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ON THE CAUCHY PROBLEM FOR MULTIDIMENSIONAL MONGE-AMPÈRE EQUATIONS

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Abstract. The Cauchy problem for Monge-Ampère equations with several variables is formulated and reduced to that for a normal system of first-order nonlinear partial differential equations. The noncharacteristic condition for the Cauchy problem of multidimensional Monge-Ampère is given. The local solvability of the noncharacteristic Cauchy problem for these equations in the class of analytic functions is proved.

1. Introduction

The classical hyperbolic Monge-Ampère equation with two variables is of the form

$$F(x,y,z,p,q,r,s,t) = Ar + Bs + Ct + D(rt - s^2) - E = 0, \tag{1}$$

where $z = z(x,y)$ is an unknown function defined for $(x,y) \in R^2$, $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$ and $t = \dfrac{\partial^2 z}{\partial y^2}$. The coefficients A, B, C, D and E are real smooth functions of (x,y,z,p,q) and satisfy the condition of hyperbolicity:

$$\Delta := B^2 - 4(AC + DE) > 0.$$

In this case the characteristic equation

$$\lambda^2 + B\lambda + (AC + DE) = 0 \tag{2}$$

has two different real roots $\lambda_1 = \lambda_1(x,y,z,p,q)$, $\lambda_2 = \lambda_2(x,y,z,p,q)$. This equation was well studied by G. Darboux and E. Goursat [1], [2].

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In this article we consider the following Monge-Ampère equations with several variables

$$\begin{vmatrix} z_{x_1x_1}+a_{11} & z_{x_1x_2}+a_{12} & \dots & z_{x_1x_n}+a_{1n} \\ z_{x_2x_1}+a_{21} & z_{x_2x_2}+a_{22} & \dots & z_{x_2x_n}+a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{x_nx_1}+a_{n1} & z_{x_nx_2}+a_{n2} & \dots & z_{x_nx_n}+a_{nn} \end{vmatrix} = 0, \tag{3}$$

where $z = z(x)$ is an unknown function of $x = (x_1, x_2, \dots, x_n)$. The coefficients a_{ij} are smooth functions of x, z and $p = (p_1, p_2, \dots, p_n)$, where $p_j = z_{x_j}$.

Proposition 1. *Suppose that $D = 1$ and equation (1) is hyperbolic. Then it can be written in the form of (3), i.e.*

$$\begin{vmatrix} z_{xx}+C & z_{xy}+\lambda_1 \\ z_{xy}+\lambda_2 & z_{yy}+A \end{vmatrix} = 0,$$

where λ_1 and λ_2 are the roots of equation (2).

Proof. Since $\lambda_1 + \lambda_2 = -B$ and $\lambda_1\lambda_2 = AC + E$, we have

$$\begin{aligned} Ar + Bs + Ct + (rt - s^2) - E &= Ar + Ct + (rt - s^2) + Bs - E \\ &= Ar + Ct + (rt - s^2) - (\lambda_1 + \lambda_2)s \\ &\quad - \lambda_1\lambda_2 + AC \\ &= (r + C)(t + A) - (s + \lambda_1)(s + \lambda_2) \\ &= \begin{vmatrix} z_{xx}+C & z_{xy}+\lambda_1 \\ z_{xy}+\lambda_2 & z_{yy}+A \end{vmatrix}. \end{aligned}$$

The assertion of the proposition immediately follows. □

Equation (1) was investigated in [1], [2] by G. Darboux and E. Goursat under the assumption that it has two independent first integrals. This equation had been also considered in [3], [4], [6], [7] by reducing it to a hyperbolic quasilinear system of first-order partial differential equations with two variables. The multi-dimensional equation (3) is more difficult to study, and it was considered in [5] by M. Tsuji in the case where it possesses n independent first integrals. Though the local existence of an analytic solution to the Cauchy problem for equation (3) has been proved by the theorem of Cauchy-Kovalevski, we are interested in the structure of the equations. In this paper we do not assume that equation (3) possesses n independent first integrals and we shall investigate it by reducing it to a normal system of first-order partial differential equations.

The outline of this paper is as follows. In Section 2 we formulate the Cauchy problem for equation (3). In Section 3 we outline an idea for solving it. In Section 4 we make a change of variables and instead of the Cauchy problem for (3) we obtain an other for a system of first-order partial differential equations. In Section 5 we reduce this system of equations to a normal one of $2n+1$ equations with $2n+1$ unknowns. In Section 6 we obtain the noncharacteristic condition for the Cauchy problem for (3). As an application, in Section 7 we apply the

Cauchy-Kovalevski theorem to obtain another proof of local solvability to the noncharacteristic Cauchy problem for equation (3) in the class of analytic functions.

A short version of this paper was published in [8]. In this paper we expose all main results with detailed proofs.

2. The Cauchy Problem

Suppose that in R^n_x there is an $(n-1)$-dimensional surface Γ that is given by equations:

$$\begin{cases} x_1 &= X^0_1(\alpha'), \\ x_2 &= X^0_2(\alpha'), \\ & \dots \\ x_n &= X^0_n(\alpha'). \end{cases} \tag{4}$$

Here and in what follows we put

$$\alpha' \equiv (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in P \subset R^{n-1}_{\alpha'}.$$

Suppose also that we are given $n+1$ functions $Z^0(\alpha')$, $P^0_j(\alpha')$, $j = 1, 2, \dots, n$.

