$$N : \mathcal{B}(R_+) \longrightarrow \mathcal{G}$$

is called an *orthogonal vector-valued measure* (o.v.m) if $N(\emptyset) = 0$, and for any disjoint sets $A_n, n = 1, 2, \ldots$ in $\mathcal{B}(R_+)$ the elements $N(A_n)$ are orthogonal and

$$N\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} N(A_n),$$

where the right-hand side is convergent in $\mathcal{G}$.

It should be noted that for every o.v.m N on $\mathcal{B}(R_+)$ the control measure μ is defined by

$$\mu(E) = \|N(E)\|^2,$$

$E \in \mathcal{B}(R_+)$, is a finite measure on R_+. Moreover, by the usual procedure one can define the following integral with values in $\mathcal{G}$:

$$\int_0^{\infty} f(u) N(du)$$

for every $f \in L^2(R_+, \mu)$.

The following simple lemma seems well-known in literature, but, for our usage, we prove it here.

Lemma 2.5. *Let N be an o.v.m on $\mathcal{B}(R_+)$ with a control measure μ. Then, for any $\varphi,\ f \in L^2(R_+, \mu)$*

$$< \varphi, f >_{L^2(R_+,\mu)} = \int_0^{\infty}\int_0^{\infty} \varphi(x) f(y) < N(dx), N(dy) > \tag{2.5}$$

and

$$\|f\|^2_{L^2(R_+,\mu)} = \int_0^{\infty}\int_0^{\infty} f(x) f(y) < N(dx), N(dy) > . \tag{2.5'}$$

Proof. First, let us note that for any $A, B \in \mathcal{B}(R_+)$

$$< N(A), N(B) >= \|N(A \cap B)\|^2$$

which implies that for any simple functions f_n and φ_n in $L^2(R_+, \mu)$ of the form

$$f_n = \sum_{j=1}^{k_n} \lambda_{jn} \chi_{A_{jn}} \quad \text{and} \quad \varphi_n = \sum_{j=1}^{k_n} \gamma_{jn} \chi_{A_{jn}}$$

where A_{jn} are disjoint sets in $B(R_+)$ and $\lambda_{jn}, \gamma_{jn}$ are real numbers, $n,\ k_n = 1, 2, \ldots,\ j = 1, 2, \ldots, k_n$ the following formula holds:

$$< f_n, \varphi_n > = \int_0^{\infty}\int_0^{\infty} f_n(x) \varphi_n(x) < N(dx), N(dy) > .$$

Thus, the formula (2.5) is true for simple funtions. Let f and φ be arbitrary functions in $L^2(R_+,\mu)$. Taking $f_n \to f$, $\varphi_n \to \varphi$, where f_n, φ_n are simple funtions, we get

$$< f_n, \varphi_n >= \int_0^\infty \int_0^\infty f_n(x)\varphi_n(x) < N(dx), N(dy) > .$$

Letting $n \to \infty$ the above formula implies the formula (2.5). In particular, the formula (2.5') holds.

Now, suppose that $\mathcal{G} = L^2(R_+,\nu)$, ν being a spectral measure of the $(\mathcal{P},\circ)$-correlated process $\{X_t\}$. Putting, for $B \in \mathcal{B}(R_+)$,

$$\Gamma(B) = \chi_B, \tag{2.6}$$

where χ_B is the indicator of B. It is evident that Γ is an o.v.m with the control measure ν. □

Lemma 2.6. *For each $f \in L^2(R_+,\nu)$*

$$\int_0^\infty f(x)\Gamma(dx) = f. \tag{2.7}$$

Proof. By (2.6), it is evident that (2.7) holds for a simple function f in $L^2(R_+,\nu)$. For general $f \in L^2(R_+,\nu)$, take a sequence of simple funtions converging to f. Then, we get (2.7) for every $f \in L^2(R_+,\nu)$.

