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GENERALIZED NULL SCROLLS IN THE n-DIMENSIONAL LORENTZIAN SPACE

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Abstract. In this paper, we define $(r+1)$-dimensional generalized null scrolls in the n-dimensional Lorentzian space R_1^n and examine their geometric invariants and characteristic properties.

1. Introduction

Ruled surfaces have an important role in Differential Geometry. The $(r+1)$-dimensional generalized ruled surfaces in the n-dimensional Euclidean space E^n are studied by Juza [7], Frank and Giering [4] and Thas [10]. Some properties of 2-dimensional ruled surfaces are given by Thas [11]. In recent years, the semi-ruled surfaces and their curvatures have been studied in the semi-Euclidean space E_ν^{n+1} (see [3]). However, these work constructed the generalized ruled surfaces as bases on spacelike curves or timelike curves in the semi-Euclidean space. Null curves have many properties very different from spacelike or timelike curves and they are very interesting and important in Differential Geometry (see [2]). Graves [5] first introduced the notion of B-scroll as bases on a null curve and a null line in the 3-dimensional Lorentzian space E_1^3.

In this paper, we introduce the notion of $(r+1)$-dimensional generalized null scrolls in the n-dimensional Lorentzian space and study their characteristic properties. To do this, we use a general Frenet equations of null curves and pseudo-orthonormal basis. We also obtaine curvatures of generalized null scrolls in the n-dimensional Lorentzian space R_1^n.

Let M be an m-dimensional Lorentzian submanifold of R_1^n. Let $\overline{\nabla}$ be a Levi-Civita connection of R_1^n and ∇ a Levi-Civita connection of M. If $X, Y \in \chi(M)$ and h is the second fundamental form of M, then we have the Gauss equation

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{1.1}$$

Let ξ be a unit normal vector field on M. Then the Weingarten equation is

$$\overline{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi, \tag{1.2}$$

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where A_ξ determines at each point a self-adjoint linear map on $T_p(M)$ and $\nabla^\perp$ is a metric connection on normal bundle of M. In this paper, A_ξ will be used for the linear map and the corresponding matrix of the linear map. From the equations (1.1) and (1.2), we have

$$\langle \overline{\nabla}_X Y, \xi \rangle = \langle h(X,Y), \xi \rangle \tag{1.3}$$

and

$$\langle \overline{\nabla}_X Y, \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{1.4}$$

Also by the equations (1.3) and (1.4),

$$\langle h(X,Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{1.5}$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ be an orthonormal basis of $\chi^\perp(M)$. Then there exist smooth functions $h^j(X,Y)$, $j = 1, \ldots, n-m$, such that

$$h(X,Y) = \sum_{j=1}^{n-m} h^j(X,Y)\xi_j \tag{1.6}$$

and furthermore we may define the mean curvature vector field H by

$$H = \sum_{j=1}^{n-m} \frac{\mathrm{trace} A_{\xi_j}}{m} \xi_j. \tag{1.7}$$

If $H(p) = 0$ for each $p \in M$, then M is said to be minimal [9].

Let ξ be a unit normal vector, then the Lipschitz-Killing curvature in the direction ξ at the point $p \in M$ is defined by

$$G(p, \xi) = \det A_\xi(p). \tag{1.8}$$

The Gauss curvature is defined by

$$G(p) = \sum_{j=1}^{n-m} G(p, \xi_j) \tag{1.9}$$

and if $G(p) = 0$ for all $p \in M$, we say that M is developable. In particular, if the Lipschitz-Killing curvature is zero for each point and each normal direction, then M is developable [11].

Following [6], we define $M(A)$ for any matrix $A = [a_{ij}]$ by

$$M(A) = \sum_{i,j} (a_{ij})^2.$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-m}\}$ be an orthonormal basis of $\chi^\perp(M)$. Then the scalar normal curvature K_N of M is defined by

$$K_N = \sum_{i,j=1}^{n-m} M(A_{\xi_i} A_{\xi_j} - A_{\xi_j} A_{\xi_i}). \tag{1.10}$$

