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ON SOME BADLY-SOLVED PROBLEMS WITH INVEXITY

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Abstract. Using invexity, D. V. Luu and N. X. Ha investigated in [4], [5], and [8] the problem

$$\begin{aligned} &\text{minimize} \quad f(x) = \inf_{\alpha \in A} f_\alpha(x), \\ &\text{subject to} \quad g(x) = \inf_{\beta \in B} g_\beta(x) \le 0, \quad x \in C. \end{aligned}$$

We show that their results concerned with this problem are wrong or too weak or redundant, while a lot of too strong and unnecessary assumptions are used. For correction, a necessary condition and a sufficient condition for local minima are given in this paper. Some comments and sufficient conditions for local minima of the other problems considered in [5] and [8] are also presented.

1. Introduction

In 1981, Hanson [7] introduced a class of differentiable functions $\phi : C \subset X \to \mathbb{R}$ satisfying

$$\phi(x) - \phi(x') \ge \langle \nabla\phi(x'), \eta(x, x')\rangle \quad \text{for all} \;\; x,\, x' \in C, \tag{1}$$

for some arbitrarily given function $\eta : C \times C \to X$, which were called *invex* by Craven [2], and proved that if all functions of a constrained minimization problem are invex with respect to a common function η, then the Kuhn-Tucker conditions become also sufficient for a (global) minimum.

To extend Hanson's result to Lipschitz functions, Reiland [12] used the Clarke [1] *generalized directional derivative* defined by

$$\phi^\circ(x; v) = \limsup_{y \to x,\, t \downarrow 0} \frac{(y + tv) - \phi(y)}{t}$$

to replace $\langle \nabla\phi(x'), \eta(x, x')\rangle$ by $\phi^\circ(x'; \eta(x, x'))$. Note that in the original definition of Hanson and Reiland X is equal to $\mathbb{R}^n$. But in this paper, X is a Banach space if not specified otherwise.

While Hanson and Reiland required the property (1) or its modification to hold everywhere in C to state a sufficient condition for global minima, it suffices to demand it locally if only local minima are considered. Concretely, a locally

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Lipschitz function $\phi : X \to \mathbb{R}$ is said to be *invex on U at $\overline{x}$ with respect to* $\eta : U \times U \to X$ if

$$\phi(x) - \phi(\overline{x}) \geq \phi^\circ(\overline{x}; \eta(x, \overline{x})) \quad \text{for all } \ x \in U \tag{2}$$

where U is a subset of X containing $\overline{x}$.

If ϕ is directionally differentiable at $\overline{x}$ in all directions, i.e., the one-sided *directional derivative*

$$\phi'(x; v) = \lim_{t \downarrow 0} \frac{\phi(x + tv) - \phi(x)}{t}$$

exists for all $v \in X$, then ϕ is called *weakly invex on U at $\overline{x}$ with respect to η* if

$$\phi(x) - \phi(\overline{x}) \geq \phi'(\overline{x}; \eta(x, \overline{x})) \quad \text{for all } \ x \in U. \tag{3}$$

From definitions it follows that $\phi'(\overline{x}; v) \leq \phi^\circ(\overline{x}; v)$. If ϕ is directionally differentiable at $\overline{x}$ in all directions and $\phi'(\overline{x}; v) = \phi^\circ(\overline{x}; v)$ for all $v \in X$, then ϕ is named *regular at $\overline{x}$* ([1], p. 39). Obviously, if ϕ is both weakly invex and regular at $\overline{x}$, then it is invex there.

What can weak invexity alone bring? To assert something concerned with optimization, one has to demand the considered function to be both weakly invex and regular at some interested point $\overline{x}$. But weak invexity along with regularity is not weaker but even stronger than invexity then.

Using the notions mentioned above, Luu and Ha investigated in [4], [5], and [8] the problem

$$(P_1) \qquad \begin{cases} \text{minimize} \quad f(x) = \inf\limits_{\alpha \in A} f_\alpha(x), \\ \text{subject to} \ \ g(x) = \inf\limits_{\beta \in B} g_\beta(x) \leq 0, \quad x \in C. \end{cases}$$

where A and B are metrizable compact topological spaces, f_α ($\alpha \in A$) and g_β ($\beta \in B$) are real-valued functions defined on a Banach space X, and $C \subset X$. But their main results, which include the weak invexity of the infimum function $\inf\limits_{\alpha \in A} f_\alpha$, a condition for the Lagrange multiplier λ to be positive, and some sufficient conditions for local minima of this problem, are wrong, as explained in Section 2.

In Section 3, we show that (P_1) is actually a simple problem, whose serious motive is still missing, and that other possibly true results concerned with necessary conditions are too weak, while a lot of strong and unnecessary assumptions are used. For correction, a necessary condition and a sufficient condition for local minima of (P_1) are presented.

To complete our discussion, some comments on other problems investigated in [5] and [8] are given in Section 4.

Our investigation shows that some auxiliary results in [5] and [8], such as the weak invexity, the generalized gradient, and the directional derivative of $\min\limits_{1 \leq i \leq m} f_i$ or of $\inf\limits_{\alpha \in A} f_\alpha$, are redundant and only misleading (to weak or wrong assertions).

2. Some essential errors

Throughout this paper, corresponding to Problem (P_1), we denote

$$M = \{x \in C : g(x) \leq 0\},$$
$$A_0(x) = \{\alpha \in A : f_\alpha(x) = f(x)\},$$
$$B_0(x) = \{\beta \in B : g_\beta(x) = g(x)\}.$$

For a nonempty subset C of a Banach space X,

$$d_C(x) = \inf\{\|x - c\| : c \in C\},$$
$$T_C(x) = \{v \in X : d_C^\circ(x; v) = 0\},$$
$$N_C(x) = \{\xi \in X^* : \langle \xi, v\rangle \leq 0 \quad \text{for all } v \in T_C(x)\}$$

are the *distance function*, the *tangent cone*, and the *normal cone* to C at $x \in X$, respectively. If $\phi : X \to \mathbb{R}$ is Lipschitz near x, then

$$\partial\phi(x) = \{\xi \in X^* : \phi^\circ(x; v) \geq \langle \xi, v\rangle \text{ for all } v \in X\}$$

is its *generalized gradient* of f at x (see [1]).

In [8], Luu and Ha considered a special case of (P_1), where $A = \{1, 2, ..., l\}$ and $B = \{1, 2, ..., m\}$, i.e.,

$$(P_{1a}) \qquad \begin{cases} \text{minimize} \quad f(x) = \min\limits_{1\leq i\leq l} f_i(x), \\ \text{subject to} \;\; g(x) = \min\limits_{1\leq j\leq m} g_j(x) \leq 0, \quad x \in C. \end{cases}$$

The next two theorems belong to the main results of [8].

Theorem 4 [8]. *Let $\bar{x}$ be a feasible point of Problem (P_{1a}). Let the functions $f_1, \ldots, f_l$, $g_1, \ldots, g_m$ be locally Lipschitz and regular at $\bar{x}$. Let the functions f and g be regular at $\bar{x}$. Assume that there are a neighborhood V of $\bar{x}$ and a function $\eta : M \times M \to T_C(\bar{x})$ such that the functions $f_1, \ldots, f_l$, $g_1, \ldots, g_m$ are weakly invex at $\bar{x}$ on $M \cap V$ with respect to the function η. Suppose, furthermore, that there exist numbers*

$$(4) \qquad \begin{aligned} &\lambda_i \geq 0, \quad i \in A_0(\bar{x}), \quad \sum_{i\in A_0(\bar{x})} \lambda_i = 1, \\ &\mu_j \geq 0, \quad j \in B_0(\bar{x}), \quad \sum_{j\in B_0(\bar{x})} \mu_j = 1, \quad \mu \geq 0, \end{aligned}$$

such that

$$(5) \qquad \begin{aligned} &0 \in \sum_{i\in A_0(\bar{x})} \lambda_i \partial f_i(\bar{x}) + \mu \sum_{j\in B_0(\bar{x})} \mu_j \partial g_j(\bar{x}) + N_C(\bar{x}), \\ &\mu \min_{1\leq j\leq m} g_j(\bar{x}) = 0. \end{aligned}$$

Then $\bar{x}$ is a local minimum of (P_{1a}).