The Cauchy problem for the equation (3) consists in looking for a solution $z(x) \in C^2$ of (3) such that

$$\begin{cases} z(x)\big|_{x=X^0(\alpha')} &= Z^0(\alpha'), \\ z_{x_j}(x)\big|_{x=X^0(\alpha')} &= P^0_j(\alpha'), \quad j = 1, 2, \dots, n, \end{cases} \tag{5}$$

where $X^0(\alpha') \equiv (X^0_1(\alpha'), X^0_2(\alpha'), \dots, X^0_n(\alpha'))$.

From (5) we obtain the following necessary conditions for the initial Cauchy data

$$\frac{\partial Z^0(\alpha')}{\partial \alpha_k} = \sum_{j=1}^{n} P^0_j(\alpha') \frac{\partial X^0_j(\alpha')}{\partial \alpha_k}, \quad k = 1, \dots, n-1, \tag{6}$$

which are assumed to be fulfilled.

3. A Solution Method for (3)

Suppose that $\{\omega_j\}_{j=0,1,\dots,n}$ are one-forms defined in

$$R^{2n+1}_{x,z,p} = \{(x_1, \dots, x_n, z, p_1, \dots, p_n)\}$$

as follows

$$\omega_0 = dz - \sum_{j=1}^{n} p_j dx_j,$$

$$\omega_j = dp_j + \sum_{k=1}^{n} a_{jk}(x, z, p) dx_k, \quad j = 1, 2, \dots, n,$$

where $a_{jk}(x, z, p)$ are the same functions in the equation (3).

The following propositions can be easily verified from the theory of differential forms. (see e.g. [9]).

Proposition 2. *Suppose that the following conditions 1) and 2) are satisfied*

1) *There is an n-dimensional C^1-surface $M \subset R^{2n+1}_{x,z,p}$ that is given by*

$$\begin{cases} z &= Z(x) \\ p_j &= P_j(x), \quad j = 1, 2, \dots, n; \end{cases}$$

2) $\omega_0 \equiv 0$ *on M, that means the form ω_0 vanishes on the tangent space to M at any $(x^0, z^0, p^0) \in M$.*

Then we have

$$P_j(x) = \frac{\partial Z(x)}{\partial x_j}, \quad j = 1, 2, \dots, n, \tag{7}$$

and consequently on M

$$dp_j = \sum_{k=1}^{n} Z_{x_j x_k}(x) dx_k, \quad j = 1, 2, \dots, n.$$

Proposition 3. *Suppose that all the conditions of Proposition 2 hold. Then we have on M*

$$\omega_j = \sum_{k=1}^{n} \Big(\frac{\partial^2 Z(x)}{\partial x_j \partial x_k} + a_{jk}(x, Z(x), Z_x(x)) \Big) dx_k,$$

and

$$\begin{aligned} &\omega_1 \wedge \omega_2 \cdots \wedge \omega_n = \\ &= \det \| Z_{x_j x_k}(x) + a_{jk}(x, Z(x), Z_x(x)) \| dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n, \end{aligned} \tag{8}$$

where the operation $\wedge$ stands for exterior product of differential forms.

From (8), in order to solve the equation (3), we must find an n-dimensional C^1-surface $M \subset R^{2n+1}_{x,z,p}$, on which the following relations hold

$$\begin{cases} \omega_0 = 0 \\ \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n = 0. \end{cases}$$

4. Change of Variables in (3)

Suppose that in equation (3) we change the variables $x = (x_1, x_2, \dots, x_n)$ into the new ones $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in such a way that

$$x_j = X_j(\alpha), \quad j = 1, 2, \dots, n \tag{9}$$

and

$$\frac{D(X_1, X_2, \dots, X_n)}{D(\alpha_1, \alpha_2, \dots, \alpha_n)} \neq 0, \tag{10}$$

where $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ is the same variables as in (4). Note that from (9) and (10), it holds locally

$$\alpha_j = \varphi_j(x), \quad j = 1, 2, \dots, n. \tag{11}$$

From Proposition 2 we obtain the following result

Proposition 4. *Let the C^1-surface $M \subset R^{2n+1}_{x,z,p}$ be given by equations:*

$$\begin{cases} x_j &= X_j(\alpha), \quad j = 1, 2, \dots, n \\ z &= Z(\alpha), \\ p_j &= P_j(\alpha), \quad j = 1, \dots, n \end{cases}$$

and the condition (10) *be fulfilled.*

Then $\omega_0 = 0$ on M if and only if

$$\frac{\partial Z(\alpha)}{\partial \alpha_k} - \sum_{\ell=1}^{n} P_\ell(\alpha) \frac{\partial X_\ell(\alpha)}{\partial \alpha_k} = 0; \quad k = 1, 2, \dots, n.$$

