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EXISTENCE OF SOLUTIONS OF GENERALIZED QUASIVARIATIONAL INEQUALITIES WITH SET-VALUED MAPS

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Abstract. This paper gives existence theorems for solutions of the problem of finding a point $(z_0, x_0, y_0) \in B(z_0, x_0) \times A(x_0) \times F(z_0, x_0, x_0)$ such that, for all $x \in A(x_0)$, $F(z_0, x_0, x) - y_0 \not\subset C(z_0, x_0, x_0)$, where A, B, C and F are set-valued maps between topological vector spaces. Our results generalize some known existence theorems for quasivariational inequalities.

1. Introduction

Let X, Y and Z be locally convex Hausdorff topological vector spaces, and $K \subset X$ and $E \subset Z$ be nonempty subsets. Let $A : K \longrightarrow 2^K$, $B : E \times K \longrightarrow 2^E$, $C : E \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ be set-valued maps with nonempty values. In this paper, we are interested in the existence of solutions of the following generalized quasivariational inequality problem with set-valued maps:

(P) Find $(z_0, x_0) \in E \times K$ such that $x_0 \in A(x_0), z_0 \in B(z_0, x_0)$ and there exists $y_0 \in F(z_0, x_0, x_0)$ such that

$$F(z_0, x_0, x) - y_0 \not\subset C(z_0, x_0, x_0), \quad \forall x \in A(x_0). \tag{1.1}$$

If $C(z_0, x_0, x_0)$ is the negative half-line and F is a (single-valued) function satisfying the condition

$$F(z_0, x_0, x_0) \geq 0, \tag{1.2}$$

then (1.1) implies that

$$F(z_0, x_0, x) \geq 0, \quad \forall x \in A(x_0),$$

i.e., (z_0, x_0) is a solution of the generalized quasivariational inequality problems investigated in [2, 7, 5]. Observe that (1.2) is an assumption often used in proving the existence of solutions of such problems (see e.g. [7, 5]).

If F is single-valued, and B does not depend on the first variable z, then Problem (P) was investigated in [6] with $C(z_0, x_0, x_0)$ being the nonempty interior of a closed convex cone and in [2] with $C(z_0, x_0, x_0)$ being the positive half-line.

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We will see in Section 3 that our main result (Theorem 3.1) includes as special cases Theorem 3.1 and Corollary 3.1 of [2], Theorem 3 of [7], Theorem 1 of [5] and Theorem 2.1 of [6].

If we additionally assume that $C(z_0, x_0, x_0) \equiv -\text{int } D(z_0, x_0)$ and $F(z_0, x_0, x_0) \subset D(z_0, x_0)$, where $D(z_0, x_0)$ is a closed convex cone with nonempty interior, then from (1.1) it follows that

$$F(z_0, x_0, x) \not\subset -\text{int } D(z_0, x_0), \quad \forall x \in A(x_0), \tag{1.3}$$

(see Theorem 3.2 of Section 3). The requirement (1.3) is considered in Corollary 1 of [10] under the additional assumption that both F and D do not depend on the first variable z. Also, it is worth noticing that, unlike Corollary 1 of [10], we do not use the pseudomonotonicity property in proving (1.3).

We will see in Section 3 that the existence of a solution of (P) is equivalent to the existence of a fixed point of a suitable set-valued map. The fixed point theorem used in this paper is due to Park [11] it will be recalled in Section 2.

2. Preliminaries

Let X be a topological space. Each subset of X can be seen as a topological space with the topology induced by the given topology of X. For $x \in X$, let us denote by $U(x), U_1(x), U_2(x), ...$ open neighbourhoods of x. The empty set is denoted by $\emptyset$.

