DUALITIES AND DIMENSIONS
OF IRREDUCIBLE REPRESENTATIONS
OF PARABOLIC SUBGROUPS OF LOW DEGREES

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Dedicated to Professor Huynh Mui on the occasion of his sixtieth birthday

Abstract. Let $GL_{n_1, \ldots, n_r}$ be a parabolic subgroup of the general linear group $GL_n$ over the prime field $\mathbb{F}_p$ of $p$ elements. A complete set of distinct irreducible modules for $\mathbb{F}_p[GL_{n_1, \ldots, n_r}]$ was explicitly constructed in [7]. In this paper, we use this construction to determine the contragredient dual module of each $\mathbb{F}_p[GL_{n_1, \ldots, n_r}]$-irreducible module and prove that its dimension can be computed via the dimensions of some $\mathbb{F}_p[GL_{n_i}]$-irreducible modules.

1. Introduction

Let $p$ be a prime number, $\mathbb{F}_p$ the finite field of $p$ elements and $GL_n$ the general linear group of all $n \times n$ invertible matrices over $\mathbb{F}_p$. Let $n_1, \ldots, n_r$ be positive integers such that $n_1 + \cdots + n_r = n$. The parabolic subgroup $GL_{n_1, \ldots, n_r}$ of $GL_n$ is defined as follows

$$GL_{n_1, \ldots, n_r} = \left\{ \begin{pmatrix} B_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_r \end{pmatrix} \in GL_n : B_i \in GL_{n_i}, 1 \leq i \leq r \right\}.$$ 

Let $\mathbb{F}_p[x_1, \ldots, x_n]$ be the commutative polynomial algebra in $n$ indeterminants $x_1, \ldots, x_n$ over $\mathbb{F}_p$. We have an action of $GL_n$ on $\mathbb{F}_p[x_1, \ldots, x_n]$ in the usual way. In other words, $\mathbb{F}_p[x_1, \ldots, x_n]$ is thought of as an $\mathbb{F}_p[GL_n]$-module, and hence an $\mathbb{F}_p[G]$-module, for each subgroup $G$ of $GL_n$. For each $1 \leq i \leq n$, the $i$-th Dickson invariant is defined as follows

$$L_i = L_i(x_1, \ldots, x_i) = \begin{vmatrix} x_1 & x_2 & \cdots & x_i \\ x_1^p & x_2^p & \cdots & x_i^p \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p_i-1} & x_2^{p_i-1} & \cdots & x_i^{p_i-1} \end{vmatrix}.$$ 

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a sequence of nonnegative integers and put $L^\beta = \prod_{i=1}^n L_i^{\beta_i}$. Denote by $H_\beta(G)$ the $\mathbb{F}_p[G]$-submodule generated by $L^\beta$. It is obvious

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that \( H_\beta(G) \) is an \( \mathbb{F}_p \)-vector space with the generators \( \{ \sigma L^\beta : \sigma \in G \} \).

**Proposition 1.1** ([7, 1.1]).

\[
\{ H_\beta(GL_{n_1, \ldots, n_r}) : \beta = (\beta_1, \ldots, \beta_n), 0 \leq \beta_i \leq p-1, 1 \leq i \leq n, \\
\beta_{n_1} \beta_{n_1+n_2} \cdots \beta_{n_1+\cdots+n_r} \neq 0 \}
\]

is a complete set of \((p-1)^r p^{n-r}\) distinct irreducible modules for the algebra \( \mathbb{F}_p[GL_{n_1, \ldots, n_r}] \) and these modules are absolutely irreducible.

For each \( 0 \leq i \leq r \), put \( N_i = n_0 + \cdots + n_i \) with \( n_0 = 0 \). Denote by \( \mathbb{F}_p^{(n_1, \ldots, n_r)} \) the set of all sequences \( (\beta_1, \ldots, \beta_n) \) such that \( 0 \leq \beta_j \leq p-1, 1 \leq j \leq n \) and \( \beta_{N_i} \neq p-1, 1 \leq i \leq r \). By noting that

\[
H_{(\beta_1, \ldots, \beta_{N_i-1}, p-1, \beta_{N_i+1}, \ldots, \beta_n)}(GL_{n_1, \ldots, n_r}) \cong H_{(\beta_1, \ldots, \beta_{N_i-1}, 0, \beta_{N_i+1}, \ldots, \beta_n)}(GL_{n_1, \ldots, n_r})
\]

for \( 1 \leq i \leq r \), we can restate the above proposition as follows.

**Proposition 1.2.** \( \{ H_\beta(GL_{n_1, \ldots, n_r}) : \beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \} \) is a complete set of \((p-1)^r p^{n-r}\) distinct irreducible modules for the algebra \( \mathbb{F}_p[GL_{n_1, \ldots, n_r}] \) and these modules are absolutely irreducible.

An immediate consequence of the proposition is the following.

**Corollary 1.3.** \( \{ H_\beta(GL_n) : \beta \in \mathbb{F}_p^{(n)} \} \) is a complete set of \((p-1)p^{n-1}\) distinct irreducible modules for the algebra \( \mathbb{F}_p[GL_n] \) and these modules are absolutely irreducible.