Let M be a Lorentzian manifold. For every $X, Y, Z, W \in \chi(M)$, the 4^{th}. order covariant tensor field

$$R(X, Y, Z, W) = \langle R(Z, W)Y, X\rangle \tag{1.11}$$

is called the Riemannian-Christoffel curvature tensor field and its value at a point $p \in M$ is called the Riemannian-Christoffel curvature of M at p. The Riemann curvature at $p \in M$ is denoted by

$$K(P) = \langle R(X, Y)Y, X\rangle\big|_p. \tag{1.12}$$

From the equations (1.1) and (1.11) we get (see [9])

$$\langle R(X, Y)Y, X\rangle = \langle h(X, X), h(Y, Y)\rangle - \langle h(X, Y), h(X, Y)\rangle. \tag{1.13}$$

Let M be an m-dimensional Lorentzian manifold and P a null plane of $T_p(M)$. Then as a real number, $K_U(P)$ defined by

$$K_U(P)\big|_p = \frac{\langle R(W, U)U, W\rangle}{\langle W, W\rangle}\bigg|_p, \quad p \in M \tag{1.14}$$

is called the null sectional curvature of P with respect to U, where W is an arbitrary non-null vector in P and U is a null vector of $T_p(M)$ [2].

Let M be an n-dimensional Lorentzian manifold and R the Riemann curvature tensor. The tensor field Ric defined by

$$Ric(X, Y) = \sum_{i=1}^{n} \varepsilon_i \langle R(e_i, X)Y, e_i\rangle \tag{1.15}$$

is called the Ricci curvature tensor field, where $\{e_1, \dots, e_n\}$ is a system of orthonormal basis of $T_p(M)$ and the value of $Ric(X, Y)$ at $p \in M$ is called the Ricci curvature.

Let M be an n-dimensional Lorentzian manifold and $\{e_1, \dots, e_n\}$ an orthonormal basis of $T_p(M)$ at $p \in M$. The scalar curvature of M is defined by

$$\mathbf{r} = \sum_{i=1}^{n} \varepsilon_i Ric(e_i, e_i) \tag{1.16}$$

or

$$\mathbf{r} = \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_i\varepsilon_j \langle R(e_j, e_i)e_i, e_j\rangle, \tag{1.17}$$

where $\varepsilon_i = \langle e_i, e_i\rangle$ so that

$$\varepsilon_i = \begin{cases} -1, & \text{if } e_i \text{ timelike}, \\ 1, & \text{if } e_i \text{ spacelike} \end{cases}$$

(see [1]).

A basis $\{X, Y, Z_1, \ldots, Z_{n-2}\}$ of an n-dimensional Lorentzian space R_1^n is called a pseudo-orthonormal basis if the following conditions are fullfilled:

$$
\begin{aligned}
\langle X, X\rangle &= \langle Y, Y\rangle = 0,\\
\langle X, Y\rangle &= -1,\\
\langle X, Z_i\rangle &= \langle Y, Z_i\rangle = 0 \quad \text{for } 1 \leq i \leq n-2,\\
\langle Z_i, Z_j\rangle &= \delta_{ij}, \quad \text{for } 1 \leq i \leq n-2
\end{aligned}
\tag{1.18}
$$

(see [8]).

2. Null curves

In this section, we recall the notion of null curve in the Lorentzian manifold [2].

Let $(M, \langle , \rangle)$ be a real $(n+2)$-dimensional Lorentzian manifold and α a smooth null curve in M locally given by

$$\alpha^i = \alpha^i(t), \quad t \in I \subset R, \quad i \in \{1, \ldots, n+2\}$$

for a coordinate neighbourhood U on α. Then the tangent vector field

$$\frac{d\alpha}{dt} = \Big(\frac{d\alpha^1}{dt}, \ldots, \frac{d\alpha^{n+2}}{dt}\Big)$$

on U satisfies the condition

$$\Big\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \Big\rangle = 0.$$

We denote by $T\alpha$ the tangent bundle of α and $T\alpha^\perp$ is defined as follows

$$T\alpha^\perp = \bigcup_{p\in\alpha} T_p\alpha^\perp; \quad T_p\alpha^\perp = \{v_p \in T_p(M); \langle v_p, \xi_p\rangle = 0\},$$

where ξ_p is null vector tangent at any $p \in \alpha$. Clearly, $T\alpha^\perp$ is a vector bundle over α of rank $(n+1)$. Since ξ_p is null, it follows that the tangent bundle $T\alpha$ is a vector subbundle of $T\alpha^\perp$, of rank 1.