Theorem 5 [8]. *Let $\bar{x}$ be a feasible point of Problem (P_{1a}) and C be a closed convex subset of X. Let the functions $f_1, \dots, f_l$, $g_1, \dots, g_m$ be* convex *and locally Lipschitz at $\bar{x}$, and the functions f and g are regular at $\bar{x}$. Assume that there are numbers λ_i, μ_j, and μ satisfying* (4) *such that* (5) *is fulfilled. Then $\bar{x}$ is a local minimum of (P_{1a}).*

Counter-example 1. Let $X = C = \mathbb{R}$, $A = \{1\}$, $B = \{1,2\}$, $f_1(x) = x$, $g_1(x) = 0$, and

$$g_2(x) = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Obviously, $f = f_1$, $g = g_1$, and g_2 are convex, and therefore, invex with respect to $\eta(x, x') = x - x'$. Moreover, $f = f_1$, $g = g_1$, and g_2 are regular everywhere. For $\bar{x} = 0$ we have $g_1(\bar{x}) = g_2(\bar{x}) = 0$, $\partial f_1(\bar{x}) = \{1\}$, $\partial g_1(\bar{x}) = \{0\}$, $\partial g_2(\bar{x}) = [-1, 0]$, and $N_C(\bar{x}) = \{0\}$, which implies for $\lambda_1 = \mu = \mu_2 = 1$ and $\mu_1 = 0$ that

$$0 \in [0,1] = \lambda_1 \partial f_1(\bar{x}) + \mu\big(\mu_1 \partial g_1(\bar{x}) + \mu_2 \partial g_2(\bar{x})\big) + N_C(\bar{x}).$$

Since all assumptions of Theorems 4 and 5 from [8] are fulfilled, each of them yields that $\bar{x}$ is a local minimum of (P_{1a}). But, in fact, no point of the feasible set

$$\{x \in C : g(x) = \min\{g_1(x), g_2(x)\} \leq 0\} = \mathbb{R}$$

can be a local minimum of $f(x) = f_1(x) = x$.

Counter-example 2. Let $X = C = \mathbb{R}$, $A = \{1,2\}$, $B = \{1\}$, $f_1(x) = x$, $g_1(x) = 0$, and

$$f_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Obviously, $f = f_1$, f_2, and $g = g_1$ are convex, and therefore, invex with respect to $\eta(x, x') = x - x'$. Moreover, $f = f_1$, f_2, and $g = g_1$ are regular everywhere. For $\bar{x} = 0$ we have $f_1(\bar{x}) = f_2(\bar{x}) = g_1(\bar{x}) = 0$, $\partial f_1(\bar{x}) = \{1\}$, $\partial f_2(\bar{x}) = [0,1]$, $\partial g_1(\bar{x}) = \{0\}$, and $N_C(\bar{x}) = \{0\}$, which implies for $\lambda_2 = \mu = \mu_1 = 1$ and $\lambda_1 = 0$ that

$$0 \in [0,1] = \lambda_1 \partial f_1(\bar{x}) + \lambda_2 \partial f_2(\bar{x}) + \mu\mu_1 \partial g_1(\bar{x}) + N_C(\bar{x}).$$

Since all the assumptions of Theorems 4 and 5 from [8] are fulfilled, each of them yields that $\bar{x}$ is a local minimum of (P_{1a}). But, in fact, no point of the feasible set

$$\{x \in C : g(x) = g_1(x) \leq 0\} = \mathbb{R}$$

can be a local minimum of $f(x) = \min\{f_1(x), f_2(x)\} = x$.

The above examples show that both Theorems 4 and 5 from [8] are already false if $|A| > 1$ or $|B| > 1$, i.e., if A or B contains more than one element.

In [4] and [5], another case of (P_1) is considered, where A and B are not necessarily finite and $C = X = \mathbb{R}^n$, i.e.,

$$(P_{1b}) \qquad \begin{cases} \text{minimize} & f(x) = \inf_{\alpha \in A} f_\alpha(x), \\ \text{subject to} & g(x) = \inf_{\beta \in B} g_\beta(x) \leq 0. \end{cases}$$

In this context, the following assertions on the weak invexity or the invexity of infimum function are stated.

Theorem 5.3 [5]. *Assume that for all $x \in V \subset \mathbb{R}^n$, $x \neq \bar{x}$, the function $\beta \mapsto g_\beta(x)$ is lower semicontinuous, the function $\beta \mapsto g_\beta(\bar{x})$ is continuous, and the set-valued mapping $(\beta, x) \mapsto -\partial g_\beta(x)$ is upper semicontinuous at $(\bar{\beta}, \bar{x})$, $\forall \bar{\beta} \in B_0(\bar{x})$. Suppose, furthermore, that for all $\bar{\beta} \in B_0(\bar{x})$, $-g_{\bar{\beta}}$ is regular at $\bar{x}$, and $g_{\bar{\beta}}$ is weakly invex on U at $\bar{x}$ with respect to the same function η. Then there exists a number $\delta > 0$ such that the function $g = \inf_{\beta \in B} g_\beta$ is weakly invex on $B(\bar{x}; \delta) \cap V$ at $\bar{x}$ with respect to η.*

Corollary 5.1 [5]. *Assume that all hypotheses of* Theorem 5.3 [5] *are satisfied. Suppose, furthermore, that the function $g = \inf_{\beta \in B} g_\beta$ is regular at $\bar{x}$. Then there exists a number $\delta > 0$ such that g is invex on $B(\bar{x}; \delta) \cap V$ at $\bar{x}$ with respect to η.*

Counter-example 3. Consider $B = [-1, 1] \subset \mathbb{R}$ and

$$g_\beta(x) = \frac{1}{2}\beta^2 - \beta x \quad (\beta \in B,\, x \in \mathbb{R}).$$

Since $g_\beta(x) \geq g_x(x) = -\dfrac{1}{2}x^2$ for all $\beta \in B$ and for any fixed $x \in \mathbb{R}$, it holds

$$(6) \qquad g(x) = \begin{cases} x + \dfrac{1}{2} & \text{if } x < -1 \\ -\dfrac{1}{2}x^2 & \text{if } -1 \leq x \leq 1 \\ -x + \dfrac{1}{2} & \text{if } x > 1. \end{cases}$$

Moreover, we have:

- B is compact.
- For every $x \in \mathbb{R}$, the mapping $\beta \mapsto g_\beta(x)$ is continuous.
- $\partial g_\beta(x) = \{-\beta\}$. Hence, the mapping $(\beta, x) \mapsto -\partial g_\beta(x)$ is continuous everywhere.
- For all $\beta \in \mathbb{R}$, $-g_\beta$ and g are regular at any $x \in \mathbb{R}$.
- For all $\beta \in \mathbb{R}$, g_β is affine and therefore weakly invex (even invex) with respect to $\eta(x, x') = x - x'$.

Thus all the assumptions of Theorem 5.3 and Corollary 5.1 from [5] are fulfilled. According to these statements, there exists $\delta > 0$ such that g is weakly invex or even invex on the open ball $B(\bar{x}, \delta)$ at $\bar{x} = 0$ with respect to $\eta(x, x') = x - x'$.