We recall here the definition of the so-called contragredient module. Let \( G \) be a finite group, \( \mathbb{K} \) an arbitrary field and \( M \) a left \( \mathbb{K}[G] \)-module. The contragredient \( M^* \) of \( M \) is the left \( \mathbb{K}[G] \)-module in which the underlying vector space is the dual space \( M^* \) of \( M \) and with the module operation given by

\[
(g \phi)(m) = \phi(g^{-1} m)
\]

for \( g \in G, \phi \in M^* \), \( m \in M \). The operation is then extended to all \( \mathbb{K}[G] \) by linearity. It is easily verified that \( M^* \) is irreducible if and only if so is \( M \).

For each \( \beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \), the contragredient module \( H_\beta^p(GL_{n_1, \ldots, n_r}) \) of \( H_\beta(GL_{n_1, \ldots, n_r}) \) is irreducible. Since \( \{ H_\beta(GL_{n_1, \ldots, n_r}) : \beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \} \) is a complete set of distinct irreducible modules for the algebra \( \mathbb{F}_p[GL_{n_1, \ldots, n_r}] \), a natural question arising here is to determine \( \beta^* \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \) so that \( H_\beta^p(GL_{n_1, \ldots, n_r}) \) is isomorphic to \( H_{\beta^*}(GL_{n_1, \ldots, n_r}) \).

In order to state the results, we need the following notations.

Let \( \beta \) be an element of \( \mathbb{F}_p^{(n_1, \ldots, n_r)} \). For each \( 1 \leq i \leq r \), denote by \( \beta(i) \) the sequence \( (\beta_{N_i-1+1}, \ldots, \beta_{N_i-1}, \sum_{k=N_i}^{n_i} \beta_k) \in \mathbb{F}_p^{(n_i)} \), where \( 0 \leq h < p-1 \) is the remainder in the division of \( h \) by \( p-1 \).

Consider the correspondence \( t \) from \( \mathbb{F}_p^{(n_1, \ldots, n_r)} \) to \( \mathbb{F}_p^{(n_1)} \times \cdots \times \mathbb{F}_p^{(n_r)} \) given by \( \beta \mapsto (\beta(1), \ldots, \beta(i)) \). We can easily check that \( t \) is a one-to-one correspondence.
For each $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{F}_p^{(k)}$ with $k$ a positive integer, let
$$\gamma^* = (\gamma_{k-1}, \gamma_{k-2}, \ldots, \gamma_1, -(\gamma_1 + \cdots + \gamma_k)).$$
We have then
$$(\gamma^*)^* = (\gamma_1, \ldots, \gamma_{k-1}, -(\gamma_{k-1} + \cdots + \gamma_1 + -(\gamma_1 + \cdots + \gamma_k))) = \gamma.$$

For each $\beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)}$, define $\beta^* \in \mathbb{F}_p^{(n_1, \ldots, n_r)}$ to be the inverse image of $(\beta(1)^*, \ldots, \beta(r)^*)$ under $t$, i.e.
$$\beta^* = t^{-1}((\beta(1)^*, \ldots, \beta(r)^*)).$$
Since $(\beta(i)^*)^* = \beta(i)$ for $1 \leq i \leq r$, it is clear that $(\beta^*)^* = \beta$. We can explicitly express $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ via $\beta = (\beta_1, \ldots, \beta_n)$ as follows
$$\beta_i^* = \begin{cases} 
\beta_{N_k-i} & \text{if } N_{k-1} + 1 \leq i < N_k, \\
-\sum_{j=N_{k-1}+1}^{N_k-1} \beta_j & \text{if } i = N_k.
\end{cases}$$

We are now ready to state the results.

**Theorem A.** Let $H_\beta^*(GL_{n_1, \ldots, n_r})$ be the contragredient module of $H_\beta(GL_{n_1, \ldots, n_r})$ for $\beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)}$. Then
$$H_\beta^*(GL_{n_1, \ldots, n_r}) \cong H_{\beta^*}(GL_{n_1, \ldots, n_r})$$
as $\mathbb{F}_p[GL_{n_1, \ldots, n_r}]$-modules.

**Theorem B.** For every $\beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)},$
$$\dim_{\mathbb{F}_p} H_\beta(GL_{n_1, \ldots, n_r}) = \prod_{i=1}^r \dim_{\mathbb{F}_p} H_{\beta(i)}(GL_{n_i}).$$

For $p = 2$ and $n = 4$, we have

**Proposition C.** The dimensions of all irreducible $\mathbb{F}_2[GL_4]$-modules are given as follows

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\dim_{\mathbb{F}<em>2} H</em>\beta(GL_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0, 0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 0, 0, 0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(0, 1, 0, 0)$</td>
<td>6</td>
</tr>
<tr>
<td>$(0, 0, 1, 0)$</td>
<td>4</td>
</tr>
<tr>
<td>$(1, 1, 0, 0)$</td>
<td>20</td>
</tr>
<tr>
<td>$(1, 0, 1, 0)$</td>
<td>14</td>
</tr>
<tr>
<td>$(0, 1, 1, 0)$</td>
<td>20</td>
</tr>
<tr>
<td>$(1, 1, 1, 0)$</td>
<td>64</td>
</tr>
</tbody>
</table>
2. Proof of Theorem A

We first recall some facts on the coefficient space of a \( K[G] \)-module \( M \). Suppose \( M \) is a \( K[G] \)-module of finite dimension. Let \( \{m_j : j \in I\} \) be a \( K \)-basis of \( M \), we have

\[
 gm_j = \sum_{i \in I} r_{i,j}(g)m_i
\]

for \( g \in G, \ j \in I, \ r_{i,j}(g) \in K \). The functions \( r_{i,j} : G \to K \) are called coefficient functions of \( V \). Denote by \( K^G \) the space of all mappings from \( G \) to \( K \). The \( K \)-space spanned by coefficient functions is a subspace of \( K^G \), called the coefficient space of \( M \). It is independent of the choice of the basis \( \{m_j\} \). We denote this space by \( cf(M) = \sum_{i,j} K r_{i,j} \).

