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ON CONVERGENCE OF MULTIPARAMETER MULTIVALUED MARTINGALES

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Abstract. The aim of this note is to prove some convergence theorems in Mosco's sense for multiparameter multivalued martingales.

1. Introduction and Preliminaries

The study of multivalued functions (or multifunctions) has been developed extensively with applications in several areas, such as mathematical economics, optimal control and decision theory. On the other hand, the theory of real-valued martingales indexed by $\mathbb{R} \times \mathbb{R}$ was initiated by Cairoli and Wash [1975]. Frangos and Sucheston [1985] introduced the concept of block martingales and proved their convergence theorems. Our main aim is to combine the ideas of these two approaches to prove some convergence theorems in Mosco's sense for multiparameter multivalued martingales.

Throughout this note we shall denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, X a separable Banach space with the dual X^* and 2^X is the set of all subsets of X. Further, we denote by $\mathcal{C}$ (resp. by $\mathcal{K}$) the family of non-empty closed convex subsets of X (resp. the family of non-empty weakly compact convex subsets of X).

For $A \in 2^X \setminus \emptyset$, $\emptyset$ being the empty set, let clA and $\overline{co}A$ denote the closure and the closed convex hull of A, respectively.

Define

$$|A| = \sup\{\|x\| : x \in A\},$$
$$d(x, A) := \inf\{\|x - y\| : y \in A\},$$
$$s(x^*, A) := \sup\{< x^*, x >: x \in A\}.$$

Given a sub-σ-field $\mathcal{A}$ of $\mathcal{F}$, a multifunction F is said to be *measurable* (resp. $\mathcal{A}$-*measurable*) if, for every open set U of X the set

$$F^-(U) := \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$$

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is a member $\mathcal{F}$ (resp. $\mathcal{A}$). A function $f : \Omega \to X$ is called a *selection* of F if, for any $\omega \in \Omega,\ f(\omega) \in F(\omega)$. A Castaing representation of F is a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable selections of F such that

$$F(\omega) = cl\{f_n(\omega) : n \in \mathbb{N}\}, \quad \forall \omega \in \Omega.$$

It is known (cf. Theorem III.9 [11]) that a multifunction F with nonempty closed values in X is measurable if it has a *Castaing representation* or, equivalently, for any $x \in X$, the real function $d(X, F(.))$ is measurable. Let $L^1(X)$ denote the Banach space of X-valued measurable functions f such that

$$\|f\|_1 = \mathbb{E}(\|f\|) < \infty.$$

Here and in the sequel we identify the elements of L^1 which are equal with probability one. In particular, we write L^1 instead of $L^1(\mathbb{R})$ and

$$L^1(X, \mathcal{A}) = \{f \in L^1(X) : f \text{ is } \mathcal{A}\text{-measurable}\}.$$

For any measurable multifunction F, we put

$$S^1(F, \mathcal{A}) := \{f \in L^1(X, \mathcal{A}) : f(\omega) \in F(\omega) \text{ a.s.}\}.$$

The multivalued *Aumann integral* of F is defined by

$$I(F) := \{Ef : f \in S^1(F, \mathcal{A})\}$$

(cf. Aumann [1]). Given a sub-σ-field $\mathcal{A} \subset \mathcal{F}$ and an integrable $\mathcal{F}$-measurable multifunction F, Hiai and Umegaki [11] proved the existence of a unique $\mathcal{A}$-measurable integrable multifunction G such that

$$S^1(G, \mathcal{A}) = cl\{\mathbb{E}(f|\mathcal{A}) : f \in S^1(F, \mathcal{F})\},$$

where the closure is taken with respect to the norm topology in $L^1(X, \mathcal{A})$. G is called the *(multivalued) conditional expectation of* F *relative to* $\mathcal{A}$ and is denoted by $\mathbb{E}(F|\mathcal{A})$. The ordering on $T = \mathbb{N}^d$ is defined as the natural one. Namely, for $s = (s_1, s_2, \dots, s_d)$ and $t = (t_1, t_2, \dots, t_n)$, we put $s \le t$ whenever $s_i \le t_i$ for each $i = 1, 2, \dots, d$. Let $A_t,\ t \in T$, be a sequence in $\mathcal{C}$. Define

$$\begin{aligned} s\text{-}\liminf A_t &:= \{x \in X : \lim_T d(x, A_t) = 0\}, \\ w\text{-}\limsup A_t &:= \{x \in X : w\text{-}\lim_S x_s = x \text{ for } x_s \in A_s \\ &\qquad \text{and some cofinal subset } S \subset T\}, \end{aligned}$$

where w-lim means the convergence in the weak topology of X. We denote by $M\text{-}\lim_T A_t = A$ the convergence of A_t to A in the Mosco's sense (cf. [14]), i.e., $w\text{-}\limsup A_t = A = s\text{-}\liminf A_t$.