But, obviously, the function g given in (6) is neither invex nor weakly invex at $\bar{x} = 0$, with respect to any η, because

$$g^\circ(0;\eta(x,0)) = g'(0;\eta(x,0)) = 0 > g(x) - g(0) \quad \text{for } x \neq 0.$$

We have seen that the conclusions of Theorem 5.3 and Corollary 5.1 from [5] are false.

Next, we show that the second part of the following theorem, which states a sufficient condition for the Lagrange multiplier λ corresponding to the objective function f to be positive, is false.

Theorem 1 [4]. *Assume that $\bar{x}$ is a local minimum of (P_{1b}), the mappings f_α $(\alpha \in A)$ and g_β $(\beta \in B)$ are Lipschitz on some open set U containing $\bar{x}$ with Lipschitz constants K_A and K_B, respectively. Suppose that*

(a) *the mappings $\alpha \mapsto f_\alpha(\bar{x})$ and $\beta \mapsto g_\beta(\bar{x})$ are continuous, and*
(b) *for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, the set-valued mappings $(\alpha, x) \mapsto -\partial f_\alpha(x)$ and $(\beta, x) \mapsto -\partial g_\beta(x)$ are upper semicontinuous at $(\bar{\alpha}, \bar{x})$.*

Then there exist $\lambda \geq 0$, $\mu \geq 0$, not both zero, and

$$(7)\qquad \begin{aligned} &\alpha_i \in A_0(\bar{x}), \quad \lambda_{\alpha_i} \geq 0, \quad i = 1, \dots, l, \quad \sum_{i=1}^{l} \lambda_{\alpha_i} = 1, \\ &\beta_j \in B_0(\bar{x}), \quad \mu_{\beta_j} \geq 0, \quad j = 1, ..., m, \quad \sum_{j=1}^{m} \mu_{\beta_j} = 1, \end{aligned}$$

such that

$$(8)\qquad \begin{aligned} &0 \in \lambda \sum_{i=1}^{l} \lambda_{\alpha_i} \partial f_{\alpha_i}(\bar{x}) + \mu \sum_{j=1}^{m} \mu_{\beta_j} \partial g_{\beta_j}(\bar{x}), \\ &\mu \inf_{\beta \in B_0(\bar{x})} g_\beta(\bar{x}) = 0. \end{aligned}$$

If, in addition, the following conditions are satisfied:

(c) *There exists $\hat{x} \in U$ such that $g(\hat{x}) = \inf_{\beta \in B} g_\beta(\hat{x}) < 0$;*
(d) *For every $\beta \in B_0(\bar{x})$, $-g_\beta$ and g are regular at $\bar{x}$, and g_β is weakly invex on U at $\bar{x}$ with respect to η;*

then $\lambda > 0$, thus one can set $\lambda = 1$.

Note that the statement "then $\lambda > 0$" in the above theorem and in other Lagrange multiplier rules means that λ *must* be positive. To prove it, one shows that all multipliers are equal to zero when $\lambda = 0$, which conflicts with the condition "not all zero". Therefore, to deny the second part of Theorem 1 [4], it suffices to give an example where (8) is fulfilled for $\lambda = 0$ and $\mu = 1$. In this case, f_α, $\alpha \in A$, may be chosen almost arbitrarily because they actually disappear then. That is why we only announce g_β, $\beta \in B$, in the next example.

Counter-example 4. Let $X = \mathbb{R}$, $B = \{1, 2\}$, $g_1(x) = 0$, and $g_2(x) = 1 - x$. It holds

$$g(x) = \min\{g_1(x), g_2(x)\} = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - x & \text{if } x > 1. \end{cases}$$

For $\bar{x} = 0$ and $U = (-2, 2)$, all assumptions with respect to g_β in (a)–(d) of the above theorem are satisfied, but (7) and (8) are fulfilled for $\lambda = 0$ and $\mu = \mu_1 = 1$. Thus, the second part of Theorem 1 [4], which asserts that λ must be positive, is false.

Theorem 3 [8] states a Lagrange multiplier rule to Problem (P_{1a}), which is similar to the first part of Theorem 1 [4]. Fortunately, no sufficient condition for $\lambda > 0$ is given there. In Counter-example 4, B is finite, and therefore, it is suitable to Problem (P_{1a}). This counter-example shows that the claim $\lambda > 0$ cannot be ensured by conditions similar to (c)–(d) in Theorem 1 [4].

In Counter-example 4, $\bar{x} = 0 \in \operatorname{int} M = \{x \in \mathbb{R} : g(x) \leq 0\} = \mathbb{R}$. Therefore, if some f has a local minimum at $\bar{x}$ then $0 \in \partial f(\bar{x})$. Thus, $\lambda > 0$ is possible, although it is not necessary. Let us come to an example where all assumptions of Theorem 1 [4] are satisfied but λ must be zero.

Counter-example 5. Let $X = \mathbb{R}$, $A = \{1\}$, $B = [0, 1] \subset \mathbb{R}$, $f_1(x) = x$, and

$$g_\beta(x) = x^2 - 3\beta x + \beta^2 \quad (\beta \in B,\, x \in \mathbb{R}).$$

Then it holds

$$g(x) = \inf_{\beta \in B} g_\beta(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -\dfrac{5}{4}x^2 & \text{if } 0 \leq x \leq \dfrac{2}{3} \\ x^2 - 3x + 1 & \text{if } x > \dfrac{2}{3} \end{cases}$$

and

$$M = \{x \in \mathbb{R} : g(x) \leq 0\} = \left[0, (3 + \sqrt{5})/2\right].$$

Obviously, $\bar{x} = 0$ is the unique local minimum of $f = f_1$ on M, and $A_0(\bar{x}) = \{1\}$ and $B_0(\bar{x}) = \{0\}$. Moreover, we have:

- A and B are compact.
- The mappings f_1 and g_β, $\beta \in B$, are Lipschitz on $U = (-1, 1)$.
- The mapping $\beta \mapsto g_\beta(\bar{x})$ is continuous.
- $\partial f_1(x) = \{1\}$ and $\partial g_\beta(x) = \{2x - 3\beta\}$. Hence, the mappings $(1, x) \mapsto -\partial f_1(x)$ and $(\beta, x) \mapsto -\partial g_\beta(x)$ are continuous everywhere.
- For $\hat{x} = 2/3 \in U$, $g(\hat{x}) = -5/9 < 0$.
- For $\beta \in B_0(\bar{x}) = \{0\}$, $-g_\beta$ and g are regular at $\bar{x}$, and g_β is convex, and therefore, weakly invex on U at $\bar{x}$ with respect to $\eta(x, x') = x - x'$.

Thus all the assumptions of Theorem 1 [4] including (a)–(d) are fulfilled. Therefore, it ensures that (8) must be fulfilled for $\lambda > 0$. But (7) and (8) yield

$$0 \in \lambda\partial f_1(0) + \mu\partial g_0(0) = \lambda\{1\} + \mu\{0\}.$$

Consequently, λ must be zero. This contradiction shows once again that the second part of Theorem 1 [4] is totally wrong.

The last essential error we intend to mention here is the following sufficient condition for optimality.