For each \( h \in G \), it follows from (2.1) that

\[
 (h^{-1}gh)m_j = \sum_{i \in I} r_{i,j}(h^{-1}gh)m_i.
\]

Acting \( h \) on the two sides of (2.2), we get

\[
 g(hm_j) = \sum_{i \in I} r_{i,j}(h^{-1}gh)(hm_i).
\]

Since \( \{m_j : j \in I\} \) is a \( K \)-basis of \( M \), so is \( \{hm_j : j \in I\} \). Hence (2.3) shows that if \( r \in cf(M) \), then \( rh \in cf(M) \), where \( rh(g) = r(h^{-1}gh) \) for each \( g \in G \).

Let \( M^* \) be the contragredient module of \( M \) and \( \{m_j^* : j \in I\} \) the dual \( K \)-basis of \( M^* \) with respect to the basis \( \{m_j : j \in I\} \) of \( M \). By the definition of \( M^* \), (2.1) leads to

\[
 gm_j^* = \sum_{i \in I} r_{j,i}(g^{-1})m_i^*.
\]

This equation implies that if \( r \in cf(M) \), then \( r^* \in cf(M^*) \), where \( r^*(g) = r(g^{-1}) \) for each \( g \in G \).

We summarize the above facts in the following lemma.

**Lemma 2.1.** Let \( M \) be a \( K[G] \)-module of finite dimension, \( M^* \) its contragredient module and \( r \in cf(M) \). Then

(i) \( rh \in cf(M) \) for each \( h \in G \),

(ii) \( r^* \in cf(M^*) \),

where \( rh(g) = r(h^{-1}gh) \) and \( r^*(g) = r(g^{-1}) \) for each \( g \in G \).

The following lemma holds for an algebraically closed field. Actually, it also holds for a splitting field of an algebra.

**Lemma 2.2 ([2, 27.8]).** Let \( K \) be a splitting field for an algebra \( A \) and \( \{M_1, \ldots, M_k\} \) a set of pairwise non-isomorphic irreducible \( A \)-modules with \( \dim_K M_r = n_r, \ 1 \leq r \leq k \). For each \( r \), consider a matrix of coefficient functions \( f^{(r)}_{i,j} : 1 \leq i, j \leq n_r \).
We introduce some abbreviated notations for minors of matrix. The minor on the rows \(k_1, \ldots, k_i\) and the columns \(j_1, \ldots, j_i\) of a matrix \(B\) is denoted by
\[ B^{(k_1 \ldots k_i)}_{(j_1 \ldots j_i)} \]
The \(i\)-th principal minor
\[ B^{(1 \ldots i)} \]
is briefly denoted by \(\det_i B\). The following lemma is entirely analogous to a result in [8].

**Lemma 2.3** (cf. [8, 2.3]). Let \(\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{F}_{p^{(n_1, \ldots, n_r)}}\) and \(B \in GL_{n_1, \ldots, n_r}\). Denote \(\det_\beta(B) = \prod_{i=1}^{n} (\det_i B)^{\beta_i}\). Then \(\det_\beta \in cf(H_\beta(GL_{n_1, \ldots, n_r}))\).

**Proof of Theorem A.** It follows from Lemma 2.1 and Lemma 2.3 that
\[ \det^* \beta \in cf(H^*_\beta(GL_{n_1, \ldots, n_r})) \]
and
\[ \det^J \beta \in cf(H^J_\beta(GL_{n_1, \ldots, n_r})) \]
for each \(J \in GL_{n_1, \ldots, n_r}\). By Lemma 2.2 and Proposition 1.2, the theorem will be proved if we can show that for a suitable choice of \(J\), \(\det^* \beta = \det^J \beta\), or equivalently, \(\det_\beta(B^{-1}) = \det_\beta(J^{-1}B)\) for each \(B \in GL_{n_1, \ldots, n_r}\).