Let $(\mathcal{F}_t, t \in T)$ be an increasing net of complete sub-algebras of $\mathcal{F}$ and $\mathcal{F}_\infty := \sigma(\bigcup_{t\in T} \mathcal{F}_t)$.

A net $F := (F_t)_{t\in T}$ of measurable $\mathcal{C}$-valued multifunctions is said to be adapted to $(\mathcal{F}_t)$ if, for any $t \in T,\ F_t$ is $\mathcal{F}_t$-measurable.

An adapted net $(F_t, \mathcal{F}_t)_{t\in T}$ is said to be a *(multivalued) martingale (submartingale, supermartingale*, resp.) if, for any $t \geq s$

$$\mathbb{E}(F_t|\mathcal{F}_s) = F_s \text{ a.s. } (\mathbb{E}(F_t|\mathcal{F}_s) \supset F_s,\ \mathbb{E}(F_t|\mathcal{F}_s) \subset F_s \ \text{ a.s, resp.}).$$

Given i and j such that $1 \leq i \leq j \leq d$ we put

$$\mathcal{F}_t^{i\to j} := \sigma\Big(\bigvee_{\{s\in\mathbb{N}^d : s_k = t_k, i\leq k\leq j\}} \mathcal{F}_s \Big).$$

In particular, if $i = j$, $\mathcal{F}_t^{i\to i}$ is denoted simply by $\mathcal{F}_t^i$. Futher, we denote the conditional expectation $\mathbb{E}(.|\mathcal{F}_s^{i\to j})$ by $\mathbb{E}_s^{i\to j}(.)$ and $\mathbb{E}(.|\mathcal{F}_s)$ by $\mathbb{E}_s$.

If $k \in \mathbb{N},\ \ 1 \leq k \leq d$, then we say that $(F_t, \mathcal{F}_t)$ is a multivalued block k-martingale (k-submartingale, k-supermartingale, resp.) if

$$\mathbb{E}_s^{1\to k} F_t = F_{(s_1,\dots,s_k,t_{k+1},\dots,t_d)},$$

$$\Big(\mathbb{E}_s^{1\to k} F_t \supset F_{(s_1,\dots,s_k,t_{k+1},\dots,t_d)},\ \mathbb{E}_s^{1\to k} F_t \subset F_{(s_1,\dots,s_k,t_{k+1},\dots,t_d)},\ \text{resp.}\Big)$$

whenever $s \leq t$. For single-valued block k-martingales, the reader is referred to [7].

Now suppose that $d = 2$. A stochastic basis $(\mathcal{F}_t)_{t\in T}$ is said to satisfy the condition (F_4) if $\mathcal{F}_t^1$ is conditionally independent of $\mathcal{F}_t^2$ given $\mathcal{F}_t$.

Let $f = (f_t)_{t\in T}$ be an adapted net of measurable functions from Ω into X. We shall say that (f_t) is a martingale selection of (F_t) (see [5], [9], [12]) if it satisties

a) $(f_t, \mathcal{F}_t)$ is an integrable X-valued martingale,

b) $\forall t \in T,\ f_t \in S^1(F_t, \mathcal{F}_t)$.

The set of all martingale selections of the net (F_t) will be denoted by $MS(F_t)$.

2. The main results

The following result shows the existence of martingale selections for two-parameter multivalued martingale. We will use the same notation T for $\mathbb{N}^2$.

Theorem 2.1. *Let $(F_t, \mathcal{F}_t)_{t\in T}$, be a martingale with values in $\mathcal{C}$. Then*

(i) *$MS(F_t) \neq \emptyset$.*

(ii) *For any $t \in T$, $Pr_s(MS(F_t))$ is dense in $S^1(F_s, \mathcal{F}_s)$, where Pr_s denotes the projection defined by $Pr_s(f) = f_s$, whenever $f = (f_t,\ t \in T)$.*

(iii) *There exists a countable subset D of $MS(F_t)$ such that, for any $s \in T$, $Pr_s(D)$ is a Castaing representation of F_s.*

For one-parameter multivalued martingale, this result was first obtained by Van Custem for weakly compact valued martingales and later, by Luu [12] for multivalued martingales with bounded values in an infinite dimentional Banach space, and by Hess for martingales with values in $\mathcal{C}$. Our method of the proof is similar to that of Coste [6].