Theorem 2 [4]. *Assume that $\bar{x}$ is a feasible point of (P_{1b}), and the mappings f_α $(\alpha \in A)$ and g_β $(\beta \in B)$ are Lipschitz on some open set U containing $\bar{x}$ with Lipschitz constants K_A and K_B, respectively. Suppose, furthermore, that:*

(a) *The mappings $\alpha \mapsto f_\alpha(\bar{x})$ and $\beta \mapsto g_\beta(\bar{x})$ are continuous, the set-valued mapping $(\alpha, x) \mapsto -\partial f_\alpha(x)$ is upper semicontinuous at $(\bar{\alpha}, \bar{x})$ $(\forall \bar{\alpha} \in A_0(\bar{x}))$, and the set-valued mapping $(\beta, x) \mapsto -\partial g_\beta(x)$ is upper semicontinuous at $(\bar{\beta}, \bar{x})$ $(\forall \bar{\beta} \in B_0(\bar{x}))$.*

(b) *$-f_{\bar{\alpha}}$ $(\forall \bar{\alpha} \in A_0(\bar{x}))$, $-g_{\bar{\beta}}$ $(\forall \bar{\beta} \in B_0(\bar{x}))$, f and g are regular at $\bar{x}$. $f_{\bar{\alpha}}$ $(\forall \bar{\alpha} \in A_0(\bar{x}))$ and $g_{\bar{\beta}}$ $(\forall \bar{\beta} \in B_0(\bar{x}))$ are weakly invex on U at $\bar{x}$ with respect to the same η.*

(c) *There exist $\mu \geq 0$, α_i, λ_{α_i}, β_j, and μ_{β_j} satisfying (7) such that (8) is fulfilled for $\lambda = 1$.*

Then $\bar{x}$ is a local minimum of (P_{1b}).

Counter-example 6. Let $X = \mathbb{R}$, $A = [-1, 1]$, $B = \{1\}$, and

$$f_\alpha(x) = \frac{1}{2}\alpha^2 - \alpha x \quad (\alpha \in A,\ x \in \mathbb{R}),$$
$$g_1(x) = -1 \quad (x \in \mathbb{R}).$$

We have

$$f(x) = \inf_{\alpha \in A} f_\alpha(x) = \begin{cases} x + \dfrac{1}{2} & \text{if } x < -1 \\ -\dfrac{1}{2}x^2 & \text{if } -1 \leq x \leq 1 \\ -x + \dfrac{1}{2} & \text{if } x > 1, \end{cases}$$

$g(x) = g_1(x) = -1$, $M = \{x \in \mathbb{R} : g(x) \leq 0\} = \mathbb{R}$, $A_0(0) = \{0\}$, $B_0(0) = \emptyset$, and $\partial f_0(0) = \{0\}$. For $\bar{x} = 0$, $\lambda = \lambda_0 = \mu_1 = 1$, and $\mu = 0$, (7) and (8) are satisfied. Moreover, it is easy to verify that all the assumptions of Theorem 2 [4] are fulfilled. Therefore, this theorem implies that $\bar{x} = 0$ is a local minimum of (P_{1b}), which is obviously not true. Hence, Theorem 2 [4] is false.

3. To solve a simple problem

After analyzing several errors in the previous section, one may think that (P_1) is a hard problem. Is that the reason why other authors do not investigate this problem? Not at all. A serious motive for research is missing in [4], [5], and [8]. Are there some practical examples for it? If we accept this problem unconditionally then, from the computational point of view, it is possibly easier and more effective to use the relation

$$\inf_{x\in D}\big(\inf_{\alpha\in A} f_\alpha(x)\big) = \inf_{\alpha\in A}\big(\inf_{x\in D} f_\alpha(x)\big)$$

to decompose the problem into several small ones with objective functions $\inf\limits_{x\in D} f_\alpha(x)$, $\alpha\in A$, which can be solved parallel or separately before combining the results. And from the theoretical point of view? It is just a poor research object, as explained in the forthcoming.

Assume that $\bar{x}\in M$ is a local minimum of (P_1), i.e., there exists a neighborhood $V\subset X$ of $\bar{x}$ such that

$$f(\bar{x}) \le f(x) \quad \text{for all} \quad x\in M\cap V. \tag{9}$$

Since $f(x) = \inf_{\alpha\in A} f_\alpha(x)$ and

$$M \supset \bigcup_{\beta\in B}\{x\in C : g_\beta(x)\le 0\} \supset \bigcup_{\bar{\beta}\in B_0(\bar{x})}\{x\in C : g_{\bar{\beta}}(x)\le 0\},$$

(9) implies

$$f_{\bar{\alpha}}(\bar{x}) \le f_{\bar{\alpha}}(x) \quad \text{in} \quad \{x\in C : g_{\bar{\beta}}(x)\le 0\}\cap V,$$

i.e., $\bar{x}$ is a local minimum of

$$(\bar{P}_{\bar{\alpha},\bar{\beta}}) \qquad \begin{cases} \text{minimize} & f_{\bar{\alpha}}(x),\\ \text{subject to} & g_{\bar{\beta}}(x)\le 0, \quad x\in C,\end{cases}$$

for all $\bar{\alpha}\in A_0(\bar{x})$ and $\bar{\beta}\in B_0(\bar{x})$. Applying a well known Lagrange multiplier rule ([1], p. 228) we obtain immediately a necessary condition for $(\bar{P}_{\bar{\alpha},\bar{\beta}})$ with locally Lipschitz functions $f_{\bar{\alpha}}$ and $g_{\bar{\beta}}$: There exist $\bar{\lambda}_{\bar{\alpha}}\ge 0$ and $\bar{\mu}_{\bar{\beta}}\ge 0$, not both zero, such that

$$\begin{aligned} &0 \in \bar{\lambda}_{\bar{\alpha}}\partial f_{\bar{\alpha}}(\bar{x}) + \bar{\mu}_{\bar{\beta}}\partial g_{\bar{\beta}}(\bar{x}) + N_C(\bar{x}),\\ &\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}(\bar{x}) = 0.\end{aligned} \tag{10}$$

Thus, just take arbitrary $\bar{\alpha}\in A_0(\bar{x})$ and $\bar{\beta}\in B_0(\bar{x})$ and set $\lambda=\bar{\lambda}_{\bar{\alpha}}$, $\mu=\bar{\mu}_{\bar{\beta}}$, $\lambda_{\bar{\alpha}}=\mu_{\bar{\beta}}=1$, and all other multipliers λ_α $(\alpha\ne\bar{\alpha})$ and μ_β $(\beta\ne\bar{\beta})$ to be zero, then we have at once the necessary conditions stated in Theorem 1 [4] and Theorem 3 [8]. The essential differences are:

- We only need the locally Lipschitz continuity of corresponding functions, and all other assumptions such as (a) and (b) in Theorem 1 [4] can be dropped.
- In our proof, no formula for the generalized gradients of $f(x)=\inf_{\alpha\in A} f_\alpha(x)$ and $g(x)=\inf_{\beta\in B} g_\beta(x)$ is needed, which is an essential and expensive tool in [4], [5], and [8].

Moreover, our result is so simple that it has almost no theoretical value and is not worth being published extra, but it is still much more stronger than the ones in [4], [5], and [8]. To explain that, let us formulate the result as follows.

Proposition 1. *Assume that $\bar{x}$ is a local minimum of (P_1) and $A_0(\bar{x})$ and $B_0(\bar{x})$ are nonempty. Then, for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, $\bar{x}$ is a local minimum of $(\bar{P}_{\bar{\alpha},\bar{\beta}})$. In particular, if $f_{\bar{\alpha}}$ and $g_{\bar{\beta}}$ are locally Lipschitz at $\bar{x}$, then there exist $\bar{\lambda}_{\bar{\alpha}} \geq 0$ and $\bar{\mu}_{\bar{\beta}} \geq 0$, not both zero, such that* (10) *is satisfied, where $\bar{\lambda}_{\bar{\alpha}} > 0$ (so one can set $\bar{\lambda}_{\bar{\alpha}} = 1$) if*

$$g_{\bar{\beta}}(\bar{x}) < 0 \quad \text{or} \quad -\partial g_{\bar{\beta}}(\bar{x}) \cap N_C(\bar{x}) = \emptyset. \tag{11}$$

The first part is shown above, and the second part saying when $\bar{\lambda}_{\bar{\alpha}} > 0$ is absolutely obvious. Two additional remarks should be mentioned here.