From the definition of M it follows that on M

$$\begin{aligned} \omega_j &= \sum_{\ell=1}^{n} \frac{\partial P_j}{\partial \alpha_\ell} d\alpha_\ell + \sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) \sum_{\ell=1}^{n} \frac{\partial X_k}{\partial \alpha_\ell} d\alpha_\ell \\ &= \sum_{\ell=1}^{n} \Big(\frac{\partial P_j}{\partial \alpha_\ell} + \sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) \frac{\partial X_k}{\partial \alpha_\ell} \Big) d\alpha_\ell. \end{aligned}$$

It is easy to see that the following indentity holds on M

$$\begin{aligned} &\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \\ &= \det \Big\| \frac{\partial P_j}{\partial \alpha_\ell} + \sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) \frac{\partial X_k}{\partial \alpha_\ell} \Big\| d\alpha_1 \wedge d\alpha_2 \wedge \cdots \wedge d\alpha_n. \end{aligned} \tag{12}$$

We note that the determinant in the right hand side of (12) is equal to 0 if the columns vectors are linearly dependent. So from (12) we arrive at the following

Proposition 5. *Suppose that the C^1-surface $M \subset R^{2n+1}_{x,z,p}$ is given by equations:*

$$\begin{cases} x_j &= X_j(\alpha), j = 1, 2, \dots, n \\ z &= Z(\alpha), \\ p_j &= P_j(\alpha), j = 1, 2, \dots, n, \end{cases}$$

where $X_j(\alpha)$, $Z(\alpha)$, $P_j(\alpha)$ satisfy the system of equations

$$\sum_{\ell=1}^{n} \frac{\partial P_j}{\partial \alpha_\ell} + \sum_{\ell=1}^{n} \sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) \frac{\partial X_k}{\partial \alpha_\ell} = 0, \quad j = 1, 2, \dots, n.$$

Then $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n = 0$ on M.

Set

$$X(\alpha) = (X_1(\alpha), ..., X_n(\alpha))\,;\ P(\alpha) = (P_1(\alpha), ..., P_n(\alpha)).$$

From equations (3), (8), (12) and from Propositions 2, 3, 4 and 5 we obtain the following.

Theorem 1. *Suppose that* $(X(\alpha), Z(\alpha), P(\alpha))$ *is a* C^2*-solution of the following system of equations:*

$$(13_1) \qquad \sum_{\ell=1}^{n} \frac{\partial P_j}{\partial \alpha_\ell} + \sum_{\ell=1}^{n}\sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) \frac{\partial X_k}{\partial \alpha_\ell} = 0, \quad j = 1, 2, \ldots, n$$

$$(13_2) \qquad \frac{\partial Z}{\partial \alpha_k} - \sum_{\ell=1}^{n} P_\ell(\alpha) \frac{\partial X_\ell}{\partial \alpha_k} = 0, \quad k = 1, 2, \ldots, n$$

and satisfies condition (10)*. Then the function*

$$(14) \qquad z(x) = Z(\varphi(x)) = Z(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)),$$

where $\varphi(x) \equiv (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))$ *are defined by* (11) *is a solution of the Monge-Ampère equation* (3)*. Moreover, we also have* $z_x(x) = P(\varphi(x))$*.*

We now formulate the Cauchy problem for the system (13).

Cauchy problem: Find $(X(\alpha), Z(\alpha), P(\alpha))$ of class C^1 that is a solution of (13) such that

$$(15_1) \qquad X_j(\alpha)\big|_{\alpha_n=0} = X_j^0(\alpha'), \quad j = 1, \ldots, n,$$

$$(15_2) \qquad Z(\alpha)\big|_{\alpha_n=0} = Z^0(\alpha'),$$

$$(15_3) \qquad P_j(\alpha)\big|_{\alpha_n=0} = P_j^0(\alpha'), \quad j = 1, \ldots, n,$$

where the functions $X_j^0(\alpha'), Z^0(\alpha'), P_j^0(\alpha')$ are given above as in (5) and satisfy the condition (6).

5. Reducing (13) to a Normal System

We first prove some lemmas.

Lemma 1. *Suppose that* $(X(\alpha), Z(\alpha), P(\alpha))$ *is a* C^2*-solution of the system* (13)*. If we set*

$$(16) \qquad \sum_{\ell=1}^{n} \frac{\partial X_i}{\partial \alpha_\ell} = g_i(\alpha), \quad i = 1, 2, \ldots, n$$

$$(17) \qquad f_j(\alpha) = -\sum_{k=1}^{n} a_{jk}(X(\alpha), Z(\alpha), P(\alpha)) g_k(\alpha), \quad j = 1, 2, \ldots, n,$$

then we have

$$(18) \qquad \sum_{\ell=1}^{n} \frac{\partial P_i}{\partial \alpha_\ell} = f_i(\alpha); \quad i = 1, 2, \ldots, n.$$

Proof. (18) follows from (13_1), (16) and (17). □

We denote

$$\vec{g}(\alpha) \equiv (g_1(\alpha), g_2(\alpha), \ldots, g_n(\alpha))^T,$$
$$\vec{f}(\alpha) \equiv (f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha))^T,$$
$$\frac{\partial P}{\partial \alpha_j} \equiv \Big(\frac{\partial P_1}{\partial \alpha_j}, \ldots, \frac{\partial P_n}{\partial \alpha_j}\Big)^T \in R^n \quad j = 1, 2, \ldots, n,$$
$$\frac{\partial X}{\partial \alpha_j} \equiv \Big(\frac{\partial X_1}{\partial \alpha_j}, \ldots, \frac{\partial X_n}{\partial \alpha_j}\Big)^T \in R^n \quad j = 1, 2, \ldots, n.$$

