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PROBABILITY CAPACITIES IN $\mathbb{R}^d$ AND THE CHOQUET INTEGRAL FOR CAPACITIES

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ABSTRACT. A notion of capacity in $\mathbb{R}^d$ is introduced and a concept of Choquet integral of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ is defined and investigated.

1. Introduction

The Choquet theorem gives a tool to specify probability measures on the measurable spaces. Namely, there exists a bijection between probability measures and capacity functionals (see Choquet [1], Matheron [7]). Therefore, it seems natural to use capacities instead of probability measures in many problems of mathematical statistics and geometric probability. Note that the notion of capacities has been investigated by several authors (see G. Choquet [1], S. Graf [2], P. J. Huber [3], P. J. Huber and V. Strassen [4], Hung T. Nguyen, Nhu T. Nguyen and T. Wang [5], J. B. Kodane and L. Wasserman [6], Matheron [7], and T. Norberg [9]). However, the probabilistic aspects of the theory of capacities have not been developed to a level comparable with the standards of measure theory.

In this note we introduce a notion of capacity, that generalizes the notion of measure in $\mathbb{R}^d$. The note can be viewed as a step toward generalizing the probability measures to capacities. In Section 2 we give the notion of capacity in $\mathbb{R}^d$ and show that the capacity theory is, in fact, a generalization of the measure theory in $\mathbb{R}^d$. Some important examples of capacities are considered. In this section, the capacity and the measure with finite support are characterized. In Section 3, we define the Choquet integral and describe some of their properties for the purpose of studying the weak topology [8]. In Section 4, we show the difference between the notion of capacity introduced in this note and the one in the sense of Graf.

2. Probability capacities in $\mathbb{R}^d$

Let $\mathcal{K}(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$, $\mathcal{G}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ denote the families of all *compact sets, closed sets, open sets* and *Borel sets* in $\mathbb{R}^d$, respectively.

Definition 2.1. A set function $T : \mathcal{B}(\mathbb{R}^d) \to [0, +\infty)$ is called a *capacity* in $\mathbb{R}^d$ if the following conditions hold:

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1. $T(\emptyset) = 0$.

2. T is alternating of infinite order: For any Borel sets A_i, $i = 1, \dots, n$, $n \geq 2$, we have

$$T\Big(\bigcap_{i=1}^{n} A_i\Big) \leq \sum_{I \in \mathcal{I}(n)} (-1)^{\#I+1} T\Big(\bigcup_{i \in I} A_i\Big), \tag{2.1}$$

where $\mathcal{I}(n) = \big\{ I \subset \{1, \dots, n\},\ I \neq \emptyset \big\}$ and $\#I$ denotes the cardinality of I.

3. $T(A) = \sup\{T(C) : C \in \mathcal{K}(\mathbb{R}^d), C \subset A\}$, for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$.

4. $T(C) = \inf\{T(G) :\ G \in \mathcal{G}(\mathbb{R}^d),\ G \supset C\}$, for any compact set $C \in \mathcal{K}(\mathbb{R}^d)$.

In comparison with the notion of measure we have the following theorem.

Theorem 2.1. *If μ is a measure defined on $\mathcal{B}(\mathbb{R}^d)$, then μ has the following property: For any Borel sets A_i, $i = 1, \dots, n$, $n \geq 2$, we have*

$$\mu\Big(\bigcap_{i=1}^{n} A_i\Big) = \sum_{I \in \mathcal{I}(n)} (-1)^{\#I+1} \mu\Big(\bigcup_{i \in I} A_i\Big). \tag{2.2}$$

Thus, a capacity is similar to a measure, except that the equality (2.2) *is replaced by the inequality* (2.1).

Proof. We will prove the theorem by induction. For Borel sets A_1, A_2, we have

$$\mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cup A_2),$$

i.e., (2.2) holds for $n = 2$. Suppose that the statement has been verified up to n, we will show that it is true for $n+1$.

Note that $\mathcal{I}(n+1) = \mathcal{I}(n) \cup \{n+1\} \cup \{(\mathcal{I}(n), n+1)\}$. Let $A = \bigcap\limits_{i=1}^{n} A_i$. Then by the induction hypothesis we have