Proceeding sucessively, suppose that $\mathcal{G} = \mathcal{H}$. Since, by Lemma 2.4, $\mathcal{H}$ and $L^2(R_+,\nu)$ are isometrically isomorphic with the isomorphism $\mathcal{J}$ given by (2.4), it follows that the set function $M : \mathcal{B}(R_+) \to \mathcal{H}$ defined by

$$M(B) = \mathcal{J}^{-1}\Gamma(B) = \mathcal{J}^{-1}(\chi_B) \tag{2.8}$$

is also an o.v.m with values in $L^2(P)$. In the sequel, we will call $M(\cdot)$ an *orthogonal stochastic measure* (o.s.m). In general, a set function $M : \mathcal{B}(R_+) \to L^2(P)$ is called an *orthogonal stochastic measure* (o.s.m), if it is an orthogonal vector-valued measure (o.v.m) with values in $L^2(P)$. □

Now we are ready to prove the following representation theorem:

Theorem 2.3. *Let $\{X_t\}$ be a $(\mathcal{P},\circ)$-correlated process. Then, there exists a unique o.s.m M on $\mathcal{B}(R_+)$ such that for each $t \geqslant 0$*

$$X_t = \int_0^\infty \Omega(tx)M(dx). \tag{2.9}$$

Conversely, for every o.s.m M the integral (2.9) defines a $(\mathcal{P},\circ)$-correlated process.

Proof. Suppose that $\{X_t\}$ is a $(\mathcal{P}, \circ)$-correlated process with the spectral measure ν and the correlation function $R(t, s)$ given by (1.3).

Let M be an o.s.m defined by formulas (2.6) and (2.8). Putting,

$$Y_t = \int_0^\infty \Omega(tx) M(dx) \tag{2.10}$$

and taking into account (2.4), (2.6) and (2.8) we get, for each $t \in R_+$,

$$\mathcal{J}(Y_t) = \int_0^\infty \Omega(tx)\Gamma(dx),$$

which, by Lemma (2.6), implies that

$$\mathcal{J}(Y_t) = \Omega_t(\cdot), \quad t \in R_+.$$

Hence and by (2.4) it follows that, for every $t \in R_+$,

$$\mathcal{J}(Y_t) = \mathcal{J}(X_t) = \Omega_t(\cdot) \tag{2.11}$$

Consequently, $Y_t = X_t$, $t \in R_+$, which together with (2.10) implies the reprentation (2.9) of $\{X_t\}$. Our further aim is to prove that the representation (2.9) is unique.

Accordingly, suppose that there is another o.s.m V such that

$$X_t = \int_0^\infty \Omega(tx) V(dx), \quad t \in R_+. \tag{2.12}$$

Let Z be a r.v. in $L^2(P)$. Then,

$$EZX_t = \int_0^\infty \Omega(tx)\tau(dx) = \int_0^\infty \Omega(tx)\gamma(dx) \tag{2.13}$$

$(t \in R_+)$, where τ and γ are signed measures on $\mathcal{B}(R_+)$ defined by

$$\tau(dx) = EZM(dx),$$
$$\gamma(dx) = EZV(dx).$$

From (2.13) it follows that for each $t \in R_+$ we get

$$\int_0^\infty\int_0^\infty \Omega(txu)\tau(dx)\sigma_\kappa(du) = \int_0^\infty\int_0^\infty \Omega(txu)\gamma(dx)\sigma_\kappa(du)$$

where σ_κ is the characteristic measure of $(\mathcal{P}, \circ)$.

Using the Fubini's Theorem for signed measures we have

$$\int_0^\infty\int_0^\infty \Omega(txu)\sigma_\kappa(du)\tau(dx) = \int_0^\infty\int_0^\infty \Omega(txu)\sigma_\kappa(du)\gamma(dx)$$

and, by (1.2), it follows that

$$\int_0^\infty exp(-t^\kappa x^\kappa)\tau(dx) = \int_0^\infty exp(-t^\kappa x^\kappa)\gamma(dx)$$

($t \in R_+$), which, by the uniqueness of the Laplace transform for signed measures, implies that $\tau = \gamma$. Since Z is arbitrary we conclude that $M = V$.