For each positive integer \(m\), define the \(m \times m\)-matrix \(J_m\) as follows
\[ J_m = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix}_{m \times m} \]
It is easily checked that \(J_m^{-1} = J_m\) and
\[ (J_m^{-1}AJ_m) \begin{pmatrix} 1 & \cdots & m-i \\ 1 & \cdots & m-i \end{pmatrix} = A \begin{pmatrix} i+1 & \cdots & m \\ i+1 & \cdots & m \end{pmatrix} \]
for \(A \in GL_m\) and \(1 \leq i \leq m\). Exercise 972 of [5] shows that
\[ A^{-1} \begin{pmatrix} 1 & \cdots & i \\ 1 & \cdots & i \end{pmatrix} = A \begin{pmatrix} i+1 & \cdots & m \\ i+1 & \cdots & m \end{pmatrix} \frac{1}{|A|} \]
where \(|A|\) is the determinant of \(A\). We have then
\[ A^{-1} \begin{pmatrix} 1 & \cdots & i \\ 1 & \cdots & i \end{pmatrix} = \frac{(J_m^{-1}AJ_m) \begin{pmatrix} 1 & \cdots & m-i \\ 1 & \cdots & m-i \end{pmatrix}}{|A|} \]
\[\det_i(A^{-1}) = \frac{\det_{m-i}(J_m^{-1}AJ_m)}{\det_m(J_m^{-1}AJ_m)}.\]

For each \(\gamma \in \mathbb{F}_p^{(m)}\), we have
\[
\gamma^* = (\gamma_{m-1}, \gamma_{m-2}, \ldots, \gamma_1, -(\gamma_1 + \cdots + \gamma_m)),
\]
and therefore
\[
\det_\gamma(A^{-1}) = \prod_{i=1}^{m} \det_i^\gamma(A^{-1})
= \prod_{i=1}^{m} \left( \frac{\det_{m-i}(J_m^{-1}AJ_m)}{\det_m(J_m^{-1}AJ_m)} \right)^{\gamma_i} \quad \text{(by (2.5))}
= \det_\gamma^* (J_m^{-1}AJ_m).
\]

Let
\[
J = \begin{pmatrix}
J_{n_1} & & 0 \\
& \ddots & \vdots \\
0 & & J_{n_r}
\end{pmatrix} \in GL_{n_1, \ldots, n_r}.
\]

We prove that \(J\) is a matrix satisfying the equality \(\det_\beta(B^{-1}) = \det_\beta^* (J^{-1}BJ)\) for each \(B \in GL_{n_1, \ldots, n_r}\).

In fact, for each \(B = \begin{pmatrix}
B_1 & & \ast \\
& \ddots & \vdots \\
0 & & B_r
\end{pmatrix} \in GL_{n_1, \ldots, n_r},\) it is clear that
\[
J^{-1}BJ = \begin{pmatrix}
J_{n_1}^{-1}B_1J_{n_1} & & \ast \\
& \ddots & \vdots \\
0 & & J_{n_r}^{-1}B_rJ_{n_r}
\end{pmatrix},
\]
and hence
\[
\det_\beta(B^{-1}) = \prod_{i=1}^{r} \det_{\beta(i)}(B_i^{-1})
= \prod_{i=1}^{r} \det_{\beta(i)^*}(J_{n_i}^{-1}B_iJ_{n_i}) \quad \text{(by (2.6))}
= \det_\beta^* (J^{-1}BJ).
\]

The theorem follows.

**Remark 2.4.** (a) By using the same arguments as above, we can prove that the contravariant module of \(H_\beta(GL_{n_1, \ldots, n_r})\) is isomorphic to \(H_\beta(GL_{n_1, \ldots, n_r})\).

The contravariant \(H_\beta^0(GL_{n_1, \ldots, n_r})\) of \(H_\beta(GL_{n_1, \ldots, n_r})\) is the left \(\mathbb{F}_p[GL_{n_1, \ldots, n_r}]\)-module in which the underlying vector space is the dual space \(H_\beta^0(GL_{n_1, \ldots, n_r})\) and with the module operation given by
\[
(B\phi)(\ell) = \phi(B^t\ell)
\]
for $B \in GL_{n_1, \ldots, n_r}$, $\phi \in H^\beta_\ell(GL_{n_1, \ldots, n_r})$, $\ell \in H_\beta(GL_{n_1, \ldots, n_r})$ and $B^t$ the transpose of $B$.

Since $\det_\beta \in cf(H_\beta(GL_{n_1, \ldots, n_r}))$, it is similar to Lemma 2.1 that

$$\det^0_\beta \in cf(H^0_\beta(GL_{n_1, \ldots, n_r})),$$

where $\det^0_\beta(B) = \det_\beta(B^t)$ for each $B \in GL_{n_1, \ldots, n_r}$.

For each $1 \leq i \leq n$, it is clear that $\det_i(B) = \det_i(B^t)$, and hence $\det_\beta(B) = \det_\beta(B^t)$ for each $B \in GL_{n_1, \ldots, n_r}$. This obviously implies that $H^0_\beta(GL_{n_1, \ldots, n_r})$ is isomorphic to $H_\beta(GL_{n_1, \ldots, n_r})$ as an $F_p[GL_{n_1, \ldots, n_r}]$-module.

(b) For the irreducible modules of the general linear group $GL_n$, we have

$$H^\beta_\beta(GL_n) \cong H^\beta_\beta(GL_n)$$

and

$$H^0_\beta(GL_n) \cong H^0_\beta(GL_n)$$

as $F_p[GL_n]$-modules, where $\beta^* = (\beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, -(\beta_1 + \cdots + \beta_n))$.