Proof of Theorem 2.1. (i) For each $(s,t) \in \mathbb{N}^2$, $s \le t$, define the map $\alpha_{st} : S^1(F_t, \mathcal{F}_t) \to S^1(F_s, \mathcal{F}_s)$ by

$$\alpha_{st}(f) = \mathbb{E}(f|\mathcal{F}_s),\ f \in S^1(F_t, \mathcal{F}_t).$$

It is easy to check that, the system $\{S^1(F_t, \mathcal{F}_t),\ \alpha_{st}\}$ is a projective system of non-empty complete subsets of $L^1(X)$.

On the other hand, for any $s \le t$

$$\begin{aligned} cl(\alpha_{st}(S^1(F_t, \mathcal{F}_t))) &= cl\{\mathbb{E}(f|\mathcal{F}_s), f \in S^1(F_t, \mathcal{F}_t)\} \\ &= S^1(F_s, \mathcal{F}_s). \end{aligned}$$

Hence and by Mittag-Leffler's Theorem (cf. [2], p.II.17) the projective limit of the above projective system is non-empty. Futher, if $f = (f_t)$ is a member of the projective limit, then for any $(s,t) \in \mathbb{N}^2$, $s \le t$

$$Pr_s((f)) = \alpha_{st} \circ Pr_t((f)).$$

Hence $f_s = \mathbb{E}(f_t|\mathcal{F}_s)$ and (f_t) is a martingale selection of (F_t).

(ii) Follows from ([2], Proposition 8, p.I.64).

(iii) It follows from (ii) that $Pr_s(MS(F_t))$ is dense in $S^1(F_s, \mathcal{F}_s)$. Moreover, since for each $s \in T$, F_s has a Castaing representation in $S^1(F_s, \mathcal{F}_s)$ there exists a countable subset $D_s \subset MS(F_t)$ and a negligible set N_s of such that for each $\omega \notin \bigcup_{t \in T} N_t$, the set $\{f_s(\omega) : (f_s) \in D_t\}$ is dense in $F_s(\omega)$. Thus, D_s is a Castaing representation of F_s. Finally, putting

$$D = \bigcup_{t \in T} D_t$$

we obtain (iii), which completes the proof. □

We shall need the following lemmas.

Lemma 2.1. *Let $(C_t) \subset \mathcal{K}$ and $C \in \mathcal{H}$. Suppose that the following two conditions are satisfied:*

(i) $d(x, C) = \lim_T d(x, C_t)$, *for any $x \in X$,*

(ii) $s(x^*, C) = \lim_T s(x^*, C_t)$ *for every $x^* \in X^*$.*

Then $M\text{-}\lim_T C_t \longrightarrow C$.

Proof. The proof is carried out in two steps:

(a) We first prove that $w\text{-}\limsup C_t \subset C$. Suppose that $x \in X \backslash C$. Then, by the Separation Theorem (cf. [4]), there exists a $x^* \in X^*$ such that

$$(x^*, x) > s(x^*, C).$$

Therefore there exists $t_0 \in T$ and $\epsilon > 0$ such that, for any $t \in T$ and $t \ge t_0$,

$$(x^*, x) > s(x^*, C_t) + \varepsilon,$$

which shows that $x \notin w - \limsup C_t$.

(b) Let $x \in C$. Then $d(x, C) = 0$. It follows from (i) that

$$\lim d(x, C_t) = 0.$$

Then there exists a net $(x_t)_{t \in T}$ such that for any t, $x_t \in C_t$ which implies that $d(x, x_t) \to 0$. Thus $x \in s - \liminf C_t$ and the lemma is proved. □

The proof of the following lemma is given in [16].

Lemma 2.2. *If F is an integrable measurable multifunction, then for any $x \in X$ and $\mathcal{G} \subset \mathcal{F}$ we have*

(i) $|\mathbb{E}(F|\mathcal{G})| \leq \mathbb{E}(|F||\mathcal{G})$ *a.s.*

(ii) $d(x, \mathbb{E}(F|\mathcal{G}) \leq \mathbb{E}[d(x, F)|\mathcal{G}]$ *a.s.*

(iii) $s(x^*, \mathbb{E}(F|\mathcal{G})) = \mathbb{E}[s(x^*, F)|\mathcal{G}]$ *a.s.*, $x^* \in X^*$.