Suppose $S(T\alpha^\perp)$ is the complementary vector subbundle to $T\alpha$ in $T\alpha^\perp$, i.e.,

$$T\alpha^\perp = T\alpha \perp S(T\alpha^\perp),$$

where $\perp$ means the orthogonal direct sum. It follows that $S(T\alpha^\perp)$ is a nondegenerate n-dimensional vector subbundle of TM. We call $S(T\alpha^\perp)$ a screen vector bundle of α. We have

$$TM\mid_\alpha = S(T\alpha^\perp) \perp S(T\alpha^\perp)^\perp, \tag{2.1}$$

where $S(T\alpha^\perp)^\perp$ is a 2-dimensional complementary orthogonal vector subbundle to $S(T\alpha^\perp)$ in $TM\mid_\alpha$ [2].

Theorem 2.1. [2]. *Let α be a null curve of a semi-Riemannian manifold $(M, \langle,\rangle)$ and $S(T\alpha^{\perp})$ a screen vector bundle of α. Then there exists a unique vector bundle E over α of rank 1 such that on each coordinate neighbourhood $U \subset \alpha$ there is a unique section $N \in \Gamma(E\mid_U)$ satisfying*

$$\langle \frac{d\alpha}{dt}, N\rangle = -1, \quad \langle N, N\rangle = \langle N, X\rangle = 0, \tag{2.2}$$

for every $X \in \Gamma(S(T\alpha^{\perp})\mid_U)$.

Now, suppose α is a null curve of an $(n+2)$-dimensional Lorentzian manifold $(M, \langle,\rangle)$. Denote by ∇ the Levi-Civita connection on M and $\dfrac{d\alpha}{dt} \equiv X$. Then the following equations can be obtained:

$$\begin{aligned}
\nabla_X X &= \lambda X + k_1 Z_1,\\
\nabla_X Y &= -\lambda Y - k_2 Z_1 - k_3 Z_2,\\
\nabla_X Z_1 &= k_2 X + k_1 Y + k_4 Z_2 + k_5 Z_3,\\
\nabla_X Z_2 &= k_3 X - k_4 Z_1 + k_6 Z_3 + k_7 Z_4,\\
&\cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\\
\nabla_X Z_{n-1} &= -k_{2n-3} Z_{n-3} - k_{2n-2} Z_{n-2} + k_{2n} Z_n,\\
\nabla_X Z_n &= -k_{2n-1} Z_{n-2} - k_{2n} Z_{n-1}
\end{aligned} \tag{2.3}$$

provided $n \geq 5$, where λ and $\{k_1, \ldots, k_{2n}\}$ are smooth functions on U and $\{Z_1, \ldots, Z_n\}$ is a certain orthonormal basis of $\Gamma(S(T\alpha^{\perp})\mid_U)$. We call $F = \{X, Y, Z_1, \ldots, Z_n\}$ a Frenet frame. It is also called a pseudo-orthonormal frame since the equations (1.18) holds on M along α with respect to the screen vector bundle $S(T\alpha^{\perp})$. The functions $\{k_1, \ldots, k_{2n}\}$ are called curvature functions of α with respect to F and the equations (2.3) are called the Frenet equations with respect to F [2].

3. Generalized null scrolls in R^n_1

Let $\alpha : I \subset R \to R^n_1$ be a smooth null curve in the n-dimensional Lorentzian space R^n_1 and $\{X, Y, Z_1, \ldots, Z_{n-2}\}$ be a pseudo-orthonormal frame along the null curve α. Let $\{Y(t), Z_1(t), \ldots, Z_{r-1}(t)\}$ be a null basis defined at each point $\alpha(t)$ of the null curve α. This system spans a subspace of the tangent space $T_{\alpha(t)}(R^n_1)$ at $\alpha(t) \in R^n_1$. This space is denoted by $W_r(t)$. It is a r-dimensional subspace of the form

$$W_r(t) = Sp\{Y(t), Z_1(t), \ldots, Z_{r-1}(t)\} \subset R^n_1.$$

$W_r(t)$ will be called a degenerate subspace and the following equalities are satisfied:

$$\begin{aligned}
\langle Y(t), Y(t)\rangle &= 0,\\
\langle Z_i(t), Z_j(t)\rangle &= \delta_{ij},\\
\langle Y(t), Z_i(t)\rangle &= 0
\end{aligned}$$

for $1 \leq i,\, j \leq r-1$.