Lemma 2. *Suppose that* $(X(\alpha), Z(\alpha), P(\alpha))$ *is a* C^2*-solution of the system* (13) *and satisfies* (16), (17). *Then we have the following necessary condition:*

$$\sum_{\ell=1}^{n} f_\ell(\alpha)\frac{\partial X_\ell}{\partial \alpha_k} = \sum_{\ell=1}^{n} g_\ell(\alpha)\frac{\partial P_\ell}{\partial \alpha_k}, \quad k = 1, 2, \ldots, n. \tag{19}$$

Proof. From (13_2) we have

$$\frac{\partial Z}{\partial \alpha_k} = \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell}{\partial \alpha_k}, \tag{20}$$

$$\frac{\partial Z}{\partial \alpha_m} = \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell}{\partial \alpha_m}. \tag{21}$$

Differentiating both sides of (20), (21) with respect to α_m and α_k, respectively, we get

$$\frac{\partial^2 Z}{\partial \alpha_k \partial \alpha_m} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial \alpha_m}\frac{\partial X_\ell}{\partial \alpha_k} + \sum_{\ell=1}^{n} P_\ell \frac{\partial^2 X_\ell}{\partial \alpha_k \partial \alpha_m}, \tag{22}$$

$$\frac{\partial^2 Z}{\partial \alpha_m \partial \alpha_k} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial \alpha_k}\frac{\partial X_\ell}{\partial \alpha_m} + \sum_{\ell=1}^{n} P_\ell \frac{\partial^2 X_\ell}{\partial \alpha_m \partial \alpha_k}. \tag{23}$$

Since $Z(\alpha), X(\alpha) \in C^2$, from (22) and (23) we get

$$\sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial \alpha_m}\frac{\partial X_\ell}{\partial \alpha_k} = \sum_{\ell=1}^{n} \frac{\partial P_\ell}{\partial \alpha_k}\frac{\partial X_\ell}{\partial \alpha_m}. \tag{24}$$

Summing both sides of (24) with respect to m from 1 to n we arrive at (19). □

Set

$$A(x, z, p) = \Big\| a_{jk}(x, z, p) \Big\|_{n \times n}.$$

From (17) it follows that

$$\vec{f}(\alpha) = -A(X(\alpha), Z(\alpha), P(\alpha))\vec{g}(\alpha). \tag{25}$$

We introduce the column-vectors

$$(26)\qquad \vec{v}_j(\alpha) \equiv \frac{\partial P}{\partial \alpha_j} + A^T(X(\alpha), Z(\alpha), P(\alpha))\frac{\partial X}{\partial \alpha_j} = (v_{j1}(\alpha), v_{j2}(\alpha), \ldots, v_{jn}(\alpha))^T \in R^n, \quad j = 1, 2, \ldots, n-1.$$

Lemma 3. *Suppose that* $(X(\alpha), Z(\alpha), P(\alpha))$ *is a* C^2*-solution of the system* (13) *and satisfies* (16) *and* (17). *Assume that the vector* $\vec{g}(\alpha) \equiv (g_1(\alpha), g_2, \ldots, g_n(\alpha))^T$ *is given by the formula*

$$(27)\qquad \vec{g}(\alpha) = \vec{v}_1(\alpha) \times \vec{v}_2(\alpha) \times \cdots \times \vec{v}_{n-1}(\alpha) \in R^n,$$

where the vectors $\vec{v}_j(\alpha)$ *are defined by* (26) *and the vector product* (27) *is defined by*

$$(28)\qquad \vec{v}_1 \times \vec{v}_2 \times \cdots \times \vec{v}_{n-1} = \begin{vmatrix} v_{11} & v_{21} & \ldots & v_{n-1,1} & \vec{e}_1 \\ v_{12} & v_{22} & \ldots & v_{n-1,2} & \vec{e}_2 \\ v_{13} & v_{23} & \ldots & v_{n-1,3} & \vec{e}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \ldots & v_{n-1,n} & \vec{e}_n \end{vmatrix} \in R^n.$$

with the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ *being unit column-vectors on coordinate axes* Ox_1, $Ox_2, \ldots, Ox_n$ *respectively. Then the condition* (19) *is equivalent to*

$$(29)\qquad \langle \vec{g}(\alpha), \vec{v}_k(\alpha) \rangle = 0, \quad k = 1, 2, \ldots, n-1, \forall \alpha,$$

where $\langle , \rangle$ *is a scalar product in* R^n *and the vectors* $\vec{v}_k(\alpha)$ *are defined by* (26).