$$\begin{aligned}
\mu\Big(\bigcap_{i=1}^{n+1} A_i\Big) &= \mu(A \cap A_{n+1}) \\
&= \mu(A) + \mu(A_{n+1}) - \mu(A \cup A_{n+1}) \\
&= \mu(A) + \mu(A_{n+1}) - \mu\Big(\big(\bigcap_{i=1}^{n} A_i\big) \cup A_{n+1}\Big) \\
&= \mu\Big(\bigcap_{i=1}^{n} A_i\Big) + \mu(A_{n+1}) - \mu\Big(\bigcap_{i=1}^{n} (A_i \cup A_{n+1})\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I} A_i\big) + \mu(A_{n+1}) - \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I}(A_i\cup A_{n+1})\big)\\
&= \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I} A_i\big) + \mu(A_{n+1}) - \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I'} A_i\big)\\
&= \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I} A_i\big) + \mu(A_{n+1}) + \sum_{I'\in\{(\mathcal{I}(n),n+1)\}} (-1)^{\#I'+1}\mu\big(\bigcup_{i\in I'} A_i\big)\\
&= \sum_{I\in\mathcal{I}(n+1)} (-1)^{\#I+1}\mu\big(\bigcup_{i\in I} A_i\big),
\end{aligned}
$$

where $I' = (I, n+1)$, $I \in \mathcal{I}(n)$. □

Definition 2.2. ([5]) We say that a set function T is *maxitive* if

$$T(A\cup B) = \max\{T(A),\ T(B)\} \quad \text{for } A, B \in \mathcal{B}(\mathbb{R}^d).$$

Proposition 2.1. *If T is the maxitive set function defined on Borel sets $\mathcal{B}(\mathbb{R}^d)$, then for any Borel sets A_i, $i = 1,\dots n$, $n \geq 2$, we have*

$$\sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1} T\big(\bigcup_{i\in I} A_i\big) = \min_i\{T(A_i)\}. \tag{2.3}$$

Proof. We will prove the proposition by induction. For any Borel sets A_1, A_2, we have

$$
\begin{aligned}
T(A_1) + T(A_2) - T(A_1\cup A_2) &= T(A_1) + T(A_2) - \max\{T(A_1),\ T(A_2)\}\\
&= \min\{T(A_1), T(A_2)\},
\end{aligned}
$$

i.e., the proposition is true for $n = 2$. Assume that the proposition is true up to n, we will prove it holds for $n+1$. For $A_i \in \mathcal{B}(\mathbb{R}^d)$, $i = 1, \dots, n+1$, without loss of generality, we may assume that

$$T(A_1) = \min_{1\leq i\leq n+1}\{T(A_i)\}; \quad T(A_{n+1}) = \max_{1\leq i\leq n+1}\{T(A_i)\}.$$

Then, by the hypothesis of induction we have

$$
\begin{aligned}
\sum_{I\in\mathcal{I}(n+1)} (-1)^{\#I+1} T\big(\bigcup_{i\in I} A_i\big) &= \sum_{I\in\mathcal{I}(n)} (-1)^{\#I+1} T\big(\bigcup_{i\in I} A_i\big) + T(A_{n+1})\\
&\quad + \sum_{I'=(I,n+1),\ I\in\mathcal{I}(n)} (-1)^{\#I'+1} T\big(\bigcup_{i\in I'} A_i\big)\\
&= T(A_1) + T(A_{n+1})\\
&\quad + (-C_n^1 + C_n^2 - \cdots + (-1)^n C_n^n) T(A_{n+1})\\
&= T(A_1)\\
&\quad + (C_n^0 - C_n^1 + C_n^2 - \cdots + (-1)^n C_n^n) T(A_{n+1})\\
&= T(A_1) = \min_{1\leq i\leq n+1}\{T(A_i)\},
\end{aligned}
$$

and the proposition is proved. □

Definition 2.3. A maxitive function T is called to be a *maxitive measure* if the conditions 1., 3., 4. of Definition 2.1. hold.

The following theorem was shown in [5]

Theorem 2.2. *Any maxitive set function is alternating of infinite order. Consequently, the class of capacity in $\mathbb{R}^d$ contains both classes: the class of measures and the class of maxitive measures.*

Proposition 2.2. *Any capacity is a non-decreasing set function on Borel sets of $\mathbb{R}^d$.*

Proof. Assume that A, B are Borel sets of $\mathbb{R}^d$ such that $A \subset B$. Let

$$\mathcal{K}(A) := \{C : \ C \in \mathcal{K}(\mathbb{R}^d), \ C \subset A\}.$$

Observe that

$$\mathcal{K}(A) \subset \mathcal{K}(B).$$

Therefore, by Definition 2.1(3.) we have

$$T(A) = \sup_{C \in \mathcal{K}(A)} \{T(C)\} \leq \sup_{C \in \mathcal{K}(B)} \{T(C)\} = T(B).$$

□

Corollary 2.1. *Let T be a capacity in $\mathbb{R}^d$. If $A \in \mathcal{B}(\mathbb{R}^d)$ with $T(A) = 0$, then*

$$T(B) = T(A \cup B) \quad \textit{for } B \in \mathcal{B}(\mathbb{R}^d).$$

Proof. By Proposition 2.2, $T(B \cup A) \geq T(B)$. On the other hand, since T is alternating of infinite order,

$$T(B \cup A) \leq T(A) + T(B) - T(A \cap B) = T(B).$$

Consequently,

$$T(B \cup A) = T(B).$$

□

Definition 2.4. By *support* of a capacity T we mean the smallest closed set $S \subset \mathbb{R}^d$ such that $T(\mathbb{R}^d \setminus S) = 0$.