Conversely, suppose that M is an o.s.m with the control measure $\nu(\cdot) = E|M(\cdot)|^2$. Let $\{X_t\}$ be defined by (2.9). Then, for any $t, s \in R_+$,

$$\begin{aligned} R(t,s) &= EX_tX_s \\ &= E\Big[\Big(\int_0^\infty \Omega(tx)M(dx)\Big)\Big(\int_0^\infty \Omega(ty)M(dy)\Big)\Big] \\ &= \int_0^\infty\int_0^\infty \Omega(tx)\Omega(ty)E[M(dx)M(dy)], \\ &= \int_0^\infty \Omega(tx)\Omega(sx)\nu(dx). \end{aligned}$$

which shows that the functions $R(t,s)$ and $R(t,0)$, $t, s \in R_+$, are bounded and continuous. Moreover, by virtue of (1.1), Lemma 2.5 and by Fubini's theorem, it follows that

$$\begin{aligned} R(t,s) &= \int_0^\infty \Big[\int_0^\infty \Omega(ux)\delta_t \circ \delta_s(du)\Big]\nu(dx) \\ &= \int_0^\infty \Big[\int_0^\infty \Omega(ux)\nu(dx)\Big]\delta_t \circ \delta_s(du) \\ &= \int_0^\infty R(u,0)\delta_t \circ \delta_s(du). \end{aligned}$$

Consequently, $\{X_t\}$ is a $(\mathcal{P}, \circ)$-correlated process. Thus, the proof is complete. □

The following theorem shows that every $(\mathcal{P}, \circ)$-correlated process can be reduced to a $(\mathcal{P}, *_\kappa)$-correlated process, where κ denotes the characteristic exponent of the convolution $\circ$.

Theorem 2.4. *For every regular Urbanik convolution $\circ$ with the characteristic measure σ_κ and for every $(\mathcal{P}, \circ)$-correlated process $\{X_t\}$ the integral*

$$Y_t = \int_0^\infty X_{ts}\sigma_\kappa(ds) \tag{2.14}$$

$(t \geqslant 0)$ exists in $L^2(P)$ and stands for a $(\mathcal{P}, *_\kappa)$-correlated process.

Proof. The existence in $L^2(P)$ of the right-hand side of (2.14) follows from the fact that $R(t,0)$ is bounded and continuous. Next, by (1.3), we have

$$\begin{aligned} EY_tY_s &= \int_0^\infty\int_0^\infty EX_{tx}X_{ty}\sigma_\kappa(dx)\sigma_\kappa(dy) \\ &= \int_0^\infty\int_0^\infty\Big[\int_0^\infty R(u,0)\delta_{tx}\circ\delta_{sy}(du)\Big]\sigma_\kappa(dx)\sigma_\kappa(dy). \end{aligned}$$

Hence and by (1.2), (2.2) and by Fubini's Theorem

$$\begin{aligned} EY_tY_s &= \int_0^\infty\int_0^\infty\Big[\int_0^\infty\Big(\int_0^\infty \Omega(uv)\nu(dv)\Big)\delta_{tx}\circ\delta_{sy}(du)\Big]\sigma_\kappa(dx)\sigma_\kappa(dy) \\ &= \int_0^\infty\Big[\int_0^\infty\int_0^\infty \Omega(txv)\Omega(syv)\sigma_\kappa(dx)\sigma_\kappa(dy)\Big]\nu(dv) \\ &= \int_0^\infty e^{-(t^\kappa+s^\kappa)v^\kappa}\nu(dv) \\ &= \int_0^\infty\Big[\int_0^\infty e^{-\lambda^\kappa v^\kappa}\delta_t *_\kappa \delta_s(d\lambda)\Big]\nu(dv) \\ &= \int_0^\infty\Big[\int_0^\infty e^{-\lambda^\kappa v^\kappa}\nu(dv)\Big]\delta_t *_\kappa \delta_s(d\lambda) \end{aligned}$$

which shows that $\{Y_t\}$ is a $(\mathcal{P}, *_\kappa)$-correlated. Thus, the proof is complete. □

Let $\{X_t\}$ be a $(\mathcal{P}, \circ)$-correlated process. We say that $\{X_t\}$ has a *finite spectrum* if its o.s.m M in Theorem 2.3 (or, equivalently, the control measure $\nu(\cdot) = E\,|M(\cdot)|^2$) is concentrated on a finite interval.

The following interpolation theorem stands for an analogue of the classical Müntz theorem (compare Thu [3]).