– In fact, Proposition 1 actually contains $|A_0(\bar{x})| \times |B_0(\bar{x})|$ independent necessary conditions in form (10), where $|A_0(\bar{x})|$ and $|B_0(\bar{x})|$ denote the cardinal numbers of $A_0(\bar{x})$ and $B_0(\bar{x})$, respectively, while corresponding theorems in [4], [5], and [8] state only one (possibly combined) necessary condition, which is, of course, not enough for investigating (P_1).
– While there is no correct sufficient condition in Theorem 3 [8] and Theorem 1 [4] for $\lambda > 0$, it is actually very easy to have it by assuming that (11) holds for at least one $\bar{\beta} \in B_0(\bar{x})$ and then taking this $\bar{\beta}$ to state (10) with $\bar{\lambda}_{\bar{\beta}} = 1$. Condition (11) is clear and simple enough, and it is irrational to replace it by another one. We have shown in Section 2 that weak invexity along with regularity and some Slater condition is not suitable to ensure $\lambda > 0$ in (8) and in Theorem 3 [8]. To guarantee $\bar{\lambda}_{\bar{\alpha}} > 0$ in (10), one could assume the invexity of $g_{\bar{\beta}}$ at $\bar{x}$, which is weaker than the weak invexity along with the regularity of $g_{\bar{\beta}}$ at $\bar{x}$, and some Slater condition like $g_{\bar{\beta}}(\hat{x}) < 0$ for some suitable $\hat{x}$. But this is an unacceptable roundabout way and invexity is not a suitable tool for this purpose, as shown by our critique in [10] to the incorrect paper [6].

We have seen that it is really simple to have necessary conditions for local minima of Problem (P_1). In comparison, the necessary conditions for (P_1) in [4] and [8] are rather weak, while many complicated assumptions and tools had been used there.

Let us explain why the sufficient conditions for local minima of (P_1) in [4] and [8] are wrong. It is a basic knowledge that sufficient conditions must contain all necessary ones. Knowing only one of them, the authors of [4] and [8] tried to use it to state sufficient conditions, which cannot lead to a correct result, of course. For instance, consider Counter-example 1 once again, where $X = C = \mathbb{R}$, $A = \{1\}$, $B = \{1, 2\}$, $f_1(x) = x$, $g_1(x) = 0$, and

$$g_2(x) = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

For $\bar{x} = 0$ we have $A_0(\bar{x}) = \{1\}$ and $B_0(\bar{x}) = \{1, 2\}$. Therefore, due to Proposition 1, there are two necessary conditions which must be satisfied at $\bar{x}$. While the condition

$$0 \in \bar{\lambda}_1 \partial f_1(\bar{x}) + \bar{\mu}_2 \partial g_2(\bar{x}) + N_C(\bar{x})$$

concerned with Problem $(\bar{P}_{1,2})$ is fulfilled for $\bar{\lambda}_1 = 1$, the other one

$$0 \in \bar{\lambda}_1 \partial f_1(\bar{x}) + \bar{\mu}_1 \partial g_1(\bar{x}) + N_C(\bar{x})$$

concerned with Problem $(\bar{P}_{1,1})$ is true only for $\bar{\lambda}_1 = 0$ because $\partial f_1(\bar{x}) = \{1\}$, $\partial g_1(\bar{x}) = \{0\}$, and $N_C(\bar{x}) = \{0\}$. Since Theorems 4 and 5 [8] require only one of them to be fulfilled for $\bar{\lambda}_1 = 1$ and ignore the other one, $\bar{x} = 0$ becomes a successful candidate although it is no local minimum.

To repair the mentioned errors, we now state a sufficient condition for local minima of (P_1).

Proposition 2. *Assume that A and B are finite (i.e., their cardinal numbers are finite), all functions f_α, $\alpha \in A$, and g_β, $\beta \in B$, are continuous, and $\bar{x}$ is a feasible point of (P_1). If, for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, $\bar{x}$ is a local minimum of $(\bar{P}_{\bar{\alpha},\bar{\beta}})$, then it is a local minimum of (P_1). In particular, $\bar{x}$ is a local minimum of (P_1) if the following conditions are satisfied for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$:*

(a) *$f_{\bar{\alpha}}$ and $g_{\bar{\beta}}$ are locally Lipschitz at $\bar{x}$.*
(b) *There exists $\bar{\mu}_{\bar{\beta}} \geq 0$ such that* (10) *is fulfilled for $\bar{\lambda}_{\bar{\alpha}} = 1$.*
(c) *There exists a neighborhood $V_{\bar{\alpha},\bar{\beta}}$ of $\bar{x}$ such that $f_{\bar{\alpha}}$ and $\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}$ are invex on $M \cap V_{\bar{\alpha},\bar{\beta}}$ at $\bar{x}$ with respect to the same function $\eta_{\bar{\alpha},\bar{\beta}} : M \times M \to T_C(\bar{x})$.*

Proof. Assume that, for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, $\bar{x}$ is a local minimum of $(\bar{P}_{\bar{\alpha},\bar{\beta}})$, i.e., there exists a neighborhood $V_{\bar{\alpha},\bar{\beta}}$ of $\bar{x}$ such that

$$f_{\bar{\alpha}}(\bar{x}) \leq f_{\bar{\alpha}}(x) \quad \text{in} \quad \{x \in C : g_{\bar{\beta}}(x) \leq 0\} \cap V_{\bar{\alpha},\bar{\beta}}. \tag{12}$$

Since A and B are finite and all functions f_α, $\alpha \in A$, and g_β, $\beta \in B$, are continuous, there exists a neighborhood V of $\bar{x}$ such that

$$\begin{aligned} & V \subset V_{\bar{\alpha},\bar{\beta}} \quad \text{for all} \;\; \bar{\alpha} \in A_0(\bar{x}),\; \bar{\beta} \in B_0(\bar{x}), \\ & f_{\bar{\alpha}}(x) < f_\alpha(x) \quad \text{for all} \;\; \bar{\alpha} \in A_0(\bar{x}),\; \alpha \in A \setminus A_0(\bar{x}),\; x \in C \cap V, \\ & g_{\bar{\beta}}(x) < g_\beta(x) \quad \text{for all} \;\; \bar{\beta} \in B_0(\bar{x}),\; \beta \in B \setminus B_0(\bar{x}),\; x \in C \cap V, \end{aligned} \tag{13}$$

which yields

$$\begin{aligned} & M \cap V = \bigcup_{\beta \in B} \{x \in C : g_\beta(x) \leq 0\} \cap V = \bigcup_{\bar{\beta} \in B_0(\bar{x})} \{x \in C : g_{\bar{\beta}}(x) \leq 0\} \cap V, \\ & f(x) = \inf_{\alpha \in A} f_\alpha(x) = \min_{\bar{\alpha} \in A_0(\bar{x})} f_{\bar{\alpha}}(x) \quad \text{for all} \;\; x \in M \cap V. \end{aligned} \tag{14}$$

It follows from (12)–(14) that

$$f_{\bar{\alpha}}(\bar{x}) \leq f_{\bar{\alpha}}(x) \quad \text{in} \quad \bigcup_{\bar{\beta} \in B_0(\bar{x})} \{x \in C : g_{\bar{\beta}}(x) \leq 0\} \cap V = M \cap V,$$

and, finally,

$$f(\bar{x}) \le \min_{\bar{\alpha}\in A_0(\bar{x})} f_{\bar{\alpha}}(x) = f(x) \quad \text{for all} \;\; x \in M \cap V,$$

i.e., $\bar{x}$ is a local minimum of (P_1).