We denote the support of a capacity T by supp T. It is easy to obtain the following

Corollary 2.2. *If T is a capacity in $\mathbb{R}^d$, then*

$$T(\operatorname{supp} T) \geq T(B) \quad \textit{for all } B \in \mathcal{B}(\mathbb{R}^d).$$

Moreover, we have

Proposition 2.3. *It holds*

$$\operatorname{supp} T = \mathbb{R}^d \setminus \bigcup\{G : \ G \in \mathcal{G}(T)\},$$

where

$$\mathcal{G}(T) = \{G \in \mathcal{G}(\mathbb{R}^d) : \ T(G) = 0\}.$$

Definition 2.5. We say that T is a *probability capacity* in $\mathbb{R}^d$ if $T(\operatorname{supp} T) = T(\mathbb{R}^d) = 1$.

Example 2.1. For any $x \in \mathbb{R}^d$, we define the set function $T_x = \delta_x$, by

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

It is clear that T_x is a capacity in $\mathbb{R}^d$. The corresponding $x \to T_x$ is one-to-one between $\mathbb{R}^d$ and the set of the probability capacities $\{T_x : \ x \in \mathbb{R}^d\} \subset \mathcal{P}$, where $\mathcal{P}$ denotes the family of all probability capacities in $\mathbb{R}^d$. Therefore, in some sense, the class of capacities in $\mathbb{R}^d$ also contains $\mathbb{R}^d$.

Example 2.2. Let $\mathbb{R}^+ = [0, \infty)$. For a finite set $A = \{(x_1, t_1), \dots, (x_k, t_k)\} \subset \mathbb{R}^d \times \mathbb{R}^+$ we define the set function T_A by

$$T_A(B) = \begin{cases} \max\{t_i : \ x_i \in B\} & \text{if } B \cap A_0 \neq \emptyset \\ 0 & \text{if } B \cap A_0 = \emptyset, \end{cases}$$

where $A_0 = \{x_1, \dots, x_k\} \subset \mathbb{R}^d$. Clearly, T_A is a capacity in $\mathbb{R}^d$. We call T_A a *capacity with finite support* and the number t_i is called the *weight* of x_i for $i = 1, \dots, k$.

If $\max\{t_i, \ i = 1, \dots, k\} = 1$, then T_A is a probability capacity. Note that supp $T_A = A_0 = \{x_1, \dots, x_k\}$.

Definition 2.6. We say that a measure T is a *probability measure* if $T(\text{supp } T) = 1$.

Note that the two capacities given in Examples 2.1 and 2.2 are maxitive measures. The capacity in the next example is a probability measure.

Example 2.3. For a finite set $A = \{(x_1, t_1), \dots, (x_k, t_k)\} \subset \mathbb{R}^d \times \mathbb{R}^+$, we define the capacity $T^A = \sum\limits_{i=1}^{k} t_i \delta_{x_i}$ by setting

$$T^A(B) = \sum_{x_i \in B \cap A_0} t_i$$

for $B \in \mathcal{B}(\mathbb{R}^d)$, where $A_0 = \{x_1, \dots, x_k\} \subset \mathbb{R}^d$. It is easy to see that T^A is a measure in $\mathbb{R}^d$. We call T^A a *measure with finite support* and the number t_i is called the *weight* of x_i for $i = 1, \dots, k$. If $\sum_{i=1}^{k} t_i = 1$, then T^A is a probability measure with finite support and supp $T^A = \{x_1, \dots, x_k\}$.

Definition 2.7. We say that a set function T is *upper semi-continuous* provided it satisfies the condition: If $C_1 \supset \dots \supset C_n \supset \dots$ is a decreasing sequence of Borel sets in $\mathbb{R}^d$ and $\bigcap_{n=1}^{\infty} C_n = C_0$, then

$$\lim_{n\to\infty} T(C_n) = T(C_0).$$

In notation: $C_n \searrow C_0 \;\Rightarrow\; T(C_n) \searrow T(C_0)$.

The following example shows that, in general, capacities are not upper semi-continuous.

Example 2.4. We define the set function $\mu :\ \mathcal{B}(\mathbb{R}^d) \to [0, 1]$ by setting

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

for $A \in \mathcal{B}(\mathbb{R}^d)$. Then μ is maxitive. Hence μ is alternating of infinite order.