3. PROOF OF THEOREM B

Let $G_i$ $(i = 1, 2)$ be finite groups and $G = G_1 \times G_2$ their direct product. Let $M_i$ be a $\mathbb{K}[G_i]$-module $(i = 1, 2)$. We equip $M_1 \otimes \mathbb{K} M_2$ with a $\mathbb{K}[G]$-module structure by setting

$$(g_1, g_2)(m_1 \otimes \mathbb{K} m_2) = g_1 m_1 \otimes \mathbb{K} g_2 m_2$$

for $g_i \in G_i$, $m_i \in M_i$, $i = 1, 2$. The operation is then extended to all $\mathbb{K}[G]$ by linearity.

Lemma 3.1 ([1, 27.15]). Let $G_i$ $(i = 1, 2)$ be finite groups and $G = G_1 \times G_2$ their direct product. Let $\{M_j : 1 \leq j \leq \nu_1\}$ and $\{N_k : 1 \leq k \leq \nu_2\}$ be respectively the complete sets of distinct irreducible modules for the algebras $\mathbb{K}[G_1]$ and $\mathbb{K}[G_2]$. Assume that $\mathbb{K}$ is a splitting field for $\mathbb{K}[G_i]$ $(i = 1, 2)$. Then

$$\{M_j \otimes \mathbb{K} N_k : 1 \leq j \leq \nu_1, 1 \leq k \leq \nu_2\}$$

is a complete set of $\nu_1 \nu_2$ distinct irreducible modules for the algebra $\mathbb{K}[G]$.

Let $GL_{n_1 \times \cdots \times n_r}$ be the subgroup of $GL_{n_1, \ldots, n_r}$ defined as follows

$$GL_{n_1 \times \cdots \times n_r} = \left\{ B = \begin{pmatrix} B_1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & B_r \end{pmatrix} \in GL_n : B_i \in GL_{n_i}, 1 \leq i \leq r \right\}.$$ 

We identify $GL_{n_1 \times \cdots \times n_r}$ with $GL_{n_1} \times \cdots \times GL_{n_r}$ by the group isomorphism given by $B \mapsto (B_1, \ldots, B_r)$.

Lemma 3.2. $\{H_\beta(GL_{n_1 \times \cdots \times n_r}) : \beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)}\}$ is a complete set of $(p-1)^r p^{n-r}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_{n_1 \times \cdots \times n_r}]$ and these modules are absolutely irreducible.
Proof. By Corollary 1.3 and Lemma 3.1, there are exactly \((p - 1)^r p^{n-r}\) distinct irreducible modules for the algebra \(F_p[GL_{n_1 \times \cdots \times n_r}]\), which is identified with the algebra \(F_p[GL_{n_1} \times \cdots \times GL_{n_r}]\). It is sufficient to prove that the modules \(H_\beta(GL_{n_1 \times \cdots \times n_r})\), for \(\beta \in F_p(n_1, n_r)\), are absolutely irreducible and distinct.

For each matrix

\[
B = \begin{pmatrix} B_1 & \cdots & B_r \end{pmatrix} \in GL_{n_1, \ldots, n_r},
\]

the matrix \(\overline{B} \in GL_{n_1 \times \cdots \times n_r}\) is defined as follows

\[
\overline{B} = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\
0 & B_r \\
\end{pmatrix}.
\]

The mapping \(B \mapsto \overline{B}\) homomorphically maps \(GL_{n_1, \ldots, n_r}\) onto \(GL_{n_1 \times \cdots \times n_r}\). It is clear that \(BL_i = \overline{B}L_i\) for \(B \in GL_{n_1, \ldots, n_r}, 1 \leq i \leq n\), and hence \(BL_\beta = \overline{B}L_\beta\) for each \(\beta \in F_p(n_1, \ldots, n_r)\).

Fix an element \(\beta \in F_p(n_1, \ldots, n_r)\). We first prove that, as \(F_p\)-spaces, \(H_\beta(GL_{n_1, \ldots, n_r})\) is the same as \(H_\beta(GL_{n_1 \times n_2})\). Indeed, the generators of the spaces \(H_\beta(GL_{n_1, \ldots, n_r})\) and \(H_\beta(GL_{n_1 \times n_2})\) are respectively

\[
S = \{ BL_\beta : B \in GL_{n_1, \ldots, n_r} \} \quad \text{and} \quad \overline{S} = \{ \overline{B}L_\beta : \overline{B} \in GL_{n_1 \times n_2} \}.
\]

It is clear that \(\overline{S}\) is a subset of \(S\). Since \(BL_\beta = \overline{B}L_\beta\) for each \(B \in GL_{n_1, \ldots, n_r}\), it follows that \(S\) is a subset of \(\overline{S}\). We have then \(S = \overline{S}\), which implies that the \(F_p\)-spaces \(H_\beta(GL_{n_1, \ldots, n_r})\) and \(H_\beta(GL_{n_1 \times n_2})\) are the same. We denote this space by \(H_\beta\) for short. We have an immediate remark that \(Bh = \overline{B}h\) for \(B \in GL_{n_1, \ldots, n_r}, h \in H_\beta\).