Corollary 2.1. *If $(F_t, \mathcal{F}_t)_{t \in \mathbb{N}^d}$ is a multivalued block martingale, then*

(i) *for any $x \in X$, $(d(x, F_t), \mathcal{F}_t)_{t \in T}$ is a block sub-martingale,*

(ii) *for any $x^* \in X$, the process $(s(x^*, F_t), \mathcal{F}_t)_{t \in T}$ is a block martingale.*

Further, let us denote by C^* a coutable dense (in the Mackey topology) subset of the closed unit ball B^* of X^* and by D^* the set of all rational linear combinations of members of C^*. It is clear that D^* is countable dense subset of X^* in the Mackey topology. Moreover, by a simple reasoning we have:

Lemma 2.3. *Let $(C_t)_{t \in T}$ be a net in $\mathcal{K}$ such that the following conditions are satisfied:*

(i) *There exists $K \in \mathcal{K}$ such that $C_t \subset K$, $\forall t \in T$.*

(ii) *For each $x^* \in D^*$ there exists the limit $\lim_T s(x^*, C_t)$.*

Then there exists $C \in \mathcal{K}$ such that for any $x^* \in X^*$

$$s(x^*, C) = \lim_T s(x^*, C_t).$$

Now, we are in a position to prove the following theorem.

Theorem 2.2. *If $(F_t, \mathcal{F}_t)_{t \in \mathbb{N}^d}$ is a multivalued block martingale which is $L \log^{d-1} L$-bounded, i.e.,*

$$\sup_t \mathbb{E}(|F_t| \log^{d-1} |F_t|) < \infty. \tag{1}$$

Then

(i) *for any $x^* \in X^*$, the block martingale $\big(s(x^*, F_t)\big)$ is convergent a.s. as $t \to \infty$.*

(ii) *if for any $t \in T$, $F_t \in \mathcal{K}$ and there exists $G \in \mathcal{K}$ such that for each $t \in T$, $F_t \subset G$, then there exists $F \in \mathcal{K}$ and a negligible subset N of Ω such that*

$$s(x^*, F(\omega)) = \lim_t s(x^*, F_t(\omega)) \quad \forall \omega \notin N, \ \forall x^* \in X^*.$$

Proof. (i) It follows from Corollary 2.1 that $(s(x^*, F_t))_{t\in T}$ is a real-valued block martingale. Futhermore, for any $x^* \in X^*$, the process $(s(x^*, F_t))_{t\in T}$ is bounded in $L\log^{d-1} L$. By Theorem 9.4.4 [7], the block-martingale $(s(x^*, F_t))_{t\in T}$ converges a.s. as $t \to \infty$. Since D^* is countable it is possible to find a negligible subset N such that $\lim\limits_T s(x^*, F_t(\omega))$ exists for any $x^* \in D^*$ and $\omega \in \Omega\backslash N$.

(ii) By Lemma 2.3, there exists a measurable multifunction F such that

$$s(x^*, F(\omega)) = \lim_T s(x^*, F_t(\omega)), \quad \forall\omega \in \Omega\backslash N,\ \forall x^* \in X^*$$

which completes the proof. □

We know that if $(\mathcal{F}_t)_{t\in T},\ T = \mathbb{N}^2$, is a stochastic basis satisfying the condition (F_4) then every uniformly integrable martingale is a block martingale. Thus we have the following

Theorem 2.3. *Suppose that X has the Radon-Nikodym property (RNP), X^* is separable and $(\mathcal{F}_t)_{t\in T}$ is a stochastic basis satisfying the condition (F_4). Assume, furthermore, that $((F_t, \mathcal{F}_t))_{t\in I}$ is a $\mathcal{K}$-valued martingale such that*

$$\sup_t \mathbb{E}(|F_t| \log |F_t|) < \infty$$

and $F_t(\omega) \subset K(\omega),\ \ \omega \in \Omega,\ \ t \in T$ for some $\mathcal{K}$-valued multifunction K. Then there exists a measurable multifunction F such that

$$M\text{-}\lim_T F_t = F \quad a.s.$$

Proof. Define

$$D := \{f \in L^1(\Omega, \mathcal{F}, \mathbb{P}, X) : \exists (f_t, \mathcal{F}_t) \in MS(F_t),\ \lim_T f_t = f\ a.s.\}$$

We shall show that D is a non-empty bounded convex and $\mathcal{F}_\infty$-decomposable subset.