Let $x \in A$. Then

$$\mu(\{x\}) = 1 = \mu(A).$$

Therefore

$$\mu(A) = \sup\{\mu(C) :\ C \in \mathcal{K}(\mathbb{R}^d), C \subset A\}.$$

Finally, if C is a compact set in $\mathbb{R}^d$, we have $\mu(C) = 1$ and $\mu(G) = 1$ for any open G containing C. Hence

$$\mu(C) = \inf\{\mu(G) :\ G \in \mathcal{G}(\mathbb{R}^d),\ G \supset C\}.$$

Thus, conditions (1) - (4) of Definition 2.1 are satisfied. We will show that μ is not upper semi-continuous. For each $n = 1, 2, \dots,$ define

$$A_n = \Big\{x \in \mathbb{R}^d :\ 0 < \|x\| < \frac{1}{n}\Big\}.$$

Then $\{A_n\}$ is a decreasing sequence in $\mathcal{B}(\mathbb{R}^d)$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. We have

$$\mu(A_n) = 1 \text{ for every } n \in \mathbb{N} \text{ and } \mu\big(\bigcap_{n=1}^{\infty} A_n\big) = 0.$$

Hence

$$\lim_{n\to\infty} \mu(A_n) = 1 > \mu\big(\bigcap_{n=1}^{\infty} A_n\big).$$

This means that μ is not upper semi-continuous. However, on compact sets the situation will be different as shown in the following theorem.

Theorem 2.3. *Any capacity is upper semi-continuous on compact sets.*

Proof. Assume that $C_1 \supset \ldots \supset C_n \supset \ldots$ is a decreasing sequence of compact sets of $\mathbb{R}^d$ and $\bigcap_{n=1}^{\infty} C_n = C_0$. Since T is a non-decreasing set function, $\lim_{n\to\infty} T(C_n)$ exists and

$$T(C_0) \le \lim_{n\to\infty} T(C_n). \tag{2.5}$$

By Definition 2.1 we have

$$T(C_0) = \inf\{T(G) :\ G \in \mathcal{G}(\mathbb{R}^d),\ C_0 \subset G\}.$$

Therefore, for a given $\varepsilon > 0$ there is $G \in \mathcal{G}(\mathbb{R}^d)$, $G \supset C_0$, such that

$$T(C_0) + \varepsilon > T(G).$$

Since $C_0 = \bigcap_{n=1}^{\infty} C_n$ is a compact set and G is a open set containing C_0, there is $n_0 \in \mathbb{N}$ such that $G \supset C_n$ for all $n \ge n_0$. By Proposition 2.2, $T(G) \ge T(C_n)$ for all $n \ge n_0$. Hence

$$T(G) \ge \lim_{n\to\infty} T(C_n).$$

Therefore

$$T(C_0) + \varepsilon > \lim_{n\to\infty} T(C_n).$$

Since ε is an arbitrarily small positive number, we have

$$T(C_0) \ge \lim_{n\to\infty} T(C_n).$$

From the latter and (2.5), the assertion follows. □

3. Choquet integral for capacities

Let T be a capacity in $\mathbb{R}^d$. Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}^+ = [0, +\infty)$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the function $f_A : \mathbb{R} \to \mathbb{R}$ defined by

$$f_A(t) = T(\{x \in A :\ f(x) \ge t\}) \quad \text{for } t \in \mathbb{R}$$

is a non-increasing function in t. Therefore we can define the *Choquet integral* $\int_A f dT$ of f with respect to T by

$$\int_A f dT = \int_0^{\infty} f_A(t) dt = \int_0^{\infty} T(\{x \in A :\ f(x) \ge t\}) dt.$$

This notion of integral is originated from Choquet (1953). One should note that the function $f_A(t) = T(\{x \in A :\ f(x) \ge t\})$ is well defined because f is measurable. Furthermore, as T is monotone, the function f_A is nonincreasing. As any nonincreasing function has an extended Riemann integral, the definition is valid. If $\int_A f dT < \infty$, then we say that f is *integrable.* In particular, for $A = \mathbb{R}^d$ we write

$$\int_{\mathbb{R}^d} f dT = \int f dT.$$

Observe that if f is bounded, then

$$\int_A f dT = \int_0^\alpha T(\{x \in A : \ f(x) \geq t\})dt, \tag{3.1}$$

where $\alpha = \sup\{f(x) : \ x \in A\}$. In fact, since $\{x \in A : \ f(x) \geq t\} = \emptyset$ for every $t > \alpha$, we have

$$T(\{x \in A : \ f(x) \geq t\}) = 0.$$

Hence

$$\begin{aligned} \int_A f dT &= \int_0^\alpha T(\{x \in A : \ f(x) \geq t\})dt + \int_\alpha^{+\infty} T(\{x \in A : \ f(x) \geq t\})dt \\ &= \int_0^\alpha T(\{x \in A : \ f(x) \geq t\})dt. \end{aligned}$$

In the general case, if $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable function, then we define

$$\int_A f dT = \int_A f^+ dT - \int_A f^- dT,$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

By using Choquet integrals we are able to evalute intergrals for the capacities defined in Examples 2.1 - 2.3.