Assume that (a) and (b) and (c) are satisfied for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$. Due to (b), we have

$$\xi + \bar{\mu}_{\bar{\beta}}\vartheta + \nu = 0 \quad \text{for some} \;\; \xi \in \partial f_{\bar{\alpha}}(\bar{x}), \quad \vartheta \in \partial g_{\bar{\beta}}(\bar{x}), \quad \nu \in N_C(\bar{x}),$$

which yields by applying

$$\phi^\circ(x; d) = \max\{\langle \chi, d\rangle : \chi \in \partial\phi(x)\},$$
$$\langle \nu, d\rangle \le 0 \quad \text{for} \;\; \nu \in N_C(\bar{x}), \; d \in T_C(\bar{x})$$

and by $\bar{\mu}_{\bar{\beta}} \ge 0$ that

$$f^\circ_{\bar{\alpha}}(\bar{x}; d) + \bar{\mu}_{\bar{\beta}} g^\circ_{\bar{\beta}}(\bar{x}; d) \ge 0 \quad \text{for all} \;\; d \in T_C(x).$$

Consequently, it follows from (c), $\bar{\mu}_{\bar{\beta}} g^\circ_{\bar{\beta}}(\bar{x}; .) = \big(\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}\big)^\circ(\bar{x}; .)$, $\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}(\bar{x}) = 0$, and $\eta_{\bar{\alpha},\bar{\beta}}(x, \bar{x}) \in T_C(\bar{x})$ that

$$\begin{aligned} f_{\bar{\alpha}}(x) - f_{\bar{\alpha}}(\bar{x}) &\ge f^\circ_{\bar{\alpha}}(\bar{x}; \eta_{\bar{\alpha},\bar{\beta}}(x, \bar{x})) \\ &\ge -\bar{\mu}_{\bar{\beta}} g^\circ_{\bar{\beta}}(\bar{x}; \eta_{\bar{\alpha},\bar{\beta}}(x, \bar{x})) \\ &\ge -\big(\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}(x) - \bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}(\bar{x})\big) \\ &= -\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}(x) \\ &\ge 0 \end{aligned}$$

holds for any x in $\{x \in C : g_{\bar{\beta}}(x) \le 0\} \cap V_{\bar{\alpha},\bar{\beta}}$, i.e., $\bar{x}$ is a local minimum of Problem $(\bar{P}_{\bar{\alpha},\bar{\beta}})$. Thus, due to the first part of this proposition, $\bar{x}$ is a local minimum of (P_1). □

The first part of proof is simple, for which (13) and (14) are essential. The second part contains only standard arguments.

It is worth mentioning that, in Theorem 4 [8], all functions f_α, $\alpha \in A$, and g_β, $\beta \in B$, are assumed to be regular and weakly invex, that means properly invex, at $\bar{x}$ with respect to the same function η. That is essentially stronger than the assumption in Proposition 2, where only $f_{\bar{\alpha}}$ and $\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}}$, with $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, in pairs are required to be invex on $M \cap V_{\bar{\alpha},\bar{\beta}}$ at $\bar{x}$ with respect to the same function $\eta_{\bar{\alpha},\bar{\beta}}$. In particular, in case $g(\bar{x}) = g_{\bar{\beta}}(\bar{x}) < 0$, (10) implies $\bar{\mu}_{\bar{\beta}} = 0$, and $\bar{\mu}_{\bar{\beta}} g_{\bar{\beta}} \equiv 0$ is invex with respect to any η. Therefore, only $f_{\bar{\alpha}}$, $\bar{\alpha} \in A_0(\bar{x})$, are required to be invex on $M \cap V_{\bar{\alpha},\bar{\beta}}$ at $\bar{x}$, each of them with respect to the own function $\eta = \eta_{\bar{\alpha}}$. By the way, the regularity of f and g at $\bar{x}$ is not demanded explicitly in Proposition 2, as done in Theorem 4 [8].

Note that the sufficient condition in Proposition 2 is no more true if A or B is infinite. To see it for infinite A, just take Counter-example 6. For infinite B, we modify Counter-example 5 as follows: Let $X = C = \mathbb{R}$, $A = \{1\}$, $B = [0,1] \subset \mathbb{R}$, $f_1(x) = -x$, and

$$g_\beta(x) = x^2 - 3\beta x + \beta^2 \quad (\beta \in B,\, x \in \mathbb{R}).$$

Then $\bar{x} = 0$ cannot be a local minimum of $f = f_1$ on

$$M = \{x \in \mathbb{R} : g(x) \le 0\} = \left[0, (3+\sqrt{5})/2\right],$$

while $A_0(\bar{x}) = \{1\}$, $B_0(\bar{x}) = \{0\}$, $\{x \in \mathbb{R} : g_0(x) \le 0\} = \{0\}$ implies that all the assumptions of Proposition 2 are fulfilled for this $\bar{x}$.

The conclusion of Proposition 2 cannot be saved for infinite A by assuming that, for all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$, $\bar{x}$ is a *strict* local minimum of $(\bar{P}_{\bar{\alpha},\bar{\beta}})$. To see this, it suffices to modify Counter-example 6 by setting $f_\alpha(x) = \frac{1}{2}\alpha^2 - \alpha x + (x-\alpha)^2$.

Thus, to obtain sufficient conditions for $\bar{x}$ to be a local minimum of (P_1) when A or B is infinite, it is not enough to consider all the subproblems $(\bar{P}_{\bar{\alpha},\bar{\beta}})$ with $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in B_0(\bar{x})$. In the examples just mentioned, the mappings $(x,\alpha) \mapsto f_\alpha(x)$ and $(x,\beta) \mapsto g_\beta(x)$ are continuously Fréchet differentiable, i.e., all analytical properties are already optimal, nevertheless, it does not help. Hence, a sufficient condition would be almost expensive as the definition (9) of local minimum. But one should not proceed to do something more before answering the question: What are the practical and theoretical motives of Problem (P_1) ?

4. Some comments on two residuary problems

The second optimization problem considered in [5] and [8] is

$$(P_2) \qquad \begin{cases} \text{minimize} & \tilde{f}(x) = \sup_{\alpha \in A} f_\alpha(x), \\ \text{subject to} & \tilde{g}(x) = \sup_{\beta \in B} g_\beta(x) \le 0, \end{cases}$$

where A and B are metrizable compact topological spaces, f_α, $\alpha \in A$, and g_β, $\beta \in B$, are real-valued functions defined on $\mathbb{R}^n$. Denote

$$\begin{aligned} \tilde{M} &= \{x \in C : \tilde{g}(x) \le 0\}, \\ \tilde{A}_0(x) &= \{\alpha \in A : f_\alpha(x) = \tilde{f}(x)\}, \\ \tilde{B}_0(x) &= \{\beta \in B : g_\beta(x) = \tilde{g}(x)\}. \end{aligned}$$

Let us mention a sufficient condition given in [5].

Theorem 6.3 [5]. *Let $\bar{x}$ be a feasible point of Problem (P_2). Suppose:*

(a) *The functions f_α $(\alpha \in A)$ and g_β $(\beta \in B)$ are Lipschitz on some open set U containing $\bar{x}$ with the same Lipschitz constants K_A and K_B, respectively.*
(b) *The mappings $\alpha \mapsto f_\alpha(\bar{x})$ and $\beta \mapsto g_\beta(\bar{x})$ are continuous, the set-valued mapping $(\alpha, x) \mapsto \partial f_\alpha(x)$ is upper semicontinuous at $(\bar{\alpha}, \bar{x})$ $(\forall \bar{\alpha} \in \tilde{A}_0(\bar{x}))$,*

and the set-valued mapping $(\beta, x) \mapsto \partial g_\beta(x)$ is upper semicontinuous at $(\bar\beta, \bar x)$ $(\forall \bar\beta \in \tilde B_0(\bar x))$.