Let \(W\) be an \(F_p[GL_{n_1 \times \cdots \times n_r}]\)-submodule of \(H_\beta\). Since \(Bw = \overline{B}w\) for \(B \in GL_{n_1, \ldots, n_r}, w \in W\), it follows that \(BW = \overline{B}W = W\). Thus \(W\) is an \(F_p[GL_{n_1, \cdots, n_r}]\)-submodule of \(H_\beta\). Then \(W\) is trivial since \(H_\beta\) is irreducible as \(F_p[GL_{n_1, \cdots, n_r}]\)-module. This establishes the irreducibility of \(H_\beta\) as \(F_p[GL_{n_1 \times \cdots \times n_r}]\)-module.

Let \(M, N\) be irreducible \(K[G]\)-modules of finite dimensions. We recall the following elementary facts:

(i) \(M\) is absolutely irreducible if and only if \(\text{Hom}_K(M, M) = K\),

(ii) \(M\) and \(N\) are distinct if and only if \(\text{Hom}_K(M, N) = 0\).

By the above facts and Proposition 1.2, in order to prove the modules \(H_\beta, \beta \in F_p(n_1, \ldots, n_r)\), are absolutely irreducible and distinct, it is sufficient to show that

\[
\text{Hom}_{F_p[GL_{n_1 \times \cdots \times n_r}]}(H_\beta, H_{\beta'}) = \text{Hom}_{F_p[GL_{n_1, \ldots, n_r}]}(H_\beta, H_{\beta'})
\]
for \( \beta, \beta' \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \). However, this equality follows immediately from the fact that \( B h = B h \) for each \( B \in GL_{n_1, \ldots, n_r}, h \in H_\beta \) and \( \beta \in \mathbb{F}_p^{(n_1, \ldots, n_r)} \). The lemma is proved.

**Proof of Theorem B.** It follows from Lemma 2.3 that \( \det \beta \in cf(H_\beta(GL_{n_1, \ldots, n_r})) \). Since the \( \mathbb{F}_p \)-spaces \( H_\beta(GL_{n_1, \ldots, n_r}) \) and \( H_\beta(GL_{n_1 x \ldots x n_r}) \) are the same and \( \det \beta(B) = \det \beta(B) \) for each \( B \in GL_{n_1, \ldots, n_r} \), we have \( \det \beta \in cf(H_\beta(GL_{n_1 x \ldots x n_r})) \).

From Lemma 2.3 it also follows that \( \det \beta(i) \in cf(H_\beta(i)(GL_{n_i})) \) for each \( 1 \leq i \leq r \). Therefore

\[
\prod_{i=1}^{r} \det \beta(i) \in cf(H_\beta(1)(GL_{n_1}) \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} H_\beta(r)(GL_{n_r})),
\]

where

\[
(\prod_{i=1}^{r} \det \beta(i))(B) = \prod_{i=1}^{r} \det \beta(i)(B_i)
\]

for \( B = (B_1, \ldots, B_r) \in GL_{n_1 \times \ldots \times n_r} \). By the definitions of \( \det \beta \) and \( \beta(i) \) we have

\[
\det \beta(B) = (\prod_{i=1}^{r} \det \beta(i))(B).
\]

This fact together with Lemmas 2.2, 3.1 and 3.2 imply that

\[
H_\beta(GL_{n_1 \times \ldots \times n_r}) \cong H_\beta(1)(GL_{n_1}) \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} H_\beta(r)(GL_{n_r})
\]
as \( \mathbb{F}_p[GL_{n_1 \times \ldots \times n_r}] \)-modules. As a result,

\[
\dim_{\mathbb{F}_p} H_\beta(GL_{n_1, \ldots, n_r}) = \dim_{\mathbb{F}_p} H_\beta(GL_{n_1 \times \ldots \times n_r}) = \prod_{i=1}^{r} \dim_{\mathbb{F}_p} H_\beta(i)(GL_{n_i}).
\]

The theorem is proved.

**Remark 3.3.** Denote by \( R(G) \) the representation ring of a group \( G \). Then it follows easily from the above proof that

\[
R(GL_{n_1, \ldots, n_r}) \cong R(GL_{n_1}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R(GL_{n_r}).
\]

### 4. Proof of Proposition C

For each \( \beta \in \mathbb{F}_2^{(n)} \), denote \( H_\beta(GL_n) \) by \( H_\beta \) for brevity. We note that

- If \( \beta = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{F}_2^{(n)} \), then \( \dim \mathbb{F}_2 H_\beta = \binom{n}{i} \) by [6, 1.4].
- If \( \beta = (1, 1, \ldots, 1, 0) \in \mathbb{F}_2^{(n)} \), then \( H_\beta \) has been known to be the Steinberg module for \( \mathbb{F}_2[GL_n] \). The dimension of the Steinberg module for \( \mathbb{F}_2[GL_n] \) is equal to the order of the Sylow 2-subgroup of \( GL_n \), namely \( 2^{\frac{n(n-1)}{2}} \).
By the above facts, in order to determine the dimensions of all irreducible \( \mathbb{F}_2[GL_4] \)-modules, we only need to compute those of \( H_{(1,1,0,0)} \), \( H_{(1,0,1,0)} \) and \( H_{(0,1,1,0)} \). However, Theorem A implies that \( \dim_{\mathbb{F}_2} H_{(1,1,0,0)} = \dim_{\mathbb{F}_2} H_{(0,1,1,0)} \), and hence we only deal with \( H_{(1,1,0,0)} \) and \( H_{(1,0,1,0)} \).