Theorem 3.1. *For $x \in \mathbb{R}^d$, let T_x be the capacity defined in Example 2.1. Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$ we have*

$$\int f dT_x = f(x) \quad \textit{for every } x \in \mathbb{R}^d. \tag{3.2}$$

Conversely, if T is a capacity in $\mathbb{R}^d$ such that for some $x \in \mathbb{R}^d$ we have

$$f(x) = \int f dT \quad \textit{for every } f \in C_0^+(\mathbb{R}^d), \tag{3.3}$$

where $C_0^+(\mathbb{R}^d)$ denotes of all continuous non-negative real valued functions with compact support in $\mathbb{R}^d$, then $T = T_x$.

Proof. The equality (3.2) can be proved as follows

$$\begin{aligned}\int f dT_x &= \int_0^{\infty} T_x(\{y \in \mathbb{R}^d : \ f(y) \geq t\})dt \\ &= \int_0^{f(x)} T_x(\{y \in \mathbb{R}^d : \ f(y) \geq t\})dt + \int_{f(x)}^{\infty} T_x(\{y \in \mathbb{R}^d : \ f(y) \geq t\})dt \\ &= \int_0^{f(x)} dt = f(x).\end{aligned}$$

To obtain second part of the theorem we now establish two claims.

Claim 3.1. Let T be the capacity defined on Borel sets of $\mathbb{R}^d$. Assume that for $C \in \mathcal{K}(\mathbb{R}^d)$, $f_C : \mathbb{R}^d \to \mathbb{R}$ is a function defined by

$$f_C(y) = \begin{cases} 1 & \text{if } y \in C \\ 0 & \text{if } y \notin C. \end{cases} \tag{3.4}$$

Then $\int f_C dT = T(C)$.

Indeed, for every $\alpha \in (0, 1]$ we have $\{y \in \mathbb{R}^d : \ f(y) \geq \alpha\} = C$. Then by (3.1) we have

$$\begin{aligned}\int f_C dT &= \int_0^1 T(\{y \in \mathbb{R}^d : \ f(y) \geq t\})dt \\ &= \int_0^1 T(C)dt = T(C).\end{aligned}$$

Claim 3.2. Under the condition (3.3), we have

$$\text{supp } T = \{x\}.$$

Indeed, if it is not the case, then $T(G) = \delta > 0$ for some open set $G \subset \mathbb{R}^d \setminus \{x\}$. By Definition 2.1 we can find a compact $C \subset G$, such that $T(C) > \dfrac{\delta}{2}$. Let $f_{C,G} : \mathbb{R}^d \to [0, 1]$ be a continuous function such that

$$f_{C,G}(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin G. \end{cases} \tag{3.5}$$

Such a function exists by the Urysohn-Tietze Theorem. Then $f_{C,G} \in \mathbb{R}^d$. Since $x \notin G$, by (3.3) we have

$$0 = f_{C,G}(x) = \int f_{C,G} dT \geq \int f_C dT = T(C) > \frac{\delta}{2} > 0,$$

a contradiction. Therefore

$$\text{supp } T = \{x\}.$$

Now we are able to complete the proof of the theorem.

By Definition 2.1, for every $\varepsilon > 0$ there is an open set $V \ni x$ such that

$$T(V) < T(\{x\}) + \varepsilon.$$

We have

$$T(\{x\}) \leq \int f_{\{x\},V} dT = f_{\{x\},V}(x) = 1 \leq T(V) < T(\{x\}) + \varepsilon,$$

which implies $T(\{x\}) = 1$, and so $T = T_x$. □

Using the arguments in the proof of Theorem 3.1 we obtain the following more general result.

Theorem 3.2. *For a compact set C in $\mathbb{R}^d$, define*

$$T_C(A) = \begin{cases} 1 & \text{if } A \cap C \neq \emptyset \\ 0 & \text{if } A \cap C = \emptyset, \end{cases}$$

with $A \in \mathcal{B}(\mathbb{R}^d)$. Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$, it holds

$$\int f dT_C = \sup\{f(x) : \ x \in C\}.$$

Conversely, if T is a capacity in $\mathbb{R}^d$ such that for some compact set $C \in \mathcal{K}(\mathbb{R}^d)$

$$\int f dT = \sup\{f(x) : \ x \in C\} \quad \text{for every } f \in C_0^+(\mathbb{R}^d), \tag{3.6}$$

then $T = T_C$.

