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A SHARP ENDPOINT ESTIMATE FOR MULTILINEAR MARCINKIEWICZ INTEGRAL OPERATOR

LIU LANZHE

ABSTRACT. We establish a sharp estimate for multilinear Marcinkiewicz integral operator. As applications, we obtain the weighted norm inequalities and $L\log L$ type estimate for the multilinear operator.

1. Introduction and results

Suppose that S^{n-1} is the unit sphere of $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip$_\gamma$ condition on $S^{n-1} (0 < \gamma \leq 1)$, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\displaystyle\int_{S^{n-1}} \Omega(x')dx' = 0.$

Fix $\lambda > 1$. Let m be a positive integer and A a function on R^n. We put

$$\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x - y| < t\}$$

and denote the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\lambda^A(f)(x) = \Big[\iint_{R_+^{n+1}} \Big(\frac{t}{t + |x-y|} \Big)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \Big]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

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and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

We write

$$F_t(f)(y) = \int_{|y-z| \le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z) dz$$

and define

$$\mu_\lambda(f)(x) = \Big(\iint_{R^{n+1}_+} \Big(\frac{t}{t+|x-y|} \Big)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \Big)^{1/2},$$

which is the Marcinkiewicz integral operator (see [16]).

Let H be the Hilbert space

$$H = \Big\{ h : ||h|| = \Big(\iint_{R^{n+1}_+} |h(t)|^2 \frac{dydt}{t^{n+3}} \Big)^{1/2} < \infty \Big\}.$$

Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ and $F_t(f)(x)$ may be viewed as mappings from $(0,+\infty)$ to H, and it is clear that

$$\mu_\lambda^A(f)(x) = \Big\| \Big(\frac{t}{t+|x-y|} \Big)^{n\lambda/2} F_t^A(f)(x,y) \Big\|,$$

$$\mu_\lambda(f)(x) = \Big\| \Big(\frac{t}{t+|x-y|} \Big)^{n\lambda/2} F_t(f)(y) \Big\|.$$

Note that when $m = 0$, μ_λ^A is just the commutator of Marcinkiewicz integral operator (see [10], [16]), while when $m > 0$, it is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1]-[5]). In [8], the authors establish a sharp estimate for some multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the multilinear Marcinkiewicz operator. Then the weighted norm inequalities and the $L \log L$ type endpoint estimate for the multilinear operator are obtained by using this sharp estimate. We point out that some of our ideas come from [8] and [11].

First, let us introduce some notations (see [7], [11], [13]). For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where and in what follows, Q will denote a cube with sides parallel to the axes, and

$$f_Q = |Q|^{-1} \int_Q f(x)dx.$$

It is well-known that

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$. For $0 < r < \infty$, we define $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x\in Q} |Q|^{-1} \int_Q |f(y)|dy.$$

We write $M_p f = (M(f^p))^{1/p}$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1 f(x) = Mf(x)$ and $M^k f(x) = M(M^{k-1}f)(x)$ when $k \geq 2$.

Let B be a Young function and $\tilde{B}$ be the complementary associated to B, we denote for a function f,

$$||f||_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B(\frac{|f(y)|}{\lambda})dy \leq 1 \right\}$$

and the maximal function by

$$M_B f(x) = \sup_{x\in Q} ||f||_{B,Q}.$$

The main Young function to be used in this paper is $B(t) = t(1 + \log^+ t)$ and its complementary $\tilde{B} = \exp t$. The corresponding maximal are denoted by $M_{L\log L}$ and $M_{\exp L}$. We have the generalized Holder's inequality

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leq ||f||_{B,Q} ||g||_{B,Q},$$

the inequality (in fact they are equivalent),

$$M_{L\log L} f(x) \leq CM^2 f(x)$$

for any $x \in R^n$ and the following inequalities, for any $b \in BMO(R^n)$,

$$||b - b_Q||_{\exp L,Q} \leq C||b||_{BMO}, \quad |b_{2^{k+1}Q} - b_{2Q}| \leq 2k||b||_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [7]).

We shall prove the following theorems in Section 3.