For each \( 1 \leq k_1 < \cdots < k_3 \leq n \), \( \sigma \in GL_n \), let \( L_{k_1}, \ldots, k_i = L_k(x_{k_1}, \ldots, x_{k_i}) \) and

\[ \sigma_{k_1, \ldots, k_i} = \sigma \left( \begin{array}{ccc} k_1 & \cdots & k_i \\ 1 & \cdots & i \end{array} \right) \]

The following formula is of basic importance

\[ \sigma L_{1,\ldots,i} = \sum_{1 \leq k_1 < \cdots < k_i \leq n} \sigma_{k_1, \ldots, k_i} L_{k_1, \ldots, k_i}. \]

**Dimension of \( H_{(1,1,0,0)} \).** We have \( H_{(1,1,0,0)} \) is an \( \mathbb{F}_2 \)-vector space generated by \( \{ \sigma(L_1 L_{1,2}) : \sigma \in GL_4 \} \). For each \( \sigma \in GL_4 \),

\[ \sigma(L_1 L_{1,2}) = \left( \sum_{1 \leq i \leq 4} \sigma_i L_i \right) \left( \sum_{1 \leq j < k \leq 4} \sigma_{j,k} L_{j,k} \right) \]

\[ = \sum_{1 \leq i < j \leq 4} T_{i,j} + \sum_{1 \leq i < j < k \leq 4} T_{i,j,k}, \]

where

\[ T_{i,j} = \sigma_i \sigma_{i,j} L_{i,j} L_{i,j}, \quad T_{i,j,k} = \sigma_i \sigma_{j,k} L_{i,j} L_{i,j} + \sigma_j \sigma_{i,k} L_{i,j} L_{i,j} + \sigma_k \sigma_{i,j} L_{i,j} L_{i,j} \]

It is clear that \( \sigma_i \sigma_{j,k} + \sigma_j \sigma_{i,k} + \sigma_k \sigma_{i,j} = 0 \) and

\[ L_i L_{j,k} + L_j L_{i,k} + L_k L_{i,j} = 0. \]

We have then

\[ T_{i,j,k} = \sigma_i \sigma_{j,k} (L_{i,j} L_{j,k} L_{j,k}) + \sigma_j \sigma_{i,k} (L_{i,j} L_{i,j} L_{i,j}) \]

\[ = \sigma_i \sigma_{j,k} L_{i,j} L_{i,j} + \sigma_j \sigma_{i,k} L_{i,j} L_{i,j}. \]

We also denote by \( T_{i,j} \) and \( T_{i,j,k} \) the \( \mathbb{F}_2 \)-vector spaces generated by the sets \( \{ L_i L_{i,j}, L_j L_{i,j} \} \) and \( \{ L_j L_{i,k}, L_i L_{j,k} \} \), respectively. Let \( T_{(1,1,0,0)} \) be the sum of these spaces. Note that if \( f \in T_{i,j}, g \in T_{i,j,k} \), then

\[ f = x_i x_j f_1(x_i, x_j), \quad g = x_i x_j x_k g_1(x_i, x_j, x_k). \]

Therefore \( T_{(1,1,0,0)} \) is the direct sum of all spaces \( T_{i,j} \) and \( T_{i,j,k} \),

\[ T_{(1,1,0,0)} = \bigoplus_{1 \leq i < j \leq 4} T_{i,j} \bigoplus \bigoplus_{1 \leq i < j < k \leq 4} T_{i,j,k}. \]

It is easy to verify that the sets \( \{ L_i L_{i,j}, L_j L_{i,j} \} \) and \( \{ L_j L_{i,k}, L_i L_{j,k} \} \) are linearly independent over \( \mathbb{F}_2 \). Hence, from (4.2), the dimension of \( T_{(1,1,0,0)} \) is

\[ \dim_{\mathbb{F}_2} T_{(1,1,0,0)} = 2 \binom{4}{2} + 2 \binom{4}{3} = 20. \]

We prove that \( H_{(1,1,0,0)} = T_{(1,1,0,0)} \), and therefore \( \dim_{\mathbb{F}_2} H_{(1,1,0,0)} = 20 \). From (4.1), it follows that \( H_{(1,1,0,0)} \subseteq T_{(1,1,0,0)} \). In order to show \( T_{(1,1,0,0)} \subseteq H_{(1,1,0,0)} \), it suffices to prove that \( H_{(1,1,0,0)} \) contains the sets \( \{ L_i L_{i,j}, L_j L_{i,j} \} \) and \( \{ L_j L_{i,k}, L_i L_{j,k} \} \).
for $1 \leq i < j \leq 4$ and $1 \leq i < j < k \leq 4$. We will prove the cases where $(i, j) = (1, 2)$ and $(i, j, k) = (1, 2, 3)$.