Proof. Let $M := \sup\{f(x) : \ x \in C\}$. Then we have

$$\begin{aligned} \int f dT_C &= \int_0^M T_C(\{x \in \mathbb{R}^d : \ f(x) \geq t\}) dt + \int_M^\infty T_C(\{x \in \mathbb{R}^d : \ f(x) \geq t\}) dt \\ &= \int_0^M T_C(\{x \in \mathbb{R}^d : \ f(x) \geq t\}) dt \\ &= \int_0^M dt = M. \end{aligned}$$

To obtain the second assertion we observe that

$$T(K) = T_C(K) \quad \text{for every } K \in \mathcal{K}(\mathbb{R}^d).$$

In fact, if $K \cap C = \emptyset$ then $T_C(K) = 0$. There is an open set $G_1 \supset K$ and $G_1 \cap C = \emptyset$. Let f_{K,G_1} be the function defined by (3.5),

$$T(K) = \int f_K dT \leq \int f_{K,G_1} dT = \sup\{f_{K,G_1}(x) :\ x \in C\} = 0.$$

If $K \cap C \neq \emptyset$ then $T_C(K) = 1$. For any $\varepsilon > 0$, there is an open set $G_2 \supset K$ such that $T(G_2) < T(K) + \varepsilon$. With f_{K,G_2} defined by (3.5) we have

$$T(K) \leq \int f_{K,G_2} dT = \sup\{f_{K,G_2}(x) :\ x \in C\} = 1 \leq T(G_2) < T(K) + \varepsilon,$$

which implies $T(K) = 1$.

Therefore, by Definition 2.1,

$$\begin{aligned} T(A) &= \sup\{T(K) :\ K \in \mathcal{K}(\mathbb{R}^d),\ K \subset A\} \\ &= \sup\{T_C(K) :\ K \in \mathcal{K}(\mathbb{R}^d),\ K \subset A\} \\ &= T_C(A) \end{aligned}$$

for every Borel set A. □

Remark 3.1. Under the condition (3.6), we have

$$\operatorname{supp} T = C.$$

Indeed, for every compact set $K \subset \mathbb{R}^d \setminus C$, let G be an open set such that $K \subset G \subset \mathbb{R}^d \setminus C$. Then we have

$$T(K) \leq \int f_{K,G} dT = \sup\{f_{K,G}(x) :\ x \in C\} = 0.$$

Hence

$$T(\mathbb{R}^d \setminus C) = \sup\{T(K);\ K \in \mathcal{K}(\mathbb{R}^d),\ K \subset \mathbb{R}^d \setminus C\} = 0.$$

Therefore, by the definition of support, we have $\operatorname{supp} T \subset C$. To obtain the reverse inclusion, we first observe that

$$\text{if } T(G) = 0 \text{ then } G \subset \mathbb{R}^d \setminus C \text{ for every Borel set } G \in \mathcal{B}(\mathbb{R}^d).$$

In fact, if $G \cap C \neq \emptyset$, let $x \in G \cap C$. For any $\varepsilon > 0$, there is an open set $V \ni x$ such that $T(\{x\}) > T(V) - \varepsilon$. Let $f_{\{x\},V}$ be the function defined by (3.5),

$$\begin{aligned} T(G) \geq T(\{x\}) > T(V) - \varepsilon &\geq \int f_{\{x\},V} dT - \varepsilon \\ &= \sup\{f_{\{x\},V}(y) :\ y \in C\} - \varepsilon = 1 - \varepsilon > 0, \end{aligned}$$

a contradiction. Hence

$$\operatorname{supp} T = \mathbb{R}^d \setminus \bigcup\{T(G);\ G \in \mathcal{G}(T)\} \supset \mathbb{R}^d \setminus (\mathbb{R}^d \setminus C) = C,$$

where

$$\mathcal{G}(T) = \{G : \ G \in \mathcal{G}(\mathbb{R}^d), \ T(G) = 0\}.$$

Theorem 3.3. *Let T_A be the capacity defined in Example 2.2, where*

$$A = \{(x_1, t_1), \dots, (x_k, t_k)\} \subset \mathbb{R}^d \times \mathbb{R}^+. \tag{3.7}$$

Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$ we have

$$\int f dT_A = \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \max\{t_{n_j} : \ j = i+1, \dots, k\}, \tag{3.8}$$

where $\{x_{n_i} : \ i = 1, \dots, k\} = \{x_i : \ i = 1, \dots, k\}$ with

$$\alpha_0 = 0 \le \alpha_1 = f(x_{n_1}) \le \alpha_2 = f(x_{n_2}) \le \dots \le \alpha_k = f(x_{n_k}). \tag{3.9}$$

This means that we reorder the indices of x_i, $i = 1, \dots, k$ to get (3.9).