Proof. We write (19) in the equivalent form:

$$(30)\qquad \langle \vec{g}(\alpha), \frac{\partial P}{\partial \alpha_k} \rangle = \langle \vec{f}(\alpha), \frac{\partial X}{\partial \alpha_k} \rangle \quad k = 1, 2, \ldots, n.$$

Setting $\vec{f} = -A(X(\alpha), Z(\alpha), P(\alpha))\vec{g}(\alpha)$ in (30) we get

$$\langle \vec{g}, \frac{\partial P}{\partial \alpha_k} \rangle = -\langle A\vec{g}, \frac{\partial X}{\partial \alpha_k} \rangle, \quad k = 1, 2, \ldots, n.$$

or

$$(31)\qquad \langle \vec{g}, \frac{\partial P}{\partial \alpha_k} + A^T \frac{\partial X}{\partial \alpha_k} \rangle = 0, \quad k = 1, 2, \ldots, n.$$

From (16), (17), (18) and (26) the equivalence of (31) and (29) follows. □

Theorem 2. *The system* (13) *can be reduced to the following system*

$$(32_1)\qquad \sum_{\ell=1}^{n} \frac{\partial X_i}{\partial \alpha_\ell} = g_i(\alpha), \quad i = 1, 2, \ldots, n,$$

$$(32_2)\qquad \sum_{\ell=1}^{n} \frac{\partial Z}{\partial \alpha_\ell} = \sum_{\ell=1}^{n} g_\ell(\alpha) P_\ell,$$

$$(32_3)\qquad \sum_{\ell=1}^{n} \frac{\partial P_i}{\partial \alpha_\ell} = f_i(\alpha) \quad i = 1, 2, \ldots, n.$$

where the vector $\vec{g}(\alpha) \equiv (g_1(\alpha), g_2(\alpha), \dots, g_n(\alpha))^T$ *is defined by* (27)*, and the vector* $\vec{f}(\alpha) \equiv (f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha))^T$ *is defined by* (25)*.*

Remark 1. From (26) and (27) it follows that all the $g_j(\alpha)$ depend on $x(\alpha), z(\alpha)$, $p(\alpha)$, on the first derivatives $\dfrac{\partial x_k}{\partial \alpha_\ell}, \dfrac{\partial p_k}{\partial \alpha_\ell}, \quad \ell = 1, 2, \dots, n-1$ and they are homogenous with respect to these derivatves of degree $(n-1)$. On the other hand, the system (32) is solvable with respect to $\dfrac{\partial x_k}{\partial \alpha_n}, \dfrac{\partial z}{\partial \alpha_n}, \dfrac{\partial p_k}{\partial \alpha_n}$, then (32) is a normal system of first-order partial differential equations. In the case $n = 2$ this system is quasilinear.

Proof. The equations (32_1) follow from (16). The equations (32_3) follow from (18), (17) and (13_1). The equation (32_2) follows from (13_2) and (32_1). In order that the condition (19) is automatically satisfied we choose $\vec{g}$ by (27). □

We now state and prove the main result of the paper.

Theorem 3. *Let the vectors* $\vec{g}(\alpha) \equiv (g_1(\alpha), g_2(\alpha), \dots, g_n(\alpha))^T$ *and* $\vec{f}(\alpha) \equiv (f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha))^T$ *be defined by* (27) *and* (25) *respectively. Suppose that* $(X(\alpha), Z(\alpha), P(\alpha))$ *is a* C^2*-solution of the Cauchy problem* (32), (15)*. Then it is also a solution of the Cauchy problem* (13), (15)*.*

We first prepare the following two lemmas. By applying the classical characteristic method to Cauchy problem for first-order partial differential equations, we can get the following

Lemma 4. *The unique solution* $u(\alpha)$ *of the Cauchy problem:*

$$\begin{cases} \sum\limits_{k=1}^{n} \dfrac{\partial u}{\partial \alpha_k} & = F(\alpha) \\ u\Big|_{\alpha_n = 0} & = u_0(\alpha') \end{cases}$$

can be written in the form

$$u(\alpha_1, \alpha_2, \dots, \alpha_n) = u_0(\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n) + \int_0^{\alpha_n} F(\alpha_1 - \alpha_n + s, \alpha_2 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s) ds.$$

We set

$$h_k(\alpha) \equiv \frac{\partial Z(\alpha)}{\partial \alpha_k} - \sum_{\ell=1}^{n} P_\ell(\alpha) \frac{\partial X_\ell(\alpha)}{\partial \alpha_k}, \quad k = 1, 2, \dots, n.$$

Lemma 5. *Suppose that* (32_2) *and*

$$h_k(\alpha) \equiv 0, \ k = 1, 2, \dots, n-1.$$

hold. Then we have

$$h_n(\alpha) \equiv 0.$$

Proof. The assumptions of Lemma 5 and (32_2) yield

$$\sum_{k=1}^{n} \frac{\partial Z}{\partial \alpha_k} - \sum_{\ell=1}^{n} P_\ell \sum_{k=1}^{n} \frac{\partial X_\ell}{\partial \alpha_k} = 0.$$

Thus,

$$\sum_{k=1}^{n} \Big(\frac{\partial Z}{\partial \alpha_k} - \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell}{\partial \alpha_k}\Big) = 0.$$

So we have

$$\sum_{k=1}^{n} h_k = 0,$$

from which the assertion of the lemma follows. □

Proof of Theorem 3. Equations (13_1) follow from (32_1) and (32_3). We need only to prove (13_2). In view of Lemma 5, we show that $h_k \equiv 0, k = 1, 2, \dots, n-1$. We will prove, for example, that $h_1(\alpha) \equiv 0$.