For a set-valued map $F : X \longrightarrow 2^Y$ between two topological spaces X and Y we denote by im F and gr F the image and graph of F :

$$\text{im } F = \bigcup_{x \in X} F(x),$$

$$\text{gr } F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

By definition F is upper semicontinuous (usc) if for any $x \in X$ and any open set $N \supset F(x)$ there exists $U(x)$ such that $N \supset F(x')$ for all $x' \in U(x)$. F is lower semicontinuous (lsc) if for any $x \in X$ and any open set N with $F(x) \cap N \neq \emptyset$ there exists $U(x)$ such that $F(x') \cap N \neq \emptyset$ for all $x' \in U(x)$. F is continuous if it is both usc and lsc. F is closed if its graph is a closed set of $X \times Y$. F is compact if im F is contained in a compact set of Y. F is acyclic if it is usc and if, for any $x \in X, F(x)$ is nonempty, compact and acyclic. Recall that a topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. It is well known that contractible spaces are acyclic; and hence, convex sets and star-shaped sets are acyclic.

We will need the following fixed point theorem due to Park [11, Theorem 7].

Theorem 2.1. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X. If $F : K \longrightarrow 2^K$ is a compact acyclic map, then F has a fixed point, i.e., there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

Lemma 2.1. [4, 8] *Let Y be a Hausdorff topological vector space, $Q \subset Y$ be a nonempty compact set and $D \subset Y$ be a closed convex cone with nonempty interior ($D \neq Y$). Then there exists $q \in Q$ such that*

$$(Q - q) \cap -intD = \emptyset.$$

3. Main result

Throughout this paper we assume that X, Y and Z are locally convex Hausdorff topological vector spaces, $K \subset X$ and $E \subset Z$ are nonempty convex subsets, $A : K \longrightarrow 2^K$ is a compact continuous map with nonempty closed values, $B : E \times K \longrightarrow 2^E$ is a compact acyclic map, and $C : E \times K \times K \longrightarrow 2^Y$ and $F : E \times K \times K \longrightarrow 2^Y$ are set-valued maps with nonempty values.

Consider the set-valued maps $T : E \times K \longrightarrow 2^K$ and $\tau : E \times K \longrightarrow 2^{E\times K}$ defined by setting

(3.1)
$$T(z,\xi) = \{x \in A(\xi) : \exists y \in F(z,\xi,x), \forall \xi' \in A(\xi),\ F(z,\xi,\xi') - y \not\subset C(z,\xi,x)\},$$

$$\tau(z,\xi) = B(z,\xi) \times T(z,\xi), \tag{3.2}$$

for each $(z,\xi) \in E \times K$. Obviously, $(z_0,x_0) \in E \times K$ is a solution of (P) if and only if it is a fixed point of the map τ. So, solving (P) is equivalent to finding a fixed point of τ.

Theorem 3.1. *Let $F : E\times K\times K \longrightarrow 2^Y$ be an usc map with compact values, and $C : E \times K \times K \longrightarrow 2^Y$ be a map with open graph such that, for all $(z,\xi) \in E \times K$, the set $T(z,\xi)$ is nonempty and acyclic. Then there exists a solution of (P).*

Proof. Let τ be defined by (3.2). As we have mentioned above, to prove the theorem it suffices to show that τ has a fixed point. The existence of such a fixed point is assured by Theorem 2.1. Indeed, we first claim that T is usc. For each $(z,\xi) \in E \times K$, the set $T(z,\xi)$ can be rewritten as

$$T(z,\xi) = T_1(z,\xi) \cap A(\xi),$$

where

$$T_1(z,\xi) = \{x \in K : \exists y \in F(z,\xi,x), \forall \xi' \in A(\xi),\ F(z,\xi,\xi') - y \not\subset C(z,\xi,x)\}.$$

Since A is usc and compact-valued, it follows from [1, Proposition 2, p.71] that T is usc if $T_1 : E \times K \longrightarrow 2^K$ is closed. To prove this property we have to show that the complement of gr T_1 in the topological space $E \times K \times K$ is open. In other words, we have to show that for any point $(\bar{z},\bar{\xi},\bar{x}) \notin$ gr T_1 there exist neighbourhoods $U(\bar{z}), U(\bar{\xi})$ and $U(\bar{x})$ such that

$$\forall (z,\xi,x) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x}) : (z,\xi,x) \notin \text{gr } T_1. \tag{3.3}$$