Theorem 1. *Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$\big(\mu_\lambda^A(f)\big)_r^{\#}(x) \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2 f(x).$$

Theorem 2. *Let $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$, $w \in A_p$. Then μ_λ^A is bounded on $L^p(w)$, that is*

$$||\mu_\lambda^A(f)||_{L^p(w)} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p(w)}.$$

Theorem 3. *Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$, $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\eta > 0$,*

$$\begin{aligned} &w\big(\{x \in R^n : \mu_\lambda^A(f)(x) > \eta\}\big) \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \int_{R^n} \frac{|f(x)|}{\eta}\Big(1 + \log^+\Big(\frac{|f(x)|}{\eta}\Big)\Big) w(x)dx. \end{aligned}$$

As in [11], Theorems 2 and 3 follow from Theorem 1 and the boundedness of μ_λ with M. So we only need to prove Theorem 1.

2. Preliminaries

We begin with some preliminary lemmas.

Lemma 1. (Kolmogorov, [7, p. 485]). *Let $0 < p < q < \infty$. For any function $f \ge 0$ we define*

$$||f||_{WL^q} = \sup_{\eta>0} \eta |\{x \in R^n : f(x) > \eta\}|^{1/q},$$
$$N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p} / ||\chi_E||_{L^r}, (1/r = 1/p - 1/q),$$

where the sup *is taken for all measurable sets E with $0 < |E| < \infty$. Then*

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 2. [3, p. 448]. *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\Big)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 3. [11, p. 165]. *Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function f and for all $\eta > 0$,*

$$w(\{y \in R^n : M^2 f(y) > \eta\}) \le$$
$$\le C\eta^{-1} \int_{R^n} |f(y)|(1 + \log^+(\eta^{-1}|f(y)|))w(y)dy.$$

Lemma 4. *Let $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$, $1 < r \le \infty$, $1/q = 1/p + 1/r$. Then μ_λ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is*

$$||\mu_\lambda^A(f)||_{L^q} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^p}.$$

Proof. Using the Minkowski inequality and noting that

$$|x - z| \le 2t,$$
$$|y - z| \ge |x - z| - t \ge |x - z| - 3t$$

when $|x - y| \le t$, $|y - z| \le t$, and

$$|x - z| \le t(1 + 2^{k+1}) \le 2^{k+2}t,$$
$$|y - z| \ge |x - z| - 2^{k+3}t$$

when $|x - y| \le 2^{k+1}t$, $|y - z| \le t$, we have

$$\begin{aligned}
&\mu_\lambda^A(f)(x) \\
&\le \int_{R^n} \Big[\iint_{R_+^{n+1}} \Big(\frac{t}{t + |x - y|}\Big)^{n\lambda} \\
&\times \Big(\frac{|\Omega(y - z)||R_{m+1}(A; x, z)||f(z)|}{|y - z|^{n-1}|x - z|^m}\Big)^2 \chi_{\Gamma(z)}(y, t)\frac{dydt}{t^{n+3}}\Big]^{1/2} dz \\
&\le C \int_{R^n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x - z|^m} \\
&\times \Big[\int_0^\infty \int_{|x-y|\le t} \Big(\frac{t}{t + |x - y|}\Big)^{n\lambda} \frac{\chi_{\Gamma(z)}(y, t)}{(|x - z| - 3t)^{2n-2}} \frac{dydt}{t^{n+3}}\Big]^{1/2} dz \\
&+ C \int_{R^n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x - z|^m} \\
&\times \Big[\int_0^\infty \sum_{k=0}^\infty \int_{2^k t<|x-y|\le 2^{k+1}t} \Big(\frac{t}{t + |x - y|}\Big)^{n\lambda} \frac{\chi_{\Gamma(z)}(y, t)}{(|x - z| - 2^{k+3}t)^{2n-2}} \frac{dydt}{t^{n+3}}\Big]^{1/2} dz
\end{aligned}$$

$$\leq C\int\limits_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\Big[\int\limits_{|x-z|/2}^{\infty}\frac{dt}{(|x-z|-3t)^{2n}}\Big]^{1/2}dz$$

$$+C\int\limits_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}$$

$$\times\Big[\sum_{k=0}^{\infty}\int\limits_{2^{-2-k}|x-z|}^{\infty}2^{-kn\lambda}(2^kt)^nt^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n}}\Big]^{1/2}dz$$

$$\leq C\int\limits_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz$$

$$+C\int\limits_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n}}dz\Big[\sum_{k=0}^{\infty}2^{kn(1-\lambda)}\Big]^{1/2}$$

$$=C\int\limits_{R_n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}}|f(z)|dz.$$