Theorem 2.5. *Let $\{X_t\}$ be a $(\mathcal{P}, *_\alpha)$-correlated process with finite spectrum. Let $0 < t_1 < t_2 < \cdots$ be numbers such that*

$$\sum_{k=1}^{\infty} \frac{1}{t_k} = \infty. \tag{2.15}$$

Then r.v.'s $X_0, X_{t_1}, X_{t_2}, \ldots$ are linearly dense in $\mathcal{H}$.

Proof. By Example 1.1 we have $\Omega(t) = \exp(-t^\alpha)$. Let $A > 0$. By Müntz theorem, functions 1, $\exp(-x^\alpha t_1^\alpha)$, $\exp(-x^\alpha t_1^\alpha), \ldots$ are linearly dense in $C([0, A])$ which together with (2.4) implies that r.v.'s $X_0, X_{t_1}, X_{t_2}, \ldots$ are linearly dense in $\mathcal{H}$. □

3. Examples of $(\mathcal{P}, \circ)$-correlated processes

Example 3.1. The simplest example of $(\mathcal{P}, \circ)$-correlated processes is an additively correlated process ξ_t, $t \in R_+$, being a continuous-parameter analogue of additively correlated sequences (cf. Thu [3], Thu-Weron [4]).

Example 3.2. In [1] Lasser and Leitner presented a direct approach to modified stationarity starting from the classical estimators for the mean $M = EX_n$, $n \in Z$ of a weakly stationary process X_n, $n \in Z$ given by

$$Y_n = \frac{1}{2n+1} \sum_{k=-n}^{n} X_k, \quad n \geqslant 0.$$

The process $\{Y_n\}$ is no longer weakly stationary. However, it is stationary with respect to Jacobi polynomials $P_n^{(1/2,-1/2)}$.

Example 3.3. Let $\{X_t\}$, $t \in R$, be a complex-valued weakly stationary process with the symmetric correlation function. Then the real part process $Y_t := \operatorname{Re} X_t$, $t \in R_+$ is a $(\mathcal{P}, *_{1,1})$-correlated process. Indeed, because of $\operatorname{Re} X_t = \frac{1}{2}(X_t + \overline{X}_t)$ and for $t, s \geqslant 0$

$$\begin{aligned} R^Y(t,s) = EY_tY_s &= \frac{1}{2}(R^Y(t+s,0) + R^Y(|t-s|,\circ)) \\ &= \int_0^\infty R^Y(u,0)\delta_t *_{1,1} \delta_s(du). \end{aligned}$$

Acknowledgement

The Authors would like to thank A. Weron and M. Yamazato for careful readings of the manuscript and pointing out some uncorrectness in our preparation which help to improve the paper. Our sincere thanks are also due to the referee for his/her opinion on the first and revised version of this paper.

References

[1] R. Lasser and M. Leitner, *Stochastic processes indexed by hypergroups I*, J. Theor. Prob. **2**, No. 3 (1989).

[2] R. Lasser, *A modification of stationarity for stochastic processes induced by orthogonal polynomials*, In Probability Measures on Groups IX (H. Heyer, Ed.), pp. 185-191. Lecture Notes Math. 1379 Berlin-Heidelberg-New York, Springer, 1989.

[3] N. V. Thu, *On additively correlated random variables*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23**, no. 7, (1975), 781-785.

[4] N. V. Thu and A. Weron, *Examples of on-stationary Banach space valued stochastic processes of second order*, Probability theory on vector spaces, pp. 171-181, Lecture Notes in Math. **656**, Springer, Berlin, 1978.

[5] V. Hösel and R. Lasser, *One-step prediction for P_n-weakly stationary processes*, Mh. Math. **113** (1992), 199-212.

[6] V. Hösel and R. Lasser, *Prediction of weakly stationary sequences*, The Annals of Probability **31** (2003), no. 1, 93-114.

[7] K. Urbanik, *Generalized convolutions*, Studia Math. **23** (1964), 217-245.

[8] K. Urbanik, *Generalized convolutions II*, Studia Math. **45** (1973), 57-70.

[9] V. E. Vol'kovich, *On symmetric stochastic convolutions*, J. Theor. Prob. **5**, No. 3 (1992), 117-430.

Institute of mathematics
Hanoi, Vietnam

Pedagogical college of Danang
Danang, Vietnam