Equivalently, we have to prove that

$$\forall (z,\xi,x) \in U(\bar{z}) \times U(\bar{\xi}) \times U(\bar{x}), \forall y \in F(z,\xi,x), \exists \widehat{\xi} \in A(\xi) : F(z,\xi,\widehat{\xi}) - y \subset C(z,\xi,x). \tag{3.4}$$

Indeed, let $(\bar{z},\bar{\xi},\bar{x}) \notin \text{gr } T_1$. Then, for any $y \in F(\bar{z},\bar{\xi},\bar{x})$, there exists $\xi' \in A(\bar{\xi})$ such that $F(\bar{z},\bar{\xi},\xi') - y \subset C(\bar{z},\bar{\xi},\bar{x})$, i.e.,

$$(\bar{z},\bar{\xi},\bar{x},F(\bar{z},\bar{\xi},\xi') - y) \subset \text{gr } C. \tag{3.5}$$

By the openess of gr C and the compactness of $F(\bar{z},\bar{\xi},\xi')$ there exist open neighbourhoods $U_{y,\xi'}(\bar{z}), U_{y,\xi'}(\bar{\xi}), U_{y,\xi'}(\bar{x})$ and $U_{y,\xi'}(0_Y)$, which depend on y and ξ', such that

$$U_{y,\xi'}(\bar{z}) \times U_{y,\xi'}(\bar{\xi}) \times U_{y,\xi'}(\bar{x}) \times (F(\bar{z},\bar{\xi},\xi') - y + U_{y,\xi'}(0_Y) + U_{y,\xi'}(0_Y)) \subset \text{gr } C, \tag{3.6}$$

where $U_{y,\xi'}(0_Y)$ is a balanced neighbourhood of the origin 0_Y of Y.

When y runs over $F(\bar{z},\bar{\xi},\bar{x})$, the open neighbourhoods $y + U_{y,\xi'}(0_Y)$ cover the compact set $F(\bar{z},\bar{\xi},\bar{x})$. Hence there exist $y_i \in F(\bar{z},\bar{\xi},\bar{x})$ and $\xi'_i \in A(\bar{\xi})$ $(i = 1,2,...,n)$ such that

$$\bigcup_{i=1}^{n}(y_i + U_{y_i,\xi'_i}(0_Y)) \supset F(\bar{z},\bar{\xi},\bar{x}).$$

By the upper semicontinuity of F there exist neighbourhoods $U_1(\bar{z})$, $U_1(\bar{\xi})$ and $U(\bar{x})$ such that

$$\forall (z,\xi,x) \in U_1(\bar{z}) \times U_1(\bar{\xi}) \times U(\bar{x}) : \bigcup_{i=1}^{n}(y_i + U_{y_i,\xi'_i}(0_Y)) \supset F(z,\xi,x). \tag{3.7}$$

Without loss of generality we may assume that

$$U_1(\bar{z}) \subset \bigcap_{i=1}^{n} U_{y_i,\xi'_i}(\bar{z}), \quad U_1(\bar{\xi}) \subset \bigcap_{i=1}^{n} U_{y_i,\xi'_i}(\bar{\xi}), \quad U(\bar{x}) \subset \bigcap_{i=1}^{n} U_{y_i,\xi'_i}(\bar{x}).$$

Using (3.6) with y_i and ξ'_i instead of y and ξ' we have

$$U_{y_i,\xi'_i}(\bar{z}) \times U_{y_i,\xi'_i}(\bar{\xi}) \times U_{y_i,\xi'_i}(\bar{x}) \times (F(\bar{z},\bar{\xi},\xi'_i) - y_i + U_{y_i,\xi'_i}(0_Y) + U_{y_i,\xi'_i}(0_Y)) \subset \text{gr } C. \tag{3.8}$$