Definition 3.1. Let α be a null curve in the n-dimensional Lorentzian space R_1^n. While the r-dimensional degenerate subspace $W_r(t)$ moves along a null curve α in R_1^n, it forms an $(r+1)$-dimensional surface. This is called the $(r+1)$-dimensional generalized null scroll in the n-dimensional Lorentzian space R_1^n and denoted by M. Then M can be expressed by the parametric equation

$$\Psi : I\,X\,R^r \to R_1^n,$$

$$(t,u) \to \Psi(t,u) = \alpha(t) + u_0 Y(t) + \sum_{i=1}^{r-1} u_i Z_i(t) \tag{3.1}$$

where $u = (u_0, u_1, \ldots, u_{r-1})$. Note that

$$\begin{aligned}\operatorname{rank}(\Psi_t, \Psi_{u_0}, \ldots, \Psi_{u_{r-1}}) &= \operatorname{rank}\Big(\alpha'(t) + u_0 Y'(t) \\ &\qquad + \sum_{i=1}^{r-1} u_i Z_i'(t), Y(t), Z_1(t), \ldots, Z_{r-1}(t)\Big) \\ &= r+1.\end{aligned}$$

It is easy to check that M is a Lorentzian submanifold.

Definition 3.2. $W_r(t)$ is called the generating space (or generating degenerate space) at the point $\alpha(t)$ of the $(r+1)$-dimensional generalized null scroll M and the null curve α is called the base curve of M.

Definition 3.3. Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n. The subspace

$$\Re(t) = Sp\{Y(t), Z_1(t), \ldots, Z_{r-1}(t), Y'(t), Z_1'(t), \ldots, Z_{r-1}'(t)\}$$

is said to be the asymptotic bundle of the generalized null scroll M.

Definition 3.4. Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n. If there exists a timelike or spacelike curve such that it meets perpendicularly to each one of the generating spaces, then this curve is called an orthogonal trajectory of the generalized null scroll M.

Definition 3.5. Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $W_r(t)$ be generating space of M. If there exists a null curve β on M such that for each $t \in I$,

$$\begin{aligned}&\langle \beta'(t), Y(t)\rangle = -1, \\ &\langle \beta'(t), Z_i(t)\rangle = 0, \quad 1 \leq i \leq r-1,\end{aligned} \tag{3.2}$$

then the null curve β is called a pseudo-orthogonal trajectory of the generalized null scroll M.

Thus we have following theorem:

Theorem 3.1. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $W_r(t)$ be generating space of M and $\alpha(t)$ be base curve of M. Then there exists a pseudo-orthogonal trajectory if and only if*

$$\sum_{i=1}^{r-1} u_i \langle Z_i', Y\rangle = 0 \quad \textit{and} \quad u_i = -\mu \int u_0(t)dt + C, \tag{3.3}$$

where $\mu = \langle Y', Z_i\rangle$, $i = 1, \dots, r-1$; $u_0, u_1 \dots, u_{r-1} \in R$.

Proof. It can be easily derived from the equation (3.2). □

4. On the curvatures of generalized null scroll

Let M be an $(r+1)$-dimensional generalized null scroll and $\alpha(t)$, $t \in I$ the base curve of M. Let $\{Y(t), Z_1(t), \dots, Z_{r-1}(t)\}$ be a null basis of the generating space $W_r(t)$. Let us choose the base curve α to be a pseudo-orthogonal trajectory of the generating spaces $W_r(t)$. Then M is given by

$$\Psi(t, u_0, u_1, \dots, u_{r-1}) = \alpha(t) + u_0 Y(t) + \sum_{i=1}^{r-1} u_i Z_i(t), \tag{4.1}$$

where $(u_0, u_1, \dots, u_{r-1}) \in R$. Let us choose $X = \Psi_*\Big(\dfrac{\partial}{\partial t}\Big)$ such that $\{X, Y, Z_1, \dots, Z_{r-1}\}$ is a pseudo-orthonormal basis of the space of vector fields $\chi(M)$. By (4.1) and (1.1), we have

$$\overline{\nabla}_{Z_i} Z_j = 0, \quad \overline{\nabla}_Y Z_i = 0, \quad \overline{\nabla}_{Z_i} Y = 0, \quad \overline{\nabla}_Y Y = 0, \tag{4.2}$$