(c) *The functions f_α $(\forall \bar\alpha \in \tilde A_0(\bar x))$ and g_β $(\forall \bar\beta \in \tilde B_0(\bar x))$ are regular at $\bar x$ and invex on U at $\bar x$ with respect to the same function η.*

(d) *There exist $\mu \geq 0$ and*

$$\alpha_i \in \tilde A_0(\bar x), \quad \lambda_{\alpha_i} \geq 0, \quad i = 1, ..., l, \quad \sum_{i=1}^{l} \lambda_{\alpha_i} = 1,$$
$$\beta_j \in \tilde B_0(\bar x), \quad \mu_{\beta_j} \geq 0, \quad j = 1, ..., m, \quad \sum_{j=1}^{m} \mu_{\beta_j} = 1,$$

such that

$$0 \in \sum_{i=1}^{l} \lambda_{\alpha_i} \partial f_{\alpha_i}(\bar x) + \mu \sum_{j=1}^{m} \mu_{\beta_j} \partial g_{\beta_j}(\bar x),$$
$$\mu \sup_{\beta \in \tilde B_0(\bar x)} g_\beta(\bar x) = 0.$$

Then $\bar x$ is a local minimum of (P_2).

In the above theorem, some too strong and unnecessary assumptions are required again. In the following, to come to the same conclusion as Theorem 6.3 [5], we drop the assumptions on the Lipschitz constants K_A and K_B for f_α $(\alpha \in A)$ and g_β $(\beta \in B)$, respectively, and on the upper continuity of $(\alpha, x) \mapsto \partial f_\alpha(x)$ and $(\beta, x) \mapsto \partial g_\beta(x)$. Moreover, we deal with a more general problem, namely

$$(P_{2a}) \qquad \begin{cases} \text{minimize} & \tilde f(x) = \sup_{\alpha \in A} f_\alpha(x), \\ \text{subject to} & \tilde g(x) = \sup_{\beta \in B} g_\beta(x) \leq 0, \quad x \in C \end{cases}$$

where A and B are arbitrary parameter sets, f_α, $\alpha \in A$, and g_β, $\beta \in B$, are real-valued functions defined on some Banach space X and $C \subset X$. Note that, in Problem (P_2), A and B are metrizable compact topological spaces, $X = \mathbb{R}^n$, and there is no condition $x \in C$.

Proposition 3. *Let $\bar x$ be a feasible point of Problem (P_{2a}). Suppose that there are $\alpha_i \in \tilde A_0(\bar x)$, $i = 1, ..., l$, $\beta_j \in \tilde B_0(\bar x)$, $j = 1, ..., m$, for some $l \geq 1$ and $m \geq 1$ such that the following properties hold true:*

(a) *The functions f_{α_i}, $i = 1, ..., l$, and g_{β_j}, $j = 1, ..., m$, are locally Lipschitz and regular at $\bar x$.*

(b) *There exist $\mu \geq 0$, λ_{α_i}, and μ_{β_j} satisfying*

$$\begin{aligned}&\lambda_{\alpha_i} \geq 0, \quad i = 1, ..., l, \quad \sum_{i=1}^{l} \lambda_{\alpha_i} = 1, \\ &\mu_{\beta_j} \geq 0, \quad j = 1, ..., m, \quad \sum_{j=1}^{m} \mu_{\beta_j} = 1,\end{aligned} \tag{15}$$

and

$$\begin{aligned}&0 \in \sum_{i=1}^{l} \lambda_{\alpha_i} \partial f_{\alpha_i}(\bar{x}) + \mu \sum_{j=1}^{m} \mu_{\beta_j} \partial g_{\beta_j}(\bar{x}) + N_C(\bar{x}), \\ &\mu \tilde{g}(\bar{x}) = 0.\end{aligned} \tag{16}$$

(c) *There exist a neighborhood V of $\bar{x}$ and a function $\eta : C \times C \to T_C(\bar{x})$ such that all functions f_{α_i}, $i = 1, ..., l$, and μg_{β_j}, $j = 1, ..., m$, are invex on $C \cap V$ at $\bar{x}$ with respect to η.*

Then $\bar{x}$ is a local minimum of (P_{2a}).

Proof. Denote

$$\bar{f}(x) = \max_{1 \leq i \leq l} f_{\alpha_i}(x), \quad \bar{g}(x) = \max_{1 \leq j \leq m} g_{\beta_j}(x).$$

Since f_{α_i} and g_{β_j} are regular at $\bar{x}$, due to Proposition 2.3.12 in [1], we have

$$\partial \bar{f}(\bar{x}) = \operatorname{co} \bigcup_{1 \leq i \leq l} \partial f_{\alpha_i}(\bar{x}), \quad \partial \bar{g}(\bar{x}) = \operatorname{co} \bigcup_{1 \leq j \leq m} \partial g_{\beta_j}(\bar{x}). \tag{17}$$

Hence, for all $d \in X$,

$$\begin{aligned}\bar{f}^{\circ}(\bar{x}; d) &= \max \left\{ \left\langle \sum_{i=1}^{l} t_i \xi_i, d \right\rangle : \ \xi_i \in \partial f_{\alpha_i}(\bar{x}), \ t_i \geq 0, \ \sum_{i=1}^{l} t_i = 1 \right\} \\ &= \max \left\{ \sum_{i=1}^{l} t_i \max\{\langle \xi_i, d \rangle : \xi_i \in \partial f_{\alpha_i}(\bar{x})\} : \ t_i \geq 0, \ \sum_{i=1}^{l} t_i = 1 \right\} \\ &= \max \left\{ \sum_{i=1}^{l} t_i f_{\alpha_i}^{\circ}(\bar{x}; d) : \ t_i \geq 0, \ \sum_{i=1}^{l} t_i = 1 \right\} = \max_{1 \leq i \leq l} f_{\alpha_i}^{\circ}(\bar{x}; d).\end{aligned}$$

Since f_{α_i} is invex on $C \cap V$ at $\bar{x}$ with respect to η and $f_{\alpha_i}(\bar{x}) = \bar{f}(\bar{x})$ for $1 \leq i \leq l$, it follows that

$$\begin{aligned}\bar{f}^{\circ}(\bar{x}; \eta(x, \bar{x})) &= \max_{1 \leq i \leq l} f_{\alpha_i}^{\circ}(\bar{x}; \eta(x, \bar{x})) \\ &\leq \max_{1 \leq i \leq l} \big(f_{\alpha_i}(x) - f_{\alpha_i}(\bar{x}) \big) \\ &= \max_{1 \leq i \leq l} f_{\alpha_i}(x) - \bar{f}(\bar{x}) = \bar{f}(x) - \bar{f}(\bar{x})\end{aligned}$$

for all $x \in C \cap V$, i.e., $\bar{f}$ is invex on $C \cap V$ at $\bar{x}$ with respect to η. Similarly, by

$$\mu\partial\bar{g}(\bar{x}) = \partial\big(\mu\bar{g}(\bar{x})\big) = \text{co} \bigcup_{1\le j\le m} \partial\mu g_{\beta_j}(\bar{x})$$

(for $\mu \ge 0$), $\mu\bar{g}$ is invex on $C \cap V$ at $\bar{x}$ with respect to η, too. On the other hand, (15), (16), and (17) yield

$$0 \in \partial\bar{f}(\bar{x}) + \mu\partial\bar{g}(\bar{x}) + N_C(\bar{x}), \quad \mu\bar{g}(\bar{x}) = 0.$$

Consequently, by the same standard argument of invexity as in the proof of Proposition 2, we have

$$\bar{f}(\bar{x}) \le \bar{f}(x) \quad \text{in} \quad \{x \in C : \bar{g}(x) \le 0\} \cap V,$$

which implies by $\bar{f}(\bar{x}) = \tilde{f}(\bar{x})$, $\bar{f}(x) \le \tilde{f}(x)$, and $\bar{g}(x) \le \tilde{g}(x)$ that

$$\tilde{f}(\bar{x}) \le \tilde{f}(x) \quad \text{in} \quad \{x \in C : \tilde{g}(x) \le 0\} \cap V,$$

i.e., $\bar{x}$ is a local minimum of (P_{2a}). □

Note that Theorem 6 [8] is contained in Proposition 3.