Applying Lemma 4 to each equation of (32) with the Cauchy data (15) we have

$$X_\ell(\alpha) = X_\ell^0(\alpha_1 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n) + \int_0^{\alpha_n} g_\ell(\alpha_1 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s)ds, \tag{33}$$

$$Z(\alpha) = Z^0(\alpha_1 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n) + \int_0^{\alpha_n} \sum_{\ell=1}^{n} P_\ell(\alpha_1 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s) \times g_\ell(\alpha_1 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s)ds, \tag{34}$$

$$P_\ell(\alpha) = P_\ell^0(\alpha_1 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n) + \int_0^{\alpha_n} f_\ell(\alpha_1 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s)ds. \tag{35}$$

We now prove that

$$\frac{\partial Z}{\partial \alpha_1} - \sum_{\ell=1}^{n} P_\ell \frac{\partial X_\ell(\alpha)}{\partial \alpha_1} \equiv 0. \tag{36}$$

We have

$$\frac{\partial X_\ell(\alpha)}{\partial \alpha_1} = \frac{\partial X_\ell^0(\alpha_1 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n)}{\partial \alpha_1} + \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \dots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds, \tag{37}$$

From (35) and (37) we have

$$\sum_{\ell=1}^{n} P_\ell(\alpha)\frac{\partial X_\ell(\alpha)}{\partial \alpha_1} == \sum_{\ell=1}^{n} P_\ell(\alpha)\big[\frac{\partial X_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial \alpha_1}$$
$$+\int_0^{\alpha_n}\frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial \alpha_1}ds\big]$$
$$=\sum_{\ell=1}^{n} P_\ell(\alpha)\int_0^{\alpha_n}\frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial \alpha_1}ds$$
$$+\sum_{\ell=1}^{n} P_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)\frac{\partial X_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial \alpha_1}$$
$$+\sum_{\ell=1}^{n}\frac{\partial X_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial \alpha_1}\times$$
$$\times\int_0^{\alpha_n} f_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds. \tag{38}$$

Let us calculate $\frac{\partial Z(\alpha)}{\partial \alpha_1}$. From (34) we have

$$\frac{\partial Z(\alpha)}{\partial \alpha_1}=\frac{\partial Z^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial \alpha_1}$$
$$+\int_0^{\alpha_n}\sum_{\ell=1}^{n}\frac{\partial P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial \alpha_1}\times$$
$$\times\, g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds$$
$$+\int_0^{\alpha_n}\sum_{\ell=1}^{n} P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)\times$$
$$\times\frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial \alpha_1}ds. \tag{39}$$

Since the condition (29) is satisfied, so is the condition (19). So for the second term of (39) we have

$$\int_0^{\alpha_n}\sum_{\ell=1}^{n}\frac{\partial P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}\times$$
$$\times\, g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds$$
$$=\int_0^{\alpha_n}\sum_{\ell=1}^{n}\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}\times$$
$$\times\, f_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds$$
$$=\sum_{\ell=1}^{n}\int_0^{\alpha_n}\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}\times$$
$$\times\, f_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds. \tag{40}$$

We set

$$F_\ell(s)=\int_{\alpha_n}^{s} f_\ell(\alpha_1-\alpha_n+t,\dots,\alpha_{n-1}-\alpha_n+t,t)dt.$$

Then

$$F'_\ell(s)=f_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)\quad\text{and}\quad F_\ell(\alpha_n)=0.$$

From (40) we get

$$\int_0^{\alpha_n}\sum_{\ell=1}^{n}\frac{\partial P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}\times$$
$$\times\, g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)ds$$
$$=\sum_{\ell=1}^{n}\int_0^{\alpha_n}\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}.F'_\ell(s)ds$$
$$=\sum_{\ell=1}^{n}\Big[\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}.F_\ell(s)\Big|_0^{\alpha_n}$$
$$-\int_0^{\alpha_n}F_\ell(s)\frac{d}{ds}\Big(\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}\Big)ds\Big]$$
$$=-\sum_{\ell=1}^{n}\Big[\frac{\partial X_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial\alpha_1}\times$$
$$\times\int_{\alpha_n}^{0} f_\ell(\alpha_1-\alpha_n+t,\dots,\alpha_{n-1}-\alpha_n+t,t)dt$$
$$-\int_0^{\alpha_n}F_\ell(s)\cdot\frac{\partial}{\partial\alpha_1}\Big(\sum_{k=1}^{n}\frac{\partial X_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_k}\Big)ds\Big]$$

$$= \sum_{\ell=1}^{n} \Big[\frac{\partial X_\ell^0(\alpha_1-\alpha_n,\dots,\alpha_{n-1}-\alpha_n)}{\partial \alpha_1} \times$$
$$\times \int_0^{\alpha_n} f_\ell(\alpha_1-\alpha_n+t,\dots,\alpha_{n-1}-\alpha_n+t,t)dt$$
$$- \int_0^{\alpha_n} F_\ell(s)\cdot \frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}ds\Big]. \tag{41}$$