$$h(Z_i, Z_j) = 0, \quad h(Z_i, Y) = 0, \quad 1 \le i, j \le r-1. \tag{4.3}$$

This means that the generating space $W_r(t)$ is totally geodesic. Also, we get

$$\overline{\nabla}_{Z_i} X = h(Z_i, X), \quad 1 \le i \le r-1 \tag{4.4}$$

$$\overline{\nabla}_Y X = h(Y, X). \tag{4.5}$$

Theorem 4.1. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n. Consider the pseudo-orthonormal basis $\{X, Y, Z_1, \dots, Z_{r-1}\}$ in a neighbourhood of a point p of M. Then the null sectional curvature $K_X(P)$ in the two-dimensional direction null plane P of M spanned by the vectors X_p and $(Z_i)_p$ is given by*

$$K_X(P) = -\langle \overline{\nabla}_{Z_i} X, \overline{\nabla}_{Z_i} X\rangle \,|_p, \quad 1 \le i \le r-1. \tag{4.6}$$

Proof. From the equation (1.14), we can write

$$K_X(P) = \frac{\langle R(Z_i, X)X, Z_i\rangle}{\langle Z_i, Z_i\rangle}\cdot$$

Since $\{X, Y, Z_1, \dots, Z_{r-1}\}$ is a pseudo-orthonormal basis, we have

$$K_X(P) = \langle R(Z_i, X)X, Z_i\rangle. \tag{4.7}$$

Also, using the equations (1.13), (4.3) and (4.4) for $p \in M$, we get

$$K_X(P) = K(Z_i, X) = -\langle \overline{\nabla}_{Z_i} X, \overline{\nabla}_{Z_i} X\rangle \,|_p, \quad 1 \le i \le r-1, \tag{4.8}$$

and this completes the proof. □

Theorem 4.2. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $\{X, Y, Z_1, \ldots, Z_{r-1}\}$ is a pseudo-orthonormal basis in a neighbourhood of a point p of M. Then the sectional curvature in the two-dimensional direction Lorentzian plane σ of M spanned by $\{X_p, Y_p\}$ $p \in M$, given by*

$$K(\sigma) = K(X, Y) = \langle \overline{\nabla}_Y X, \overline{\nabla}_Y X \rangle \mid_p . \tag{4.9}$$

Proof. It can be similarly proved as Theorem 4.1. □

Corollary 4.1. *An $(r+1)$-dimensional generalized null scroll M in R_1^n is developable if and only if each sectional curvatures of M is identically zero, i.e.,*

$$K(X, Z_i) \equiv K(X, Y) \equiv 0, \quad 1 \leq i \leq r-1.$$

Proof. It is obvious from the equations (1.12), (1.13) and (4.3). □

Suppose that the vector field system $\{\xi_1, \xi_2, \ldots, \xi_{n-r-1}\}$ is an orthonormal basis of $T_pM^\perp$ at $p \in M$. Then

$$\{X, Y, Z_1, \ldots, Z_{r-1}, \xi_1, \xi_2, \ldots, \xi_{n-r-1}\}$$

is a pseudo-orthonormal basis of $T_p(R_1^n)$ at $p \in M$. Thus, the equations of the derivative can be written as follows:

$$\begin{aligned}
\overline{\nabla}_X \xi_j &= a_{00}^j X + b_{00}^j Y + \sum_{t=1}^{r-1} c_{0t}^j Z_t + \sum_{q=1}^{n-r-1} e_{0q}^j \xi_q, \\
\overline{\nabla}_Y \xi_j &= a_{10}^j X + b_{10}^j Y + \sum_{t=1}^{r-1} c_{1t}^j Z_t + \sum_{q=1}^{n-r-1} e_{1q}^j \xi_q, \\
\overline{\nabla}_{Z_i} \xi_j &= a_{1i}^j X + b_{1i}^j Y + \sum_{t=1}^{r-1} d_{it}^j Z_t + \sum_{q=1}^{n-r-1} f_{iq}^j \xi_q,
\end{aligned} \tag{4.10}$$

where $1 \leq i \leq r-1$, $1 \leq j \leq n-r-1$. If we consider the Weingarten equation (1.2) and the equations (4.10), then the matrix A_{ξ_j} that corresponds to the linear mapping is

$$A_{\xi_j} = -\begin{bmatrix}
a_{00}^j & b_{00}^j & c_{01}^j & c_{02}^j & \cdots & c_{0(r-1)}^j \\
0 & b_{10}^j & 0 & 0 & \cdots & 0 \\
0 & b_{11}^j & 0 & 0 & \cdots & 0 \\
0 & b_{12}^j & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & b_{1(r-1)}^j & 0 & 0 & \cdots & 0
\end{bmatrix} \tag{4.11}$$

and this means that $\det A_{\xi_j} = 0$ if $r > 1$, from which we obtain following corollary.