Let
\[
\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
\tau_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

We have then
\[
\begin{align*}
L_1L_{1,2} &= \tau_1(L_1L_{1,2}), \\
L_2L_{1,2} &= \tau_2(L_1L_{1,2}), \\
L_1L_{2,3} &= \tau_2(L_1L_{1,2}) + \tau_3(L_1L_{1,2}) + \tau_5(L_1L_{1,2}), \\
L_2L_{1,3} &= \tau_1(L_1L_{1,2}) + \tau_4(L_1L_{1,2}) + \tau_6(L_1L_{1,2}).
\end{align*}
\]

Since $H_{(1,1,0,0)}$ is the $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2}) : \sigma \in GL_4\}$, it follows from the above equations that $\{L_1L_{1,2}, L_2L_{1,2}\}$ and $\{L_1L_{2,3}, L_2L_{1,3}\}$ are contained in $H_{(1,1,0,0)}$. \hfill \square

**Dimension of $H_{(1,0,1,0)}$.** $H_{(1,0,1,0)}$ is an $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2,3}) : \sigma \in GL_4\}$. For each $\sigma \in GL_4$,
\[
\sigma(L_1L_{1,2,3}) = \left( \sum_{1 \leq i \leq 4} \sigma_i L_i \right) \left( \sum_{1 \leq j < k < l \leq 4} \sigma_{j,k,l} L_{j,k,l} \right)
\]
\[
= \sum_{1 \leq i < j < k \leq 4} T_{i,j,k} + T_{1,2,3,4},
\]

where
\[
T_{i,j,k} = \sigma_i \sigma_{i,j,k} L_i L_{i,j,k} + \sigma_j \sigma_{i,j,k} L_i L_{i,j,k} + \sigma_k \sigma_{i,j,k} L_k L_{i,j,k},
\]
\[
T_{1,2,3,4} = \sigma_1 \sigma_{2,3,4} L_1 L_{2,3,4} + \sigma_2 \sigma_{1,3,4} L_2 L_{1,3,4} + \sigma_3 \sigma_{1,2,4} L_3 L_{1,2,4} + \sigma_4 \sigma_{1,2,3} L_4 L_{1,2,3}.
\]

Since
\[
\sigma_1 \sigma_{2,3,4} + \sigma_2 \sigma_{1,3,4} + \sigma_3 \sigma_{1,2,4} + \sigma_4 \sigma_{1,2,3} = 0
\]
and
\[
L_1L_{2,3,4} + L_2L_{1,3,4} + L_3L_{1,2,4} + L_4L_{1,2,3} = 0,
\]
we have
\[
T_{1,2,3,4} = (\sigma_1 \sigma_{2,3,4} + \sigma_3 \sigma_{1,2,4})(L_1L_{2,3,4} + L_4L_{1,2,3})
\]
\[
+ (\sigma_2 \sigma_{1,3,4} + \sigma_3 \sigma_{1,2,4})(L_2L_{1,3,4} + L_4L_{1,2,3}).
\]
We also denote by $T_{i,j,k}$ and $T_{1,2,3,4}$ the $\mathbb{F}_2$-vector spaces generated by the sets 
$\{L_iL_{i,j,k}, L_jL_{i,j,k}, L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{1,3,4} + L_4L_{1,2,3}\}$, respectively. Let $T_{(1,0,1,0)}$ be the sum of these spaces. It is clear that

$$
T_{(1,0,1,0)} = \bigoplus_{1 \leq i < j < k \leq 4} T_{i,j,k} \oplus T_{1,2,3,4}.
$$

Since the sets $\{L_iL_{i,j,k}, L_jL_{i,j,k}, L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{1,3,4} + L_4L_{1,2,3}\}$ are linearly independent over $\mathbb{F}_2$, it follows from (4.4) that

$$
\dim_{\mathbb{F}_2} T_{(1,1,0,0)} = 3 \binom{4}{3} + 2 = 14.
$$

We finally prove that $H_{(1,0,1,0)} = T_{(1,0,1,0)}$, and hence $\dim_{\mathbb{F}_2} H_{(1,1,0,0)} = 14$. It is sufficient to show that the sets $\{L_iL_{i,j,k}, L_jL_{i,j,k}, L_kL_{i,j,k}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{1,3,4} + L_4L_{1,2,3}\}$ are contained in $H_{(1,0,1,0)}$. We only consider the case $(i, j, k) = (1, 2, 3)$.

Let

$$
\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\tau_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
$$

We have

$$
L_1L_{1,2,3} = \tau_1(L_1L_{1,2,3}), \quad L_2L_{1,2,3} = \tau_2(L_1L_{1,2,3}), \quad L_3L_{1,2,3} = \tau_3(L_1L_{1,2,3}),
$$

$$
L_1L_{2,3,4} + L_4L_{1,2,3} = \tau_1(L_1L_{1,2,3}) + \tau_4(L_1L_{1,2,3}) + \tau_5(L_1L_{1,2,3}), \quad L_2L_{1,3,4} + L_4L_{1,2,3} = \tau_2(L_1L_{1,2,3}) + \tau_4(L_1L_{1,2,3}) + \tau_6(L_1L_{1,2,3}).
$$

Since $H_{(1,0,1,0)}$ is the $\mathbb{F}_2$-vector space generated by $\{\sigma(L_1L_{1,2,3}) : \sigma \in GL_4\}$, it follows from the above equations that $H_{(1,0,1,0)}$ contains the sets $\{L_1L_{1,2,3}, L_2L_{1,2,3}, L_3L_{1,2,3}\}$ and $\{L_1L_{2,3,4} + L_4L_{1,2,3}, L_2L_{1,3,4} + L_4L_{1,2,3}\}$.

The proposition is proved.

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