Also, since F is usc there exist neighbourhoods $U_2(\bar{z}), U_2(\bar{\xi})$ and $U(\xi'_i)$ such that

$$\forall i = 1,2,...,n, \ \forall (z,\xi,\eta) \in U_2(\bar{z}) \times U_2(\bar{\xi}) \times U(\xi'_i) : F(z,\xi,\eta) \subset F(\bar{z},\bar{\xi},\xi'_i) + U_{y_i,\xi'_i}(0_Y). \tag{3.9}$$

Observe that $A(\bar{\xi}) \cap U(\xi'_i) \neq \emptyset$ since $\xi'_i \in A(\bar{\xi}) \cap U(\xi'_i)$. By the lower semicontinuity of A there exists a neighbourhood $U_3(\bar{\xi})$ such that

$$\forall i = 1,2,...,n, \ \forall \xi \in U_3(\bar{\xi}) : \quad A(\xi) \cap U(\xi'_i) \neq \emptyset. \tag{3.10}$$

Setting

$$U(\bar{z}) = \bigcap_{i=1}^{2} U_i(\bar{z}), \quad U(\bar{\xi}) = \bigcap_{i=1}^{3} U_i(\bar{\xi}),$$

we claim that (3.3) holds. In other words, taking $(z,\xi,x) \in U(\bar z)\times U(\bar\xi)\times U(\bar x)$ and $y \in F(z,\xi,x)$ we must find $\widehat{\xi} \in A(\xi)$ satisfying (3.4).

By (3.7) there exist $y_i \in F(\bar z,\bar\xi,\bar x)$ and $\xi_i' \in A(\bar\xi)$ such that

$$y \in y_i + U_{y_i,\xi_i'}(0_Y).$$

Since $\xi \in U(\bar\xi) \subset U_3(\bar\xi)$ we can find $\widehat{\xi} \in A(\xi)$ such that $\widehat{\xi} \in U(\xi_i')$ (see (3.10)).

Now, using (3.9) with $\eta = \widehat{\xi}$ we get

$$\begin{aligned} F(z,\xi,\widehat{\xi}) - y &\subset F(\bar z,\bar\xi,\xi_i') - y_i + y_i - y + U_{y_i,\xi_i'}(0_Y) \\ &\subset F(\bar z,\bar\xi,\xi_i') - y_i + U_{y_i,\xi_i'}(0_Y) + U_{y_i,\xi_i'}(0_Y). \end{aligned} \tag{3.11}$$

On the other hand,

$$(z,\xi,x) \in U(\bar z)\times U(\bar\xi)\times U(\bar x) \subset U_{y_i,\xi_i'}(\bar z)\times U_{y_i,\xi_i'}(\bar\xi)\times U_{y_i,\xi_i'}(\bar x).$$

Hence, by (3.8) and (3.11) we have

$$(z,\xi,x,F(z,\xi,\widehat{\xi}) - y) \subset \mathrm{gr}\ C,$$

i.e., (3.4) holds, as desired.

Thus T_1 is closed, hence T is usc.

Observe now that τ defined by (3.2) is usc with nonempty compact values since it is the product of the usc maps B and T with nonempty compact values (see [1, Proposition 7, p.73]). Observe also that for each $(z,\xi) \in E\times K$, the set $\tau(z,\xi)$ is acyclic since it is the product of two acyclic sets (see the Künneth formula in [9]). Thus, τ is acyclic. In addition, since im $\tau \subset$ im $B \times$ im A and A and B are compact maps, τ is a compact map. We have seen that all the assumptions of Theorem 2.1 are satisfied for τ. Therefore, τ has a fixed point, i.e., (P) has a solution. □