Corollary 4.2. *If $r > 1$, then the Lipschitz-Killing curvature of the $(r+1)$-dimensional generalized null scroll M is zero at each point in each normal direction.*

Theorem 4.3. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $\{X, Y, Z_1, \ldots, Z_{r-1}\}$ a pseudo-orthonormal basis of $\chi(M)$. Then the Ricci curvature of M is in the direction of the vector fields Y and Z_j, $1 \leq i \leq r-1$, satisfies, respectively,*

$$Ric(Y, Y) = -\sum_{j=1}^{n-r-1} (b_{10}^j)^2, \tag{4.12}$$

$$Ric(Z_i, Z_i) = -\sum_{j=1}^{n-r-1} (b_{1i}^j)^2, \tag{4.13}$$

where b_{10}^j and b_{1i}^j are the components of the matrix A_{ξ_j}.

Proof. Using the equations (1.3), (1.4), (1.13) and (1.15), we get the equations (4.12) and (4.13). □

Corollary 4.3. *The Ricci curvature of the $(r+1)$-dimensional generalized null scroll M in R_1^n in the direction of the vector field X is given by*

$$Ric(X, X) = \sum_{i=1}^{r-1} Ric(Z_i, Z_i) + Ric(Y, Y). \tag{4.14}$$

Theorem 4.4. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $\{X, Y, Z_1, \ldots, Z_{r-1}\}$ a pseudo-orthonormal basis of $\chi(M)$. Then the scalar curvature of M is equal to twice Ricci curvature in the direction of the vector field X.*

Proof. By the equation (1.16) the scalar curvature of M can be expressed by

$$\mathbf{r} = Ric(X, X) + Ric(Y, Y) + \sum_{i=1}^{r-1} Ric(Z_i, Z_i).$$

Using Corollary 4.3 we obtain

$$\mathbf{r} = 2Ric(X, X). \tag{4.15}$$

This completes the proof. □

By (4.12), (4.13), (4.14), (4.15), we obtain the following corollary.

Corollary 4.4. *The scalar curvature of M is given by*

$$\mathbf{r} = -\left[\sum_{i=1}^{r-1} \sum_{j=1}^{n-r-1} (b_{1i}^j)^2 + \sum_{j=1}^{n-r-1} (b_{10}^j)^2\right]. \tag{4.16}$$

Theorem 4.5. *Let M be an $(r+1)$-dimensional generalized null scroll in R_1^n and $\{X, Y, Z_1, \ldots, Z_{r-1}\}$ be a pseudo-orthonormal basis of $\chi(M)$ and X be the tangent vector of the base curve of M. Then the mean curvature of M is*

$$H = -\frac{2}{r+1} h(X, Y). \tag{4.17}$$

Proof. By the equations (4.4), (4.5), (4.10) and (4.11) we see that

$$h(X,Y)=\sum_{j=1}^{n-r-1} b_{10}^{j}\xi_j, \tag{4.18}$$

$$\operatorname{trace}(A_{\xi_j})=-2b_{10}^{j}. \tag{4.19}$$

Therefore, we can write

$$h(X,Y)=-\frac{1}{2}\sum_{j=1}^{n-r-1}(\operatorname{trace}A_{\xi_j})\xi_j. \tag{4.20}$$

Thus, from the equation (1.7) we obtain

$$H=-\frac{2}{r+1}h(X,Y).$$

□

Hence we get following corollary.

Corollary 4.5. *$(r+1)$-dimensional generalized null scroll M is minimal if and only if $h(X,Y)=0$.*

Theorem 4.6. *Let $\{Y,Z_1,\ldots,Z_{r-1}\}$ be a null basis for the generating space of an $(r+1)$-dimensional generalized null scroll M and X be a tangent vector field to the base curve. Suppose that the pseudo-orthogonal trajectory of the generating space is the base curve of M. Then the minimal generalized null scroll M is totally geodesic if and only if X is an asymptotic vector field and the conjugate to each vector field Z_i, $1\le i\le r-1$.*