By Theorem 2.1, there exists a martingale selection $(f_t, \mathcal{F}_t)_{t\in T}$ of $(F_t, \mathcal{F}_t)_{t\in T}$. Since

$$\sup_t \mathbb{E}(\|f_t\| \log \|f_t\|) \le \sup_t \mathbb{E}(|F_t| \log |F_t|) \le \infty,$$

X has RPN and $(\mathcal{F}_t)$ satisfies the condition (F_4) we infer, by virtue of Theorem 9.4.4 in [7], that the martingale (f_t) converges a.s to $f \in L^1(X)$. Thus D is non-empty.

It is clear that D is convex. It remains to prove that D is $\mathcal{F}_\infty$-decomposable. By Lemma 2.4 [5], it suffices to show that for any $A \in \bigcup\limits_T \mathcal{F}_t$, and $f, g \in D$, we have $f1_A + g1_{\overline{A}} \in D$, where $\overline{A} = \Omega\backslash A$.

Let $(f_t, \mathcal{F}_t)_{t\in T}$ and $(g_t, \mathcal{F}_t)_{t\in T}$ be martingale selections of (F_t) such that

$$f = \lim_T f_t, \quad \text{and} \quad g = \lim_T g_t\ a.s.$$

Since $A \in \bigcup_T \mathcal{F}_t$ there exists $t_0 \in T$ such that $A \in \mathcal{F}_{t_0}$. For $t \geq t_0$, we put

$$h_t = f_t 1_A + g_t 1_{\overline{A}}.$$

For $t_0 \not\leq t$, choose $t_1 \in T$ such that $t \vee t_0 \leq t_1$ and put

$$h_t = \mathbb{E}(h_{t_1} | \mathcal{F}_t).$$

Then $(h_t)_{t \in T}$ is the martingale selection of $(F_t)_{t \in T}$ and, by Theorem 9.4.4 [7], there exists h such that

$$h = \lim_T h_t \quad (a.s.)$$

and $f 1_A + g 1_{\overline{A}} = h \in D$. In the other words, D is $\mathcal{F}_\infty$-decomposable. Since clD is non-empty closed and decomposable there exists a measurable multifunction F such that

$$\overline{D} = S^1_F(\mathcal{F})$$

and $F(\omega) \in \mathcal{K}$ a.s. (cf. Theorem 3.1 [10]).

Our further aim is to prove that $M\text{-}\lim F_t = F$.

(a) We first show that $F \subset s\text{-}\liminf_t F_t$. Note that for every $f \in clD$ there exists a sequence $(f^n) \subset D$ such that $f^n \xrightarrow{L^1} f$. Passing to a subsequence if neccesary, one may assume that $f^n \to f$ (a.s). Moreover, for each $n = 1, 2, \ldots$ one can find a martingale selection $(f^n_t)_{t \in T} \in MS(F_t)$ such that $f^n = \lim_T f^n_t$ a.s. From this equality and the following trivial inequality

$$d(f^n(\omega), F_t(\omega)) \leq d(f^n(\omega), f^n_t(\omega)) \ (a.s)$$

it follows that

$$\lim_t d(f^n(\omega), F_t(\omega)) = 0 \ (a.s).$$

Letting $t \to \infty$ and $n \to \infty$ in the inequality

$$d(f(\omega), F_t(\omega)) \leq d(f(\omega), f^n(\omega)) + d(f^n(\omega), F_t(\omega))$$

we have

$$\lim_t d(f(\omega), F_t(\omega)) = 0 \ a.s.$$

Thus $f \in s\text{-}\liminf_t F_t$ a.s. which means that $F \subset s\text{-}\liminf_t F_t$ a.s.

(b) Next, by Theorem 2.2 there exists $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$ and a measurable multifunction H such that

$$s(x^*, H(\omega)) = \lim_t s(x^*, F_t(\omega))$$

for every $\omega \in \Omega \backslash N$ and any $x^* \in X$.

According to the proof of Lemma 2.3 we have $w\text{-}\limsup_t F_t \subset H$ (a.s.) which together with Theorem 2.8 [5] gives

$$M\text{-} \lim_{n \to \infty} F_{(n,n)} = H = F \ a.s.$$

Thus $w\text{-}\limsup_t F_t \subset F$ a.s. The proof is complete $\square$

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