Theorem 3.2. *In addition to the assumptions of Theorem 3.1, assume that for each $(z,\xi) \in E\times K$, $C(z,\xi,\xi) = -int\ D(z,\xi)$ and $F(z,\xi,\xi) \subset D(z,\xi)$, where $D(z,\xi)$ is a convex cone with nonempty interior. Then there exists $(z_0,x_0) \in E\times K$ such that $(z_0,x_0) \in B(z_0,x_0)\times A(x_0)$ and*

$$F(z_0,x_0,x) \not\subset -int\ D(z_0,x_0), \quad \forall x \in A(x_0).$$

Proof. By Theorem 3.1 there exists a solution of (P), denoted by (z_0,x_0). Let us prove that this point satisfies the conclusion of Theorem 3.2. Indeed, otherwise $F(z_0,x_0,x) \subset -\mathrm{int}\ D(z_0,x_0)$ for some $x \in A(x_0)$. From this we get

$$\begin{aligned} F(z_0,x_0,x) - y_0 &\subset -\mathrm{int}\ D(z_0,x_0) - D(z_0,x_0) \\ &\subset -\mathrm{int}\ D(z_0,x_0), \end{aligned}$$

a contradiction to (1.1) with $C(z_0,x_0,x_0) = -\mathrm{int}\ D(z_0,x_0)$. □

Remark 3.1. When both maps F and D do not depend on the first variable z, Theorem 3.2 is established in Corollary 1 of [10] under some pseudomonotonicity property of F.

From Lemma 2.1 it follows that $T(z, \xi)$ is nonempty if the following condition is satisfied: for each $(z, \xi, x) \in E \times K \times K$, $F(z, \xi, \cdot)$ is usc and $C(z, \xi, x) = -\text{int } D(z, \xi)$ where $D(z, \xi) \neq Y$ is a closed convex cone with nonempty interior. This remark together with Theorems 3.1 and 3.2 yields the following corollary.

Corollary 3.1. *Let the map $(z, \xi) \in E \times K \mapsto \text{int } D(z, \xi)$ have open graph where, for all $(z, \xi) \in E \times K$, $D(z, \xi) \neq Y$ is a closed convex cone with nonempty interior. Let $F : E \times K \times K \longrightarrow 2^Y$ be an usc map with compact values such that, for any $(z, \xi) \in E \times K$, the set*

$$(3.12) \quad T(z, \xi) = \{x \in A(\xi) : \exists y \in F(z, \xi, x), \forall \xi' \in A(\xi) \quad F(z, \xi, \xi') - y \not\subset -\text{int } D(z, \xi)\}$$

is acyclic. Then there exists $(z_0, x_0, y_0) \in E \times K \times Y$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$, $y_0 \in F(z_0, x_0, x_0)$ and

$$F(z_0, x_0, x) - y_0 \not\subset -\text{int } D(z_0, x_0), \quad \forall x \in A(x_0).$$

If, in addition, $F(z, \xi, \xi) \subset D(z, \xi)$ for all $(z, \xi) \in E \times K$, then there exists $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$ and

$$F(z_0, x_0, x) \not\subset -\text{int } D(z_0, x_0), \quad \forall x \in A(x_0).$$

Remark 3.2. Corollary 3.1 extends Theorem 1 in [5] and Theorem 2.1 in [6] to the set-valued case.

Before giving a sufficient condition for the set (3.12) to be acyclic let us introduce the following definition which is a generalization of the notion of proper quasiconcavity [3] to the set-valued case. Let $a \subset X$ be a convex subset, $D \subset Y$ be a convex cone and $f : a \longrightarrow 2^Y$ be a set-valued map. We say that f is properly D-quasiconcave on a if for all $\gamma \in (0, 1)$, $x_i \in a$, $y_i \in f(x_i)$ $(i = 1, 2)$ there exists $y \in f(\gamma x_1 + (1 - \gamma)x_2)$ such that

$$\text{either } y_1 \in y - D \text{ or } y_2 \in y - D.$$

Corollary 3.2. *Let the map $(z, \xi) \in E \times K \mapsto \text{int } D(z, \xi)$ have open graph, where for all $(z, \xi) \in E \times K$, $D(z, \xi) \neq Y$ is a closed convex cone with nonempty interior. Let $A(\xi)$ be convex for all $\xi \in K$. Let $F : E \times K \times K \longrightarrow 2^Y$ be an usc map with compact values such that, for all $(z, \xi) \in E \times K$, $F(z, \xi, \cdot)$ is properly $[-D(z, \xi)]$-quasiconcave on $A(\xi)$. Then there exists $(z_0, x_0, y_0) \in E \times K \times Y$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$, $y_0 \in F(z_0, x_0, x_0)$ and*

$$F(z_0, x_0, x) - y_0 \not\subset -\text{int } D(z_0, x_0), \quad \forall x \in A(x_0).$$