Thus, the lemma follows from [4], [5]. □

3. Proof of Theorem 1

Fix a cube $Q=Q(x_0,l)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and

$$\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha.$$

Then $R_m(A;x,y)=R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}=D^\alpha A-(D^\alpha A)_{\tilde{Q}}$ for $|\alpha|=m$. We write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}F_t^A(f)(x,y)&=\int\limits_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m}f(z)dz\\&=\int\limits_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m}f_2(z)dz\\&\quad+\int\limits_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}f_1(z)dz\\&\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\limits_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\frac{(x-z)^\alpha}{|x-z|^m}D^\alpha\tilde{A}(z)f_1(z)dz,\end{aligned}$$

then

$$
\begin{aligned}
&|\mu_\lambda^A(f)(x) - \mu_\lambda^{\tilde{A}}(f_2)(x_0)| = \\
&= \Big| \| (\frac{t}{t+|x-y|})^{n\lambda/2} F_t^A(f)(x,y) \| - \| (\frac{t}{t+|x_0-y|})^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x,y) \| \Big| \\
&\le \Big\| (\frac{t}{t+|x-y|})^{n\lambda/2} F_t\Big(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1 \Big)(y) \Big\| \\
&\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \Big\| (\frac{t}{t+|x-y|})^{n\lambda/2} F_t\Big(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \Big)(y) \Big\| \\
&\quad + \Big\| (\frac{t}{t+|x-y|})^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x,y) - (\frac{t}{t+|x_0-y|})^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0,y) \Big\| \\
&= A(x) + B(x) + C(x).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big(\frac{1}{|Q|} \int_Q \big| \mu_\lambda^A(f)(x) - \mu_\lambda^{\tilde{A}}(f_2)(x_0) \big|^r dx \Big)^{1/r} \\
&\le \Big(\frac{C}{|Q|} \int_Q A(x)^r dx \Big)^{1/r} + \Big(\frac{C}{|Q|} \int_Q B(x)^r dx \Big)^{1/r} + \Big(\frac{C}{|Q|} \int_Q C(x)^r dx \Big)^{1/r} \\
&= I + II + III.
\end{aligned}
$$

Now, let us estimate I, II and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$
R_m(\tilde{A};x,y) \le C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.
$$

Thus, by Lemma 1 and the weak type (1,1) of μ_λ (see [6], [14]), we obtain

$$
\begin{aligned}
I &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} |Q|^{-1} \frac{||\mu_\lambda(f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} |Q|^{-1} ||\mu_\lambda(f_1)(f_1)||_{WL^1} \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} |\tilde{Q}|^{-1} \int_{\tilde{Q}} |f(y)| dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(\tilde{x}).
\end{aligned}
$$

Similarly, we can estimate II as follows

$$
\begin{aligned}
II &\le C\sum_{|\alpha|=m}|Q|^{-1}\frac{||\mu_\lambda(D^\alpha\tilde{A}f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \le C\sum_{|\alpha|=m}|Q|^{-1}||\mu_\lambda(D^\alpha\tilde{A}f_1)||_{WL^1}\\
&\le C\sum_{|\alpha|=m}|\tilde{Q}|^{-1}\int_{\tilde{Q}}|D^\alpha\tilde{A}(y)||f(y)|dy \le C\sum_{|\alpha|=m}||D^\alpha A||_{\exp L,\tilde{Q}}||f||_{L\log L,\tilde{Q}}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M_{L\log L}f(\tilde{x}) \le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2f(\tilde{x}).
\end{aligned}
$$

To estimate III, we write

$$
\begin{aligned}
&\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x,y)-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\\
&=\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda/2}\Big[\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\Big]\frac{\Omega(y-z)R_m(\tilde{A};x,z)f_2(z)}{|y-z|^{n-1}}dz\\
&\quad+\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda/2}\frac{\Omega(y-z)f_2(z)}{|x_0-z|^m|y-z|^{n-1}}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)]dz\\
&\quad+\int_{|y-z|\le t}\Big[\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda/2}-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\lambda/2}\Big]\times\\
&\quad\times\frac{\Omega(y-z)R_m(\tilde{A};x_0,z)}{|y-z|^{n-1}|x_0-z|^m}f_2(z)dz\\
&\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\le t}\Big[\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda/2}\frac{(x-z)^\alpha}{|x-z|^m}\\
&\quad-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\lambda/2}\frac{(x_0-z)^\alpha}{|x_0-z|^m}\Big]\frac{\Omega(y-z)D^\alpha\tilde{A}(z)}{|y-z|^{n-1}}f_2(z)dz\\
&=III_1+III_2+III_3+III_4.
\end{aligned}
$$