Proof. Let $\{X,Y,Z_1,\ldots,Z_{r-1}\}$ be a pseudo-orthonormal basis of $\chi(M)$ and for each $U,V\in\chi(M)$ we can write

$$U=a_0X+b_0Y+\sum_{i=1}^{r-1}b_iZ_i,$$
$$V=a_1X+c_0Y+\sum_{j=1}^{r-1}c_jZ_j$$

and from now

$$\begin{aligned}h(X,Y)=&\,a_0a_1h(X,X)+(a_0c_0+b_0a_1)h(X,Y)+b_0c_0h(Y,Y)\\&+\sum_{i=1}^{r-1}(a_0c_i+a_1b_i)h(X,Z_i)+\sum_{i=1}^{r-1}(b_0c_i+c_0b_i)h(Y,Z_i)\\&+\sum_{i,j=1}^{r-1}b_ic_ih(Z_i,Z_j).\end{aligned} \tag{4.21}$$

If M is totally geodesic, then h is identically zero [2]. So we have

$$h(X,X)=0,$$
$$h(X,Z_i)=0, \quad 1\le i\le r-1.$$

This means that X is an asymptotic vector field and conjugate to each vector field Z_i [2].

If $h(X,X)=0$ and $h(X,Z_i)=0$, $1\le i\le r-1$, then from the equation (4.21) we find $h(X,Y)=0$ and this completes the proof. □

From (4.18), (4.19) and (4.20), we obtain

$$\frac{(r+1)^2}{4}\,\|H\|^2=\sum_{j=1}^{n-r-1}(b^j_{10})^2. \tag{4.22}$$

Theorem 4.7. *The scalar normal curvature of an $(r+1)$-dimensional null scroll M is given by*

$$\begin{aligned}K_N={}&\frac{(r+1)^2}{2}H^2\sum_{i=1}^{n-r-1}\sum_{t=1}^{r-1}(c^i_{0t})^2-\Big(\mathbf{r}+\frac{1}{2}(r+1)^2H^2\Big)\Big(\frac{(r+1)^2}{4}H^2\Big)\\&+2\sum_{i,j=1}^{n-r-1}\sum_{t=1}^{r-1}\big((c^i_{0t}b^j_{1t})^2-c^i_{0t}b^j_{1t}c^j_{0t}b^i_{1t}-c^j_{0t}b^i_{10}c^i_{0t}b^j_{10}-b^i_{1t}b^j_{10}b^j_{1t}b^i_{10}\big)\\&+2\sum_{i,j=1}^{n-r-1}\sum_{t\neq k=1}^{r-1}(c^i_{0t}b^j_{1t}c^i_{0k}b^j_{1k}+c^i_{0t}b^j_{1t}c^j_{0k}b^i_{1k}),\end{aligned}\tag{4.23}$$

where H and $\mathbf{r}$ are the mean curvature vector field and scalar curvature of M, respectively and c^j_{0t}, b^j_{10}, b^j_{1t}, $1\le t\le r-1$, $1\le j\le n-r-1$ are the elements of the matrix A_{ξ_j}.

Proof. It can be easily proved from the equation (1.10) and by the some calculations. □

Corollary 4.6. *The scalar normal curvature of a minimal $(r+1)$-dimensional generalized null scroll is given by*

$$\begin{aligned}K_N=2\Big\{&\sum_{i,j=1}^{n-r-1}\sum_{t=1}^{r-1}[(c^i_{0t}b^j_{1t})^2-c^i_{0t}b^j_{1t}c^j_{0t}b^i_{1t}-c^j_{0t}b^i_{10}c^i_{0t}b^j_{10}-b^i_{1t}b^j_{10}b^j_{1t}b^i_{10}]\\&+\sum_{i,j=1}^{n-r-1}\sum_{t\neq k=1}^{r-1}(c^i_{0t}b^j_{1t}c^i_{0k}b^j_{1k}+c^i_{0t}b^j_{1t}c^j_{0k}b^i_{1k})\Big\}\end{aligned}\tag{4.24}$$

Proof. If M is minimal, $H=0$ and so the corollary is clear. □

Corollary 4.7. *If the generalized null scroll M is minimal and totally geodesic, then the scalar normal curvature of M is identically zero.*

Proof. Since M is minimal and totally geodesic, A_{ξ_j} is the zero map for each $j = 1, \ldots, n-r-1$. So the scalar normal curvature of M is identically zero. This completes the proof. □

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