Conversely, if T is a capacity in $\mathbb{R}^d$ such that (3.8) *holds for any measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$, then $T = T_A$, where A is given by* (3.7).

Proof. Observe that

$$\int f dT_A = \int_0^\infty T_A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) dt$$
$$= \int_0^\alpha T_A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) dt,$$

where $\alpha = \alpha_k = \max\{f(x_i) : \ i = 1, \dots, k\}$.

By the definition of T_A in Example 2.2, $T_A(\{y \in \mathbb{R}^d : \ f(y) \ge t\})$ is a step function in t given by

$$T_A(\{x : \ f(x) \ge t\}) = \max\{t_{n_j} : \ j = i+1, \dots, k\} \quad \text{for } t \in (\alpha_i, \alpha_{i+1}],$$

where $\alpha_i = f(x_{n_i})$, $i = 0, 1, \dots, k$ are chosen to satisfy (3.9).

It follows that

$$\int f dT_A = \int_0^\alpha T_A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) dt$$
$$= \sum_{i=0}^{k-1} \int_{\alpha_i}^{\alpha_{i+1}} \max\{t_{n_j} : \ j = i+1, \dots, k\} dt$$
$$= \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \max\{t_{n_j} : \ j = i+1, \dots, k\},$$

so (3.8) is valid.

Conversely, to have $T = T_A$ we need to show that

$$T_A(K) = T(K) \quad \text{for any } K \in \mathcal{K}(\mathbb{R}^d).$$

Assume that f_K is defined as in (3.4). Without loss of gerality we may assume that

$$f_K(x_1) \leq f_K(x_2) \leq \cdots \leq f_K(x_k).$$

We put

$$\alpha_i = f_K(x_i) \quad \text{for } i = 1, \dots, k$$

and consider the following two cases:

a) $A_0 \cap K = \emptyset$. Then $T_A(K) = 0$ and

$$T(K) = \int f_K dT = \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \max\{t_j : \ j = i+1, \dots, k\} = 0.$$

b) $A_0 \cap K \neq \emptyset$. Let i be the smallest number that $x_i \in K \cap A_0$. Then

$$f_K(x_i) = 1 \leq f_K(x_j) \quad \text{for every } j = i+1, \dots, k.$$

Hence

$$x_j \in A_0 \cap K \quad \text{for every } j = i+1, \dots, k.$$

Therefore

$$\alpha_1 = \cdots = \alpha_{i-1} = 0 \text{ and } \alpha_i = \cdots \alpha_k = 1.$$

By the definition of T_A we have

$$\begin{aligned} T_A(K) &= \max\{t_i : \ x_i \in A_0 \cap K\} \\ &= \max\{t_j : \ j = i, \dots, k\}. \end{aligned}$$

By Claim 3.1 we have

$$\begin{aligned} T(K) &= \int f_K dT \\ &= \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \max\{t_j : \ j = i, \dots, k\} \\ &= (\alpha_i - \alpha_{i-1}) \max\{t_j : \ j = i, \dots, k\} \\ &= \max\{t_j : \ j = 1, \dots, k\} \\ &= T_A(K). \end{aligned}$$

□

Theorem 3.4. *Let* $T^A = \sum_{i=1}^{k} t_i \delta_{x_i}$ *be the capacity defined in Example 2.3, where*

$$A = \{(x_1, t_1), \dots, (x_k, t_k)\} \subset \mathbb{R}^d \times \mathbb{R}^+.$$

Then for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int f dT^A = \sum_{i=1}^{k} t_i f(x_i).$$

Proof. Without loss of generality we may assume that

$$f(x_1) \le f(x_2) \le \cdots \le f(x_k).$$

For every $i = 1, \ldots, k$, let

$$a_i := \sum_{n=i}^{k} t_n.$$

Then

$$T^A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) = a_{i+1} \quad \text{for } t \in (f(x_i), f(x_{i+1})].$$

Therefore

$$\begin{aligned}
\int f dT^A &= \int_0^{f(x_k)} T^A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) dt \\
&= \sum_{i=0}^{k-1} \int_{f(x_i)}^{f(x_{i+1})} T^A(\{y \in \mathbb{R}^d : \ f(y) \ge t\}) dt \\
&= \sum_{i=0}^{k-1} \int_{f(x_i)}^{f(x_{i+1})} a_{i+1} dt = \sum_{i=0}^{k-1} a_{i+1}(f(x_{i+1}) - f(x_i)) \\
&= \sum_{i=1}^{k-1} (a_i - a_{i+1}) f(x_i) + a_k f(x_k) = \sum_{i=1}^{k} t_i f(x_i),
\end{aligned}$$

where $f(x_0) = 0$. □

4. Capacities in the sense of Graf

In [2] Graf has introduced the class of capacities as follows.