Note that $|x-z|\sim|x_0-z|$ for $x\in Q$ and $z\in R^n\setminus\tilde{Q}$. By Lemma 3 and the following inequality (see [15])

$$
|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO},\quad \text{for } Q_1\subset Q_2,
$$

we know that, for $x\in Q$ and $z\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{A};x,z)| &\le C|x-z|^m\sum_{|\alpha|=m}\big(||D^\alpha A||_{BMO}+|(D^\alpha A)_{\tilde{Q}(x,z)}-(D^\alpha A)_{\tilde{Q}}|\big)\\
&\le Ck|x-z|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO}.
\end{aligned}
$$

For III_1, by the condition on Ω and arguments similar to those in the proof of Lemma 4, we get

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q ||III_1||dx &\le \frac{C}{|Q|}\int\limits_Q\Big(\int\limits_{R^n\setminus\tilde{Q}}\frac{|x-x_0|}{|x_0-z|^{m+n+1}}|R_m(\tilde{A};x,z)||f(z)|dz\Big)dx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^\infty k\int\limits_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|Q|^{1/n}}{|x_0-z|^{n+1}}|f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k}\frac{1}{|2^k\tilde{Q}|}\int\limits_{2^k\tilde{Q}}|f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k}M(f)(\tilde{x})\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x}).\end{aligned}$$

For III_2, observe (see [3]) that

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-z)^\beta.$$

Therefore, using Lemma 3 we get

$$\begin{aligned}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)| &\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-z|^{|\beta|}||D^\alpha A||_{BMO}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0||x-z|^{m-1}.\end{aligned}$$

Thus, analysis similar to that in the proof of Lemma 4 implies that

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q ||III_2||dx &\le \frac{C}{|Q|}\int_Q\Big(\int\limits_{R^n\setminus\tilde{Q}}\frac{|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n}}|f(z)|dz\Big)dx\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^\infty\int\limits_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|Q|^{1/n}}{|x_0-z|^{n+1}}|f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty 2^{-k}\frac{1}{|2^k\tilde{Q}|}\int\limits_{2^k\tilde{Q}}|f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x}).\end{aligned}$$

For III_3, by the inequality $a^{1/2}-b^{1/2}\le(a-b)^{1/2}$ for $a\ge b>0$, as in the proof of Lemma 4 and III_1, we obtain

$$\frac{1}{|Q|}\int_Q ||III_3||dx$$
$$\le \frac{C}{|Q|}\int_Q\int_{R^n}\Big(\int_{R^{n+1}_+}\Big[\frac{t^{n\lambda/2}|x-x_0|^{1/2}\chi_{\Gamma(z)}(y,t)|R_m(\tilde{A};x_0,z)||f_2(z)|}{(t+|x-y|)^{(n\lambda+1)/2}|y-z|^{n-1}|x_0-z|^m}\Big]^2\frac{dydt}{t^{n+1}}\Big)^{1/2}dzdx$$
$$\le \frac{C}{|Q|}\int_Q\int_{R^n}|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|\Big(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2}}\Big)^{1/2}dzdx$$
$$\le \frac{C}{|Q|}\int_Q\int_{R^n}|f_2(z)|R_m(\tilde{A};x_0,z)|\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}dzdx$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k/2}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|dz$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x}).$$

For III_4, as in the proof of III_1 and III_3, we get

$$\begin{aligned}||III_4|| &\le C\sum_{|\alpha|=m}\int_{R^n}\Big(\frac{|x-x_0|}{|x_0-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}\Big)|D^\alpha\tilde{A}(y)||f_2(z)|dz\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty(2^{-k}+2^{-k/2})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)||D^\alpha A(z)-(D^\alpha A)_{\tilde{Q}}|dz\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2})\times\\
&\quad\times\big(||D^\alpha A||_{\exp L,2^k\tilde{Q}}||f||_{LlogL,2^k\tilde{Q}}+||D^\alpha A||_{BMO}M(f)(\tilde{x})\big)\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2})||D^\alpha A||_{BMO}M_{L\log L}(f)(\tilde{x})\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}).\end{aligned}$$

Thus

$$III\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

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Department of Applied Mathematics
Hunan University
Changsha, 410082, P.R.China

E-mail address: lanzheliu263.net