It is our intension to present the above complete proof without using any preliminary results on the invexity and the generalized directional derivative of the functions $\bar{f}(x) = \max_{1\le i\le l} f_{\alpha_i}(x)$ and $\bar{g}(x) = \max_{1\le j\le m} g_{\beta_j}(x)$, just to show the simplicity of Problem (P_{2a}). Although A and B may be infinite, to prove the above sufficient condition for local minima of (P_{2a}), we do not need to know the invexity, the generalized gradient, and the generalized directional derivative of the supremum function of infinitely many functions.

Another optimization problem considered in [5] is

$$(P_3) \qquad \begin{cases} \text{minimize} \quad f(x) = \inf_{\alpha\in A} f_\alpha(x), \\ \text{subject to} \quad \tilde{g}(x) = \sup_{\beta\in B} g_\beta(x) \le 0, \end{cases}$$

where f_α, $\alpha \in A$, and g_β, $\beta \in B$, are real-valued functions defined on $\mathbb{R}^n$. By the same arguments as in Section 3, one can see that the necessary optimality condition given in Theorem 6.1 [5] for Problem (P_3) is weak and incomplete. The assumption on the upper semicontinuity of the mapping $(\alpha, x) \mapsto -\partial f_\alpha(x)$ is too strong and unnecessary. One can drop it and get a stronger and more complete necessary optimality condition which is similar to Proposition 1 in Section 3. Note that, for (P_3), the constraint $\tilde{g}(x) = \sup_{\beta\in B} g_\beta(x) \le 0$ cannot be decomposed into the independent constraints $g_\beta(x) \le 0$, as it was done for $g(x) = \inf_{\beta\in B} g_\beta(x) \le 0$ in Section 3.

Since no sufficient condition for local minima of (P_3) was given in [5], let us state it now for the following more general problem

$$(P_{3a})\qquad \begin{cases} \text{minimize} \quad f(x) = \inf_{\alpha\in A} f_\alpha(x), \\ \text{subject to} \;\; \tilde{g}(x) = \sup_{\beta\in B} g_\beta(x) \le 0, \quad x \in C, \end{cases}$$

where A is a finite set, B is an arbitrary parameter set, f_α, $\alpha \in A$, and g_β, $\beta \in B$, are real-valued functions defined on some Banach space X, and $C \subset X$.

Proposition 4. *Assume that f_α, $\alpha \in A$, are continuous, and $\bar{x}$ is a feasible point of (P_{3a}). If, for all $\bar{\alpha} \in A_0(\bar{x}) = \{\alpha \in A : f_\alpha(\bar{x}) = f(\bar{x})\}$, $\bar{x}$ is a local minimum of the problem*

$$(\tilde{P}_{\bar{\alpha}})\qquad \begin{cases} \text{minimize} \quad f_{\bar{\alpha}}(x), \\ \text{subject to} \;\; \tilde{g}(x) \le 0, \quad x \in C, \end{cases}$$

then it is a local minimum of (P_{3a}). In particular, $\bar{x}$ is a local minimum of (P_{3a}) if the following conditions are satisfied:

(a) $\tilde{B}_0(x) = \{\beta \in B : g_\beta(x) = \tilde{g}(x)\}$ *is nonempty.*

(b) *For all $\bar{\alpha} \in A_0(\bar{x})$ and $\bar{\beta} \in \tilde{B}_0(\bar{x})$, $f_{\bar{\alpha}}$ and $g_{\bar{\beta}}$ are locally Lipschitz at $\bar{x}$, and $g_{\bar{\beta}}$ is regular at $\bar{x}$.*

(c) *For all $\bar{\alpha} \in A_0(\bar{x})$,*

  (c1) *there exist $\mu \ge 0$, $m_{\bar{\alpha}} \ge 1$ and*

$$\beta_j \in \tilde{B}_0(\bar{x}), \quad \mu_{\beta_j} \ge 0, \quad j = 1, ..., m_{\bar{\alpha}}, \quad \sum_{j=1}^{m_{\bar{\alpha}}} \mu_{\beta_j} = 1$$

  such that

$$0 \in \partial f_{\bar{\alpha}}(\bar{x}) + \mu \sum_{j=1}^{m_{\bar{\alpha}}} \mu_{\beta_j} \partial g_{\beta_j}(\bar{x}) + N_C(\bar{x}),$$

$$\mu\tilde{g}(\bar{x}) = 0;$$

  (c2) *there exist a neighborhood $V_{\bar{\alpha}}$ of $\bar{x}$ and a function $\eta_{\bar{\alpha}} : C \times C \to T_C(\bar{x})$ such that all the functions $f_{\bar{\alpha}}$ and μg_{β_j}, $j = 1, ..., m_{\bar{\alpha}}$, are invex on $C \cap V_{\bar{\alpha}}$ at $\bar{x}$ with respect to $\eta_{\bar{\alpha}}$.*

Proof. When considering $\tilde{g}$ as a unique constraint function, (P_{3a}) becomes a special case of (P_1). Therefore, the first part of Proposition 4 follows directly from the first part of Proposition 2, i.e., if $\bar{x}$ is a local minimum of $(\tilde{P}_{\bar{\alpha}})$ for all $\bar{\alpha} \in A_0(\bar{x})$, then it is a local minimum of (P_{3a}). Note that the assumption on the continuity of g_β, $\beta \in B$, in Proposition 2 is only needed for ensuring the first half of (14), which remains true without the continuity assumption if B consists of just one element. Therefore, the continuity of $\tilde{g}$ is not needed here.

Since $(\tilde{P}_{\bar{\alpha}})$ is a special case of (P_{2a}), due to Proposition 3, the conditions (a), (b), and (c) yield that for all $\bar{\alpha} \in A_0(\bar{x})$, $\bar{x}$ is a local minimum of $(\tilde{P}_{\bar{\alpha}})$. Hence, the first part of Proposition 4 implies that $\bar{x}$ is a local minimum of (P_{3a}). □

Corresponding to the comment at the end of Section 3, we observe that the conclusion of Proposition 4 is not true in general if A is allowed to be infinite.

Note that if $\tilde{g}(\bar{x}) < 0$ then $\mu\tilde{g}(\bar{x}) = 0$ implies $\mu = 0$ and, therefore, $\mu g_{\beta_j} \equiv 0$ is invex with respect to any η. Thus, the invexity requirement in (c) of Propositions 3 and 4 becomes weaker in this case. This explains why we demand the invexity of μg_{β_j} instead of g_{β_j}.

5. Concluding remarks

In this paper, we only discuss some essential errors and the weakness of [4], [5], and [8], and ignore other errors possibly given in the proof of the assertions which seem to be true. Our main concern is that there are a lot of serious errors related to invexity and its generalizations. In [10] and [11], we have analyzed some essential errors of [6] and [9].

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