If, in addition, $F(z, \xi, \xi) \subset D(z, \xi)$ for all $(z, \xi) \in E \times K$, then there exists $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(x_0)$ and

$$F(z_0, x_0, x) \not\subset -\text{int } D(z_0, x_0), \quad \forall x \in A(x_0).$$

Proof. By Corollary 3.1, all we have to prove is the convexity of the set (3.12). Let $x_i \in T(z, \xi)$ $(i = 1, 2)$ and $\mu \in (0, 1)$. We must show that $x' := \mu x_1 + (1 - \mu)x_2 \in$

$T(z,\xi)$. Since $x_i \in T(z,\xi)$, we have $x_i \in A(\xi)$, and there exists $y_i \in F(z,\xi,x_i)$ such that, for all $\xi' \in A(\xi)$,

$$F(z,\xi,\xi') - y_i \not\subset -\text{int } D(z,\xi) \ (i = 1,2).$$

Obviously, $x' \in A(\xi)$ since $A(\xi)$ is convex. Also, since $F(z,\xi,\cdot)$ is properly $[-D(z,\xi)]$-quasiconcave on $A(\xi)$ there exists $y' \in F(z,\xi,x')$ such that $\widehat{y} \in y' + D(z,\xi)$ where $\widehat{y} \in \{y_1,y_2\}$. We now claim that $x' \in T(z,\xi)$ and hence, $T(z,\xi)$ is a convex set. More precisely, we claim that $y' \in F(z,\xi,x')$ is a point such that, for all $\xi' \in A(\xi)$,

$$F(z,\xi,\xi') - y' \not\subset -\text{int } D(z,\xi).$$

Indeed, otherwise there exists $\xi' \in A(\xi)$ such that

$$F(z,\xi,\xi') - y' \subset -\text{int } D(z,\xi),$$

which implies that

$$\begin{aligned} F(z,\xi,\xi') - \widehat{y} &\subset (y' - \widehat{y}) - \text{int } D(z,\xi) \\ &\subset -D(z,\xi) - \text{int } D(z,\xi) \\ &\subset -\text{int } D(z,\xi). \end{aligned}$$

This contradicts the condition $F(z,\xi,\xi') - \widehat{y} \not\subset -\text{int } D(z,\xi)$ which is valid since $\widehat{y} \in \{y_1,y_2\}$. □

Remark 3.3. Corollary 3.2 includes as special cases Theorem 3.1, Corollary 3.1 in [2] and Theorem 3 in [7].

Remark 3.4. Corollary 3.2 fails to hold if A is not assumed to have closed values. This can be illustrated by the following example.

Example 3.1. Let us consider Problem (P) with $X = Y = Z = \mathbb{R}$, $D(z,\xi) \equiv \mathbb{R}_+, K = E = [0,1]$, $F(z,\xi,x) = \{\langle z, x-\xi\rangle\}, A(x) \equiv (0,1]$ and $B(z,\xi) \equiv \{1\}$. Then all the assumptions of Corollary 3.2 are satisfied, but there does not exist $(z_0,x_0) \in B(z_0,x_0) \times A(x_0)$ such that

$$F(z_0,x_0,x) \geq F(z_0,x_0,x_0), \ \forall x \in A(x_0).$$

Indeed, if such a point exists then we have $z_0 = 1, x_0 \in (0,1]$ and $\langle z_0, x-x_0\rangle \geq 0$, i.e., $x \geq x_0$ for all $x \in (0,1]$. This is impossible.

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