Now we consider the third term in (39). Setting

$$G_\ell(s) = \int_0^{s} \frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}ds,$$

we have

$$G'_\ell(s) = \frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}$$

and

$$G_\ell(0) = 0.$$

Since $F_\ell(\alpha_n) = G_\ell(0) = 0$, we have

$$\int_0^{\alpha_n} \sum_{\ell=1}^{n} P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)\times$$
$$\times \frac{\partial g_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s)}{\partial\alpha_1}ds$$
$$= \sum_{\ell=1}^{n} \int_0^{\alpha_n} P_\ell(\alpha_1-\alpha_n+s,\dots,\alpha_{n-1}-\alpha_n+s,s).G'_\ell(s)ds$$

$$= \sum_{\ell=1}^{n} P_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s).G_\ell(s)\big|_0^{\alpha_n}$$
$$- \int_0^{\alpha_n} \sum_{\ell=1}^{n} G_\ell(s) \cdot \frac{d}{ds}\big(P_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)\big)ds$$
$$= \sum_{\ell=1}^{n} \ P_\ell(\alpha). \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds$$
$$- \int_0^{\alpha_n} G_\ell(s). \sum_{k=1}^{n} \frac{\partial P_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_k} ds$$
$$= \sum_{\ell=1}^{n} \ P_\ell(\alpha). \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds$$
$$- \int_0^{\alpha_n} \sum_{\ell=1}^{n} G_\ell(s).f_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)ds$$
$$= \sum_{\ell=1}^{n} \ P_\ell(\alpha). \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds$$
$$- \int_0^{\alpha_n} \sum_{\ell=1}^{n} G_\ell(s).F'_\ell(s)ds$$
$$= \sum_{\ell=1}^{n} \ P_\ell(\alpha). \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds$$
$$+ \int_0^{\alpha_n} \sum_{\ell=1}^{n} G'_\ell(s).F_\ell(s)ds. \tag{42}$$

From (39), (41) and (42) we have

$$\frac{\partial Z(\alpha)}{\partial \alpha_1} = \frac{\partial Z^0(\alpha_1 - \alpha_n, \ldots, \alpha_{n-1} - \alpha_n)}{\partial \alpha_1}$$
$$+ \sum_{\ell=1}^{n} \Big[\frac{\partial X_\ell^0(\alpha_1 - \alpha_n, \ldots, \alpha_{n-1} - \alpha_n)}{\partial \alpha_1} \times$$
$$\times \int_0^{\alpha_n} f_\ell(\alpha_1 - \alpha_n + t, \ldots, \alpha_{n-1} - \alpha_n + t, t)dt$$

$$
\begin{aligned}
&- \int_0^{\alpha_n} F_\ell(s) \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds \Big] \\
&+ \sum_{\ell=1}^{n} P_\ell(\alpha) \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds \\
&+ \int_0^{\alpha_n} \sum_{\ell=1}^{n} G'_\ell(s).F_\ell(s) ds \\
= &\frac{\partial Z^0(\alpha_1 - \alpha_n, \ldots, \alpha_{n-1} - \alpha_n)}{\partial \alpha_1} \\
&+ \sum_{\ell=1}^{n} \frac{\partial X^0_\ell(\alpha_1 - \alpha_n, \ldots, \alpha_{n-1} - \alpha_n)}{\partial \alpha_1} \times \\
&\times \int_0^{\alpha_n} f_\ell(\alpha_1 - \alpha_n + t, \ldots, \alpha_{n-1} - \alpha_n + t, t) dt \\
&+ \sum_{\ell=1}^{n} P_\ell(\alpha) \int_0^{\alpha_n} \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1} ds,
\end{aligned}
\tag{43}
$$

since

$$
G'_\ell(s) = \frac{\partial g_\ell(\alpha_1 - \alpha_n + s, \ldots, \alpha_{n-1} - \alpha_n + s, s)}{\partial \alpha_1}.
$$

From (43), (6) and (38) we obtain (36). □

6. Non-characteristic condition

It is obvious that the change of variables is locally not degenerate if

$$
\frac{D(X_1, X_2, \ldots, X_n)}{D(\alpha_1, \alpha_2, \ldots, \alpha_n)} \Big|_{\alpha_n = 0} \neq 0. \tag{44}
$$

We investigate now conditions under which the condition (44) is fulfilled. We set according to (27)

$$
\begin{aligned}
\vec{g}^0(\alpha') &= (g^0_1(\alpha'), g^0_2(\alpha'), \ldots, g^0_n(\alpha'))^T \\
&\equiv \vec{g}(\alpha)\big|_{\alpha_n = 0} \\
&= \vec{v}^0_1(\alpha') \times \vec{v}^0_2(\alpha') \times \cdots \times \vec{v}^0_{n-1}(\alpha'),
\end{aligned}
\tag{45}
$$

where the vector product in (45) is defined by (28), and according to (26)

$$\begin{aligned}\vec{v}_j^0(\alpha') &= (v_{j1}^0(\alpha'), v_{j2}^0(\alpha'), \dots, v_{jn}^0(\alpha'))^T \\ &\equiv \vec{v}_j(\alpha)\big|_{\alpha_n=0} \\ &= \frac{\partial P^0(\alpha')}{\partial \alpha_j} + A^T(X^0(\alpha'), Z^0(\alpha'), P^0(\alpha'))\frac{\partial X^0(\alpha')}{\partial \alpha_j}, \\ &\qquad j = 1, 2, \dots, n-1.\end{aligned} \tag{46}$$