Definition 4.1. Let $(U, \mathcal{U})$ is a measurable space, i.e. U is a set and $\mathcal{U}$ is a σ-field of subsets of U. A map $\nu : \ \mathcal{U} \to \mathbb{R}^+$ is called a *capacitiy* if the following conditions hold

1. $\nu(\emptyset) = 0$
2. For $A, B \in \mathcal{U}, \ \nu(A \cup B) \le \nu(A) + \nu(B)$
3. For $A, B \in \mathcal{U}, \ A \subset B$ implies $\nu(A) \le \nu(B)$
4. $\nu\big(\bigcup_{n=1}^{\infty} A_n\big) = \lim_{n \to \infty} \nu(A_n)$ for any increasing sequence $\{A_n\} \subset \mathcal{U}$.

The following example shows that the class of capacities in the sense of Graf is different from the class of capacities in $\mathbb{R}^d$ introduced in Section 2.

Example 4.1. ([5]) We consider the monotone set-function $\nu: \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by

$$\nu(A) = \begin{cases} 0 & \text{if } A \cap \mathbb{N} = \emptyset \\ \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in A \cap \mathbb{N}\} & \text{if } A \cap \mathbb{N} \neq \emptyset \end{cases}$$

for $A \in \mathcal{B}(\mathbb{R})$, where $\mathbb{N}$ denotes the set of all positive integer numbers.

It was shown in [5] that ν is not alternating of infinite order. Therefore, ν is not a capacity in $\mathbb{R}$. However, ν is a capacity in the sense of Graf. Indeed, we have

1. $\nu(\emptyset) = 0$
2. For $A, B \in \mathcal{B}(\mathbb{R})$ consider the following cases.

Case 1. $A \cap \mathbb{N} = \emptyset$ and $B \cap \mathbb{N} \neq \emptyset$. Then

$$\begin{aligned} \nu(A \cup B) &= \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in (A \cup B) \cap \mathbb{N}\}\} \\ &= \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in B \cap \mathbb{N}\}\} \\ &= \nu(B) = \nu(A) + \nu(B). \end{aligned}$$

Case 2. $A \cap \mathbb{N} \neq \emptyset$ and $B \cap \mathbb{N} \neq \emptyset$. Then

$$\begin{aligned} \nu(A \cup B) &= \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in (A \cup B) \cap \mathbb{N}\}\} \\ &\le \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in A \cap \mathbb{N}\}\} \\ &\quad + \min\{1, 2^{-1} \sum\{x^{-1}: \ x \in B \cap \mathbb{N}\}\} \\ &= \nu(A) + \nu(B). \end{aligned}$$

3. Let $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ be a increasing sequence of sets in $\mathcal{B}(\mathbb{R})$. Putting

$$A = \bigcup_{n=1}^{\infty} A_n,$$

we will show that

$$\nu(A) = \lim_{n \to \infty} \nu(A_n).$$

Let

$$\mu(A) = \begin{cases} 0 & \text{if } A \cap \mathbb{N} = \emptyset \\ 2^{-1} \sum\{x^{-1}: \ x \in A \cap \mathbb{N}\} & \text{if } A \cap \mathbb{N} \neq \emptyset \end{cases}$$

for $A \in \mathcal{B}(\mathbb{R})$. Then μ is a measure on $\mathcal{B}(\mathbb{R})$ and $\nu(A) \le \mu(A)$ for every $A \in \mathcal{B}(\mathbb{R})$. It is straightforward to check that $\mu(A) = \nu(A)$ if $\mu(A) \le 1$ or $\nu(A) < 1$. Consider two cases:

Case 1. There exists $n_0 \in \mathbb{N}$ such that $\nu(A_{n_0}) = 1$. Then $\nu(A_n) = 1$ for every $n \geq n_0$. Hence

$$\nu(A) = 1 = \lim_{n\to\infty} \nu(A_n).$$

Case 2. $\nu(A_n) < 1$ for every $n \in \mathbb{N}$. Then

$$\nu(A_n) = \mu(A_n) \text{ for every } n \in \mathbb{N}.$$

Therefore

$$\lim_{n\to\infty} \nu(A_n) = \lim_{n\to\infty} \mu(A_n) = \mu(A) \leq 1.$$

Consequently,

$$\nu(A) = \mu(A) = \lim_{n\to\infty} \nu(A_n).$$

Thus, the conditions (1) - (4) of Definition 4.1 are satisfiied. So ν is a capacity in the sense of Graf.

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