Proposition 6. *Suppose*

$$\begin{vmatrix} \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_1} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_1} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_{n-1}} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_{n-1}} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_{n-1}} \\ g_1^0(\alpha') & g_2^0(\alpha') & \dots & g_n^0(\alpha') \end{vmatrix} \neq 0, \quad \forall \alpha', \tag{47}$$

where the vector $\vec{g}^0(\alpha') = (g_1^0(\alpha'), g_2^0(\alpha'), \dots, g_n^0(\alpha'))^T$ *is defined by* (45). *Then the condition* (44) *is fulfilled.*

Proof. From (44) and (32_1) it follows that

$$\frac{D(X_1, X_2, \dots, X_n)}{D(\alpha_1, \alpha_2, \dots, \alpha_n)}\Big|_{\alpha_n=0} = \begin{vmatrix} \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_1} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_1} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_{n-1}} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_{n-1}} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_{n-1}} \\ \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_n} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_n} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_n} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_1} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_1} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_{n-1}} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_{n-1}} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_{n-1}} \\ -\sum\limits_{\ell=1}^{n-1} \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_\ell} + g_1^0(\alpha') & -\sum\limits_{\ell=1}^{n-1} \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_\ell} + g_2^0(\alpha') & \dots & -\sum\limits_{\ell=1}^{n-1} \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_\ell} + g_n^0(\alpha') \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_1} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_1} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial X_1^0(\alpha')}{\partial \alpha_{n-1}} & \dfrac{\partial X_2^0(\alpha')}{\partial \alpha_{n-1}} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial \alpha_{n-1}} \\ g_1^0(\alpha') & g_2^0(\alpha') & \dots & g_n^0(\alpha') \end{vmatrix}. \tag{48}$$

Then (44) follows from (47) and (48). □

Proposition 6 leads us to the following definition of *noncharacteristic condition* for the Cauchy problem (3), (5).

Definition 1. We say that the Cauchy problem (3), (5) is non-characteristic if the following condition holds:

$$\begin{vmatrix} \dfrac{\partial X_1^0(\alpha')}{\partial\alpha_1} & \dfrac{\partial X_2^0(\alpha')}{\partial\alpha_1} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial\alpha_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial X_1^0(\alpha')}{\partial\alpha_{n-1}} & \dfrac{\partial X_2^0(\alpha')}{\partial\alpha_{n-1}} & \dots & \dfrac{\partial X_n^0(\alpha')}{\partial\alpha_{n-1}} \\ g_1^0(\alpha') & g_2^0(\alpha') & \dots & g_n^0(\alpha') \end{vmatrix} \neq 0, \quad \forall\alpha', \tag{49}$$

where the vector $X^0(\alpha') = (X_1^0(\alpha'), X_2^0(\alpha'), \dots, X_n^0(\alpha'))$ is given in the initial condition (5) and $\vec{g}^0(\alpha') = (g_1^0(\alpha'), g_2^0(\alpha'), \dots, g_n^0(\alpha'))^T$ is given by (45).

Remark 2. We have the following remark on geometric interpretation of the noncharacteristic condition (49). Consider in R_x^n the surface Γ that is defined by equations (4). Then the vectors

$$\frac{\partial X^0(\alpha')}{\partial\alpha_k} = \Big(\frac{\partial X_1^0(\alpha')}{\partial\alpha_k}, \dots, \frac{\partial X_n^0(\alpha')}{\partial\alpha_k}\Big)^T, \quad k = 1, 2, \dots, n$$

are tangent to the surface Γ at the point $X^0(\alpha') = (X_1^0(\alpha'), \dots, X_n^0(\alpha'))$.

Thus the noncharacteristic condition (49) says that the vector $\vec{g}^0(\alpha')$, defined by (45) at the point $X^0(\alpha') = (X_1^0(\alpha'), \dots, X_n^0(\alpha')) \in \Gamma$, is not tangent to the surface Γ.

7. The Solvability of the Cauchy Problem in the Class of Analytic Functions

Applying the well-known Cauchy-Kovalevski theorem to the Cauchy problem (32), (15), we get the following consequence on local solvability of the Cauchy problem for the Monge-Ampère equation (3), (5) in the class of analytic functions.

Theorem 4. *Suppose that the functions $a_{ij}(x, z, p)$, $X_j^0(\alpha')$, $Z^0(\alpha')$, $P_j^0(\alpha')$, $j = 1, 2, \dots, n$ are analytic, and satisfy the conditions* (6), (49). *Then the Cauchy problem* (3), (5) *possesses locally a unique analytic solution $z